Preface

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Format

You can access this textbook for free in web view or PDF through OpenStax.org, and for a low cost in print.

About *University Physics*

University Physics is designed for the two- or three-semester calculus-based physics course. The text has been developed to meet the scope and sequence of most university physics courses and provides a foundation for a career in mathematics, science, or engineering. The book provides an important opportunity for students to learn the core concepts of physics and understand how those concepts apply to their lives and to the world around them.

Due to the comprehensive nature of the material, we are offering the book in three volumes for flexibility and efficiency.

Coverage and scope

Our *University Physics* textbook adheres to the scope and sequence of most two- and three-semester physics courses nationwide. We have worked to make physics interesting and accessible to students while maintaining the mathematical rigor inherent in the subject. With this objective in mind, the content of this textbook has been developed and arranged to provide a logical progression from fundamental to more advanced concepts, building upon what students have already learned and emphasizing connections between topics and between theory and applications. The goal of each section is to enable students not just to recognize concepts, but to work with them in ways that will be useful in later courses and future careers. The organization and pedagogical features were developed and vetted with feedback from science educators dedicated to the project.

VOLUME I

Unit 1: Mechanics

Unit 2: Waves and Acoustics

Chapter 6: Photons and Matter Waves
Chapter 7: Quantum Mechanics
Chapter 8: Atomic Structure
Chapter 9: Condensed Matter Physics
Chapter 10: Nuclear Physics
Chapter 11: Particle Physics and Cosmology

Pedagogical foundation

Throughout *University Physics* you will find derivations of concepts that present classical ideas and techniques, as well as modern applications and methods. Most chapters start with observations or experiments that place the material in a context of physical experience. Presentations and explanations rely on years of classroom experience on the part of long-time physics professors, striving for a balance of clarity and rigor that has proven successful with their students. Throughout the text, links enable students to review earlier material and then return to the present discussion, reinforcing connections between topics. Key historical figures and experiments are discussed in the main text (rather than in boxes or sidebars), maintaining a focus on the development of physical intuition. Key ideas, definitions, and equations are highlighted in the text and listed in summary form at the end of each chapter. Examples and chapter-opening images often include contemporary applications from daily life or modern science and engineering that students can relate to, from smart phones to the internet to GPS devices.

Assessments that reinforce key concepts

In-chapter **Examples** generally follow a three-part format of Strategy, Solution, and Significance to emphasize how to approach a problem, how to work with the equations, and how to check and generalize the result. Examples are often followed by **Check Your Understanding** questions and answers to help reinforce for students the important ideas of the examples. **Problem-Solving Strategies** in each chapter break down methods of

approaching various types of problems into steps students can follow for guidance. The book also includes exercises at the end of each chapter so students can practice what they've learned.

- **Conceptual questions** do not require calculation but test student learning of the key concepts.
- **Problems** categorized by section test student problem-solving skills and the ability to apply ideas to practical situations.
- **Additional Problems** apply knowledge across the chapter, forcing students to identify what concepts and equations are appropriate for solving given problems. Randomly located throughout the problems are **Unreasonable Results** exercises that ask students to evaluate the answer to a problem and explain why it is not reasonable and what assumptions made might not be correct.
- **Challenge Problems** extend text ideas to interesting but difficult situations.

Answers for selected exercises are available in an **Answer Key** at the end of the book.

Additional resources

Student and instructor resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, PowerPoint slides, and answer and solution guides for instructors and students. Instructor resources require a verified instructor account, which you can apply for when you log in or create your account on OpenStax.org. Take advantage of these resources to supplement your OpenStax book.

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Dr. Jeff Sanny earned a BS in Physics from Harvey Mudd College in 1974 and a PhD in Solid State Physics from the University of California–Los Angeles in 1980. He joined the faculty at Loyola Marymount University in the fall of 1980. During his tenure, he has served as department Chair as well as Associate Dean. Dr. Sanny enjoys teaching introductory physics in particular. He is also passionate about providing students with research experience and has directed an active undergraduate student research group in space physics for many years.

William Moebs, Formerly of Loyola Marymount University

Dr. William Moebs earned a BS and PhD (1959 and 1965) from the University of Michigan. He then joined their staff as a Research Associate for one year, where he continued his doctoral research in particle physics. In 1966, he accepted an appointment to the Physics Department of Indiana Purdue Fort Wayne (IPFW), where he served as Department Chair from 1971 to 1979. In 1979, he moved to Loyola Marymount University (LMU), where he served as Chair of the Physics Department from 1979 to 1986. He retired from LMU in 2000. He has published research in particle physics, chemical kinetics, cell division, atomic physics, and physics teaching.

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Introduction
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Electric charges exist all around us. They can cause objects to be repelled from each other or to be attracted to each other. (credit: modification of work by Sean McGrath)

Back when we were studying Newton’s laws, we identified several physical phenomena as forces. We did so based on the effect they had on a physical object: Specifically, they caused the object to accelerate. Later, when we studied impulse and momentum, we expanded this idea to identify a force as any physical phenomenon that changed the momentum of an object. In either case, the result is the same: We recognize a force by the effect that it has on an object.

In Gravitation, we examined the force of gravity, which acts on all objects with mass. In this chapter, we begin the study of the electric force, which acts on all objects with a property called charge. The electric force is much stronger than gravity (in most systems where both appear), but it can be a force of attraction or a force of repulsion, which leads to very different effects on objects. The electric force helps keep atoms together, so it is of fundamental importance in matter. But it also governs most everyday interactions we deal with, from chemical interactions to biological processes.

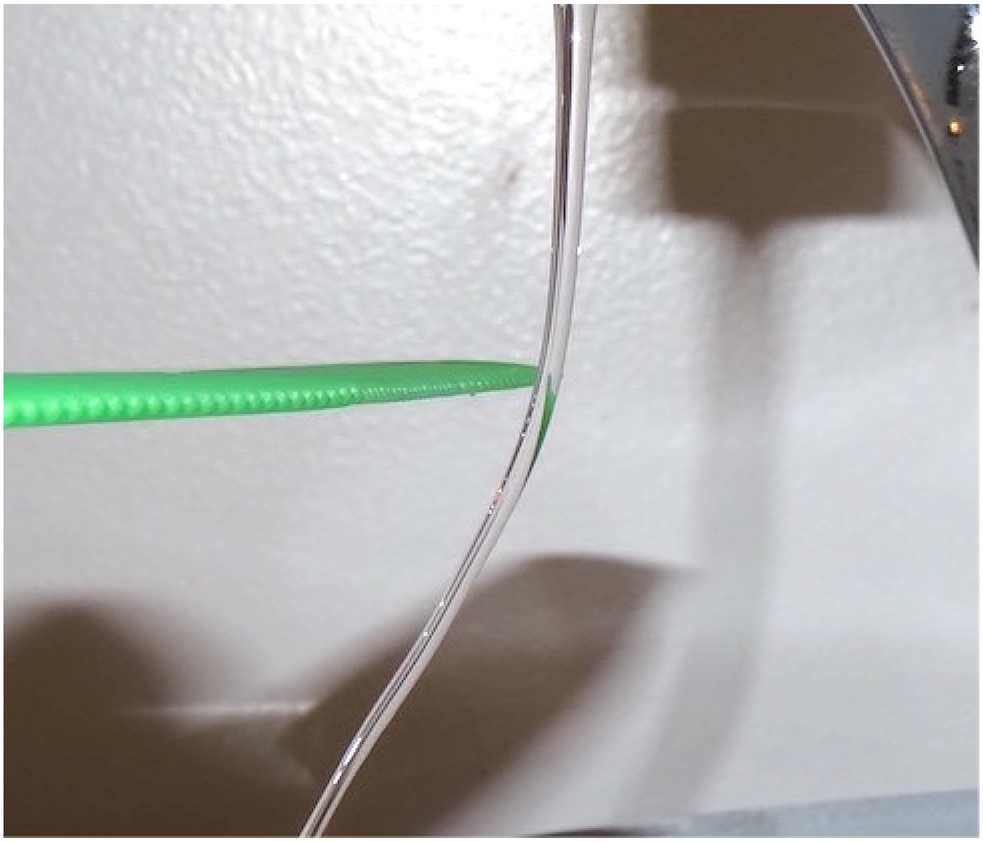
Electric Charge
By the end of this section, you will be able to:

- Describe the concept of electric charge
- Explain qualitatively the force electric charge creates

You are certainly familiar with electronic devices that you activate with the click of a switch, from computers to cell phones to television. And you have certainly seen electricity in a flash of lightning during a heavy thunderstorm. But you have also most likely experienced electrical effects in other ways, maybe without realizing that an electric force was involved. Let’s take a look at some of these activities and see what we can learn from them about electric charges and forces.

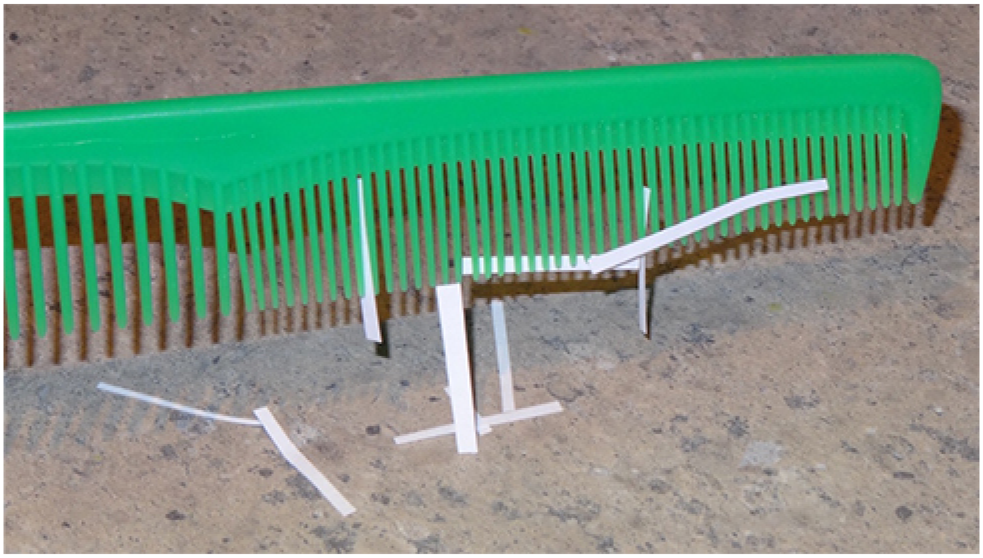
Discoveries

You have probably experienced the phenomenon of **static electricity**: When you first take clothes out of a dryer, many (not all) of them tend to stick together; for some fabrics, they can be very difficult to separate. Another example occurs if you take a woolen sweater off quickly—you can feel (and hear) the static electricity pulling on your clothes, and perhaps even your hair. If you comb your hair on a dry day and then put the comb close to a thin stream of water coming out of a faucet, you will find that the water stream bends toward (is attracted to) the comb ([link]).



An electrically charged comb attracts a stream of water from a distance. Note that the water is not touching the comb. (credit: Jane Whitney)

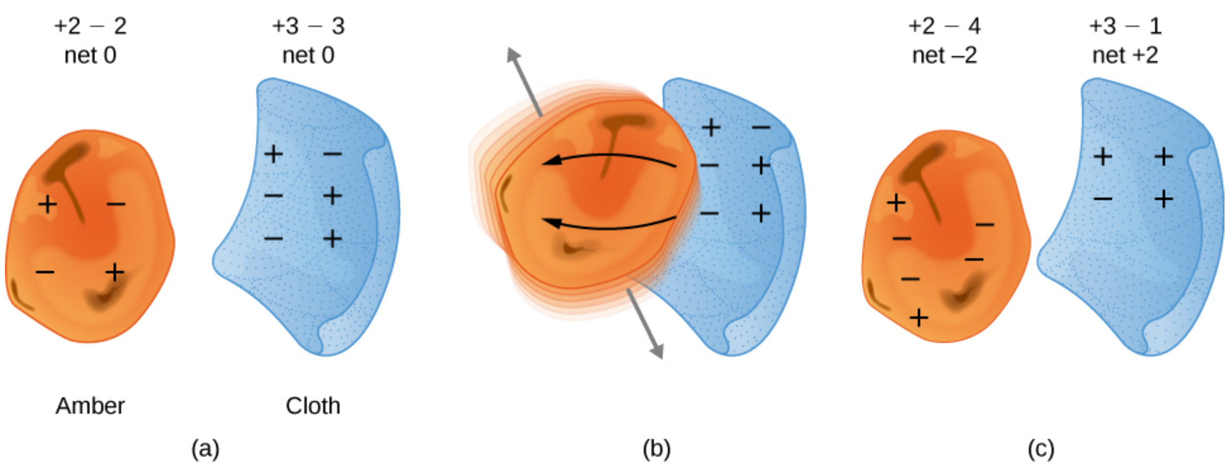
Suppose you bring the comb close to some small strips of paper; the strips of paper are attracted to the comb and even cling to it ([link]). In the kitchen, quickly pull a length of plastic cling wrap off the roll; it will tend to cling to most any nonmetallic material (such as plastic, glass, or food). If you rub a balloon on a wall for a few seconds, it will stick to the wall. Probably the most annoying effect of static electricity is getting shocked by a doorknob (or a friend) after shuffling your feet on some types of carpeting.



After being used to comb hair, this comb attracts small strips of paper from a distance, without physical contact. Investigation of this behavior helped lead to the concept of the electric force. (credit: Jane Whitney)

Many of these phenomena have been known for centuries. The ancient Greek philosopher Thales of Miletus (624–546 BCE) recorded that when amber (a hard, translucent, fossilized resin from extinct trees) was vigorously rubbed with a piece of fur, a force was created that caused the fur and the amber to be attracted to each other ([link]). Additionally, he found that the rubbed amber would not only attract the fur, and the fur attract the amber, but they both could affect other (nonmetallic) objects, even if not in contact with those objects ([link]).

Borneo amber is mined in Sabah, Malaysia, from shale-sandstone-mudstone veins. When a piece of amber is rubbed with a piece of fur, the amber gains more electrons, giving it a net negative charge. At the same time, the fur, having lost electrons, becomes positively charged. (credit: "Sebakoamber"/Wikimedia Commons)



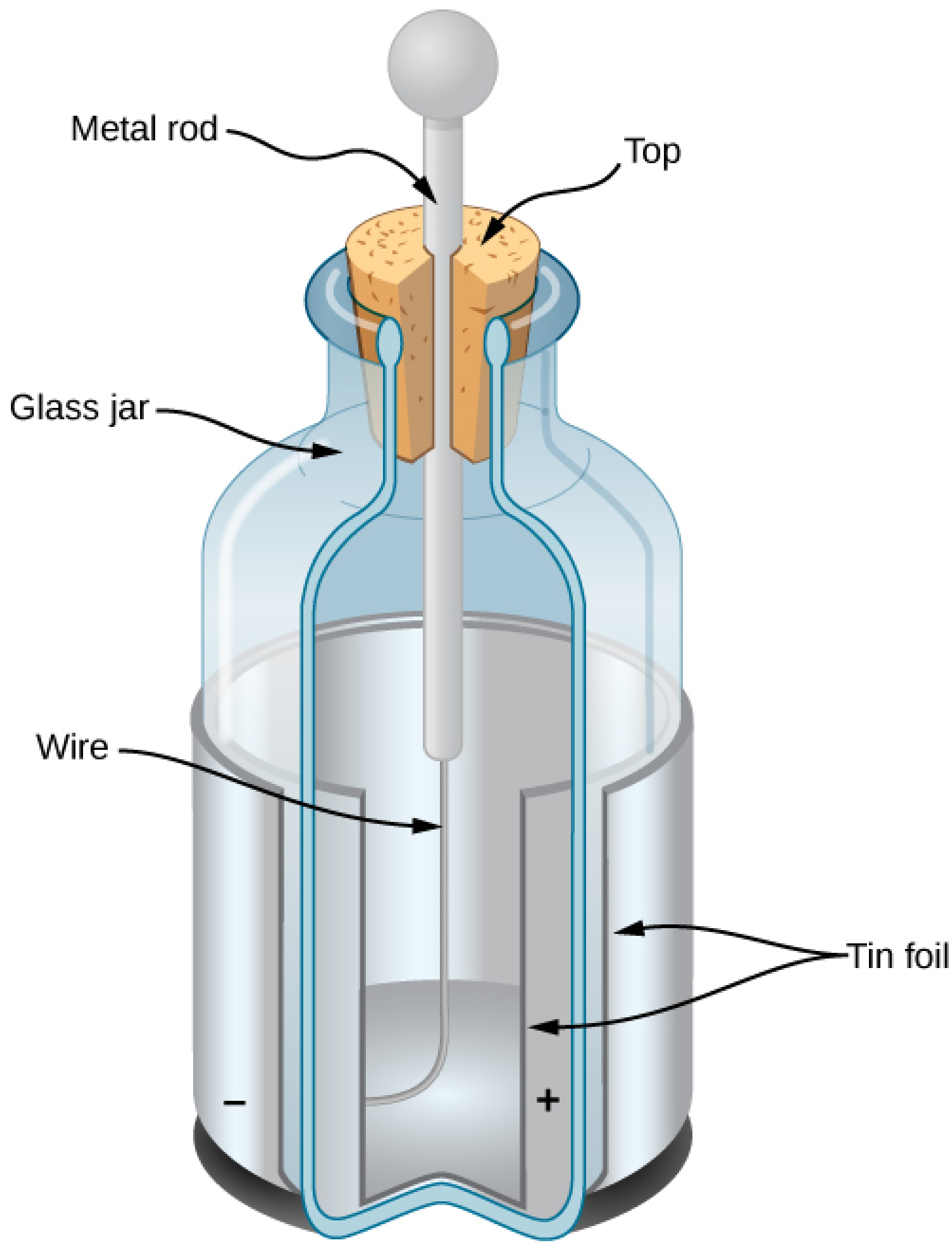
When materials are rubbed together, charges can be separated, particularly if one material has a greater affinity for electrons than another. (a) Both the amber and cloth are originally neutral, with equal positive and negative charges. Only a tiny fraction of the charges are involved, and only a few of them are shown here. (b) When rubbed together, some negative charge is transferred to the amber, leaving the cloth with a net positive charge. (c) When separated, the amber and cloth now have net charges, but the absolute value of the net positive and negative charges will be equal.

The English physicist William Gilbert (1544–1603) also studied this attractive force, using various substances. He worked with amber, and, in addition, he experimented with rock crystal and various precious and semi-precious gemstones. He also experimented with several metals. He found that the metals never exhibited this force, whereas the minerals did. Moreover, although an electrified amber rod would attract a piece of fur, it would repel another electrified amber rod; similarly, two electrified pieces of fur would repel each other.

This suggested there were two types of an electric property; this property eventually came to be called **electric charge**. The difference between the two types of electric charge is in the directions of the electric forces that each type of charge causes: These forces are repulsive when the same type of charge exists on two interacting objects and attractive when the charges are of opposite types. The SI unit of electric charge is the **coulomb** (C), after the French physicist Charles-Augustin de Coulomb (1736–1806).

The most peculiar aspect of this new force is that it does not require physical contact between the two objects in order to cause an acceleration. This is an example of a so-called “long-range” force. (Or, as James Clerk Maxwell later phrased it, “action at a distance.”) With the exception of gravity, all other forces we have discussed so far act only when the two interacting objects actually touch.

The American physicist and statesman Benjamin Franklin found that he could concentrate charge in a “Leyden jar,” which was essentially a glass jar with two sheets of metal foil, one inside and one outside, with the glass between them ([link]). This created a large electric force between the two foil sheets.



A Leyden jar (an early version of what is now called a capacitor) allowed experimenters to store large amounts of electric charge. Benjamin Franklin used such a jar to demonstrate that lightning behaved exactly like the electricity he got from the equipment in his laboratory.

Franklin pointed out that the observed behavior could be explained by supposing that one of the two types of charge remained motionless, while the other type of charge flowed from one piece of foil to the other. He further suggested that an excess of what he called this “electrical fluid” be called “positive electricity” and the deficiency of it be called “negative electricity.” His suggestion, with some minor modifications, is the model we use today. (With the experiments that he was able to do, this was a pure guess; he had no way of actually determining the sign of the moving charge. Unfortunately, he guessed wrong; we now know that the charges that flow are the ones Franklin labeled negative, and the positive charges remain largely motionless. Fortunately, as we’ll see, it makes no practical or theoretical difference which choice we make, as long as we stay consistent with our choice.)

Let’s list the specific observations that we have of this **electric force**:

- The force acts without physical contact between the two objects.
- The force can be either attractive or repulsive: If two interacting objects carry the same sign of charge, the force is repulsive; if the charges are of opposite sign, the force is attractive. These interactions are referred to as **electrostatic repulsion** and **electrostatic attraction**, respectively.
- Not all objects are affected by this force.
- The magnitude of the force decreases (rapidly) with increasing separation distance between the objects.

To be more precise, we find experimentally that the magnitude of the force decreases as the square of the distance between the two interacting objects increases. Thus, for example, when the distance between two interacting

objects is doubled, the force between them decreases to one fourth what it was in the original system. We can also observe that the surroundings of the charged objects affect the magnitude of the force. However, we will explore this issue in a later chapter.

Properties of Electric Charge

In addition to the existence of two types of charge, several other properties of charge have been discovered.

- **Charge is quantized.** This means that electric charge comes in discrete amounts, and there is a smallest possible amount of charge that an object can have. In the SI system, this smallest amount is $e \equiv 1.602 \times 10^{-19}$ C. No free particle can have less charge than this, and, therefore, the charge on any object—the charge on all objects—must be an integer multiple of this amount. All macroscopic, charged objects have charge because electrons have either been added or taken away from them, resulting in a net charge.
- **The magnitude of the charge is independent of the type.** Phrased another way, the smallest possible positive charge (to four significant figures) is $+1.602 \times 10^{-19}$ C, and the smallest possible negative charge is -1.602×10^{-19} C; these values are exactly equal. This is simply how the laws of physics in our universe turned out.
- **Charge is conserved.** Charge can neither be created nor destroyed; it can only be transferred from place to place, from one object to another. Frequently, we speak of two charges "canceling"; this is verbal shorthand. It means that if two objects that have equal and opposite charges are physically close to each other, then the (oppositely directed) forces they apply on some other charged object cancel, for a net force of zero. It is important that you understand that the charges on the objects by no means disappear, however. The net charge of the universe is constant.
- **Charge is conserved in closed systems.** In principle, if a negative charge disappeared from your lab bench and reappeared on the Moon, conservation of charge would still hold. However, this never happens. If the total charge you have in your local system on your lab bench is changing, there will be a measurable flow of charge into or out of the

system. Again, charges can and do move around, and their effects can and do cancel, but the net charge in your local environment (if closed) is conserved. The last two items are both referred to as the **law of conservation of charge**.

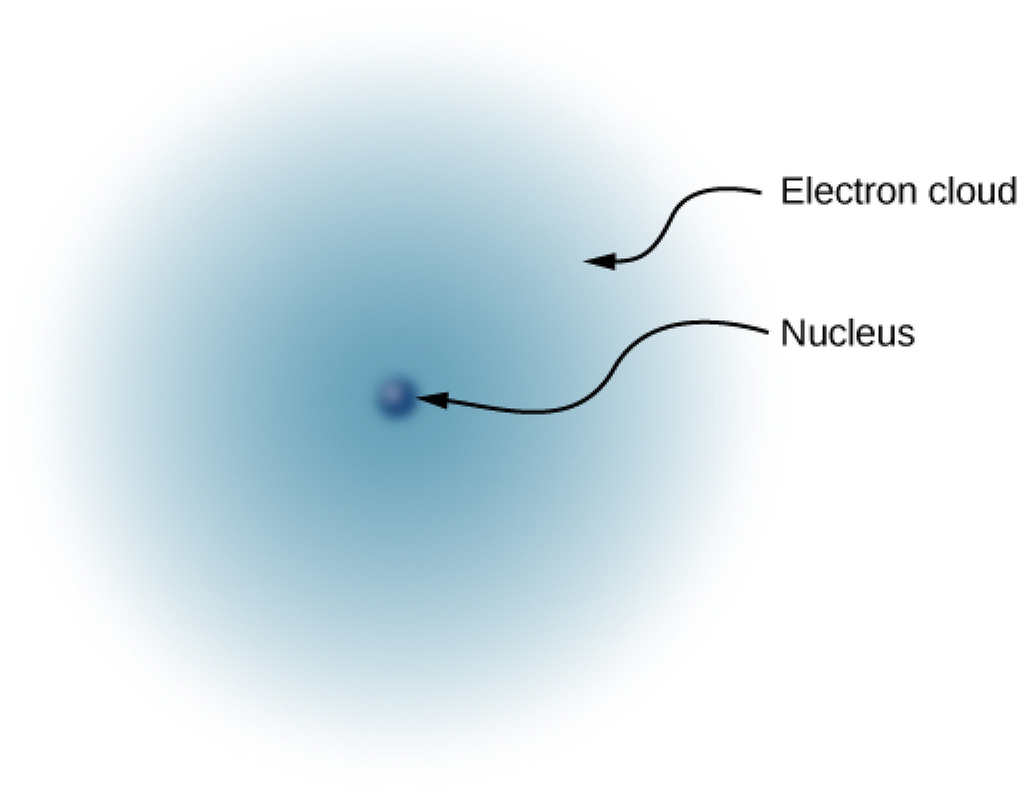
The Source of Charges: The Structure of the Atom

Once it became clear that all matter was composed of particles that came to be called atoms, it also quickly became clear that the constituents of the atom included both positively charged particles and negatively charged particles. The next question was, what are the physical properties of those electrically charged particles?

The negatively charged particle was the first one to be discovered. In 1897, the English physicist J. J. Thomson was studying what was then known as *cathode rays*. Some years before, the English physicist William Crookes had shown that these "rays" were negatively charged, but his experiments were unable to tell any more than that. (The fact that they carried a negative electric charge was strong evidence that these were not rays at all, but particles.) Thomson prepared a pure beam of these particles and sent them through crossed electric and magnetic fields, and adjusted the various field strengths until the net deflection of the beam was zero. With this experiment, he was able to determine the charge-to-mass ratio of the particle. This ratio showed that the mass of the particle was much smaller than that of any other previously known particle—1837 times smaller, in fact. Eventually, this particle came to be called the **electron**.

Since the atom as a whole is electrically neutral, the next question was to determine how the positive and negative charges are distributed within the atom. Thomson himself imagined that his electrons were embedded within a sort of positively charged paste, smeared out throughout the volume of the atom. However, in 1908, the New Zealand physicist Ernest Rutherford showed that the positive charges of the atom existed within a tiny core—called a nucleus—that took up only a very tiny fraction of the overall volume of the atom, but held over 99% of the mass. (See Linear Momentum and Collisions.) In addition, he showed that the negatively charged electrons perpetually orbited about this nucleus, forming a sort of

electrically charged cloud that surrounds the nucleus ([link]). Rutherford concluded that the nucleus was constructed of small, massive particles that he named **protons**.

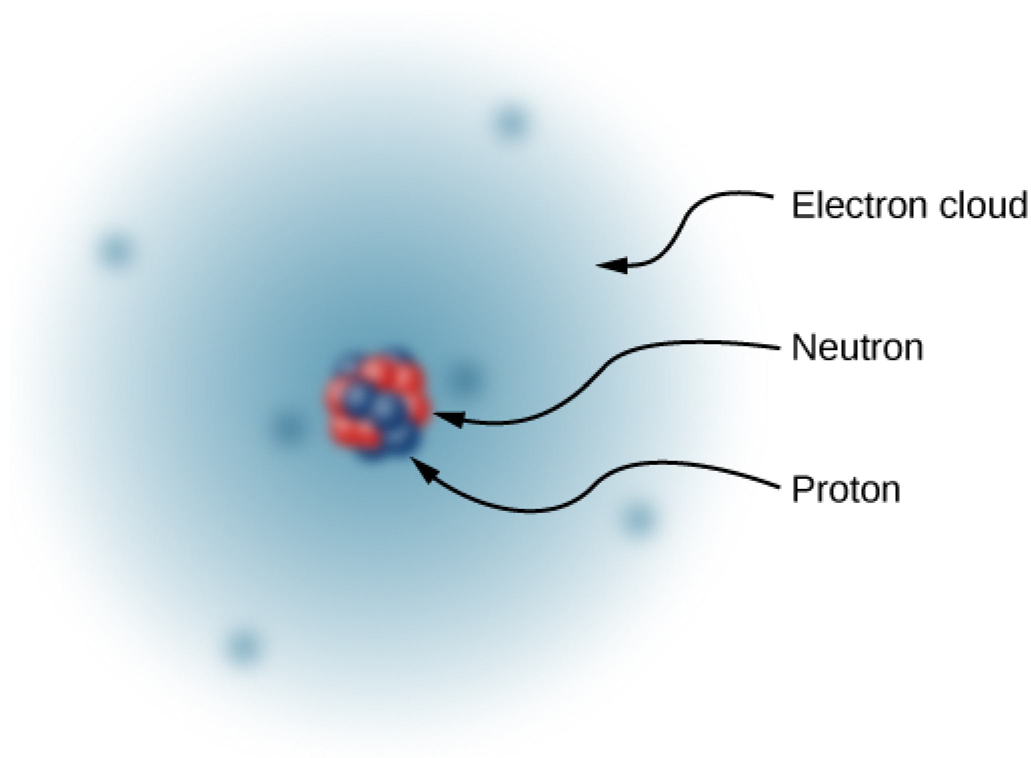


This simplified model of a hydrogen atom shows a positively charged nucleus (consisting, in the case of hydrogen, of a single proton), surrounded by an electron “cloud.” The charge of the electron cloud is equal (and opposite in sign) to the charge of the nucleus, but the electron does not have a definite location in space; hence, its representation here is as a cloud. Normal macroscopic amounts of matter contain immense numbers of atoms and molecules, and, hence, even greater numbers of individual negative and positive charges.

Since it was known that different atoms have different masses, and that ordinarily atoms are electrically neutral, it was natural to suppose that different atoms have different numbers of protons in their nucleus, with an equal number of negatively charged electrons orbiting about the positively charged nucleus, thus making the atoms overall electrically neutral. However, it was soon discovered that although the lightest atom, hydrogen, did indeed have a single proton as its nucleus, the next heaviest atom—helium—has twice the number of protons (two), but *four* times the mass of hydrogen.

This mystery was resolved in 1932 by the English physicist James Chadwick, with the discovery of the **neutron**. The neutron is, essentially, an electrically neutral twin of the proton, with no electric charge, but (nearly) identical mass to the proton. The helium nucleus therefore has two neutrons along with its two protons. (Later experiments were to show that although the neutron is electrically neutral overall, it does have an internal charge *structure*. Furthermore, although the masses of the neutron and the proton are *nearly* equal, they aren't exactly equal: The neutron's mass is very slightly larger than the mass of the proton. That slight mass excess turned out to be of great importance. That, however, is a story that will have to wait until our study of modern physics in Nuclear Physics.)

Thus, in 1932, the picture of the atom was of a small, massive nucleus constructed of a combination of protons and neutrons, surrounded by a collection of electrons whose combined motion formed a sort of negatively charged "cloud" around the nucleus ([link]). In an electrically neutral atom, the total negative charge of the collection of electrons is equal to the total positive charge in the nucleus. The very low-mass electrons can be more or less easily removed or added to an atom, changing the net charge on the atom (though without changing its type). An atom that has had the charge altered in this way is called an **ion**. Positive ions have had electrons removed, whereas negative ions have had excess electrons added. We also use this term to describe molecules that are not electrically neutral.



The nucleus of a carbon atom is composed of six protons and six neutrons. As in hydrogen, the surrounding six electrons do not have definite locations and so can be considered to be a sort of cloud surrounding the nucleus.

The story of the atom does not stop there, however. In the latter part of the twentieth century, many more subatomic particles were discovered in the nucleus of the atom: pions, neutrinos, and quarks, among others. With the exception of the photon, none of these particles are directly relevant to the study of electromagnetism, so we defer further discussion of them until the chapter on particle physics (Particle Physics and Cosmology).

A Note on Terminology

As noted previously, electric charge is a property that an object can have. This is similar to how an object can have a property that we call mass, a property that we call density, a property that we call temperature, and so on.

Technically, we should always say something like, “Suppose we have a particle that carries a charge of $3\ \mu\text{C}$.” However, it is very common to say instead, “Suppose we have a $3\text{-}\mu\text{C}$ charge.” Similarly, we often say something like, “Six charges are located at the vertices of a regular hexagon.” A charge is not a particle; rather, it is a *property* of a particle. Nevertheless, this terminology is extremely common (and is frequently used in this book, as it is everywhere else). So, keep in the back of your mind what we really mean when we refer to a “charge.”

Summary

- There are only two types of charge, which we call positive and negative. Like charges repel, unlike charges attract, and the force between charges decreases with the square of the distance.
- The vast majority of positive charge in nature is carried by protons, whereas the vast majority of negative charge is carried by electrons. The electric charge of one electron is equal in magnitude and opposite in sign to the charge of one proton.
- An ion is an atom or molecule that has nonzero total charge due to having unequal numbers of electrons and protons.
- The SI unit for charge is the coulomb (C), with protons and electrons having charges of opposite sign but equal magnitude; the magnitude of this basic charge is $e \equiv 1.602\ \times\ 10^{-19}\,\text{C}$
- Both positive and negative charges exist in neutral objects and can be separated by bringing the two objects into physical contact; rubbing the objects together can remove electrons from the bonds in one object and place them on the other object, increasing the charge separation.
- For macroscopic objects, negatively charged means an excess of electrons and positively charged means a depletion of electrons.
- The law of conservation of charge states that the net charge of a closed system is constant.

Conceptual Questions

Exercise:

Problem:

There are very large numbers of charged particles in most objects. Why, then, don’t most objects exhibit static electricity?

Solution:

There are mostly equal numbers of positive and negative charges present, making the object electrically neutral.

Exercise:

Problem:

Why do most objects tend to contain nearly equal numbers of positive and negative charges?

Exercise:

Problem:

A positively charged rod attracts a small piece of cork. (a) Can we conclude that the cork is negatively charged? (b) The rod repels another small piece of cork. Can we conclude that this piece is positively charged?

Solution:

a. yes; b. yes

Exercise:

Problem:

Two bodies attract each other electrically. Do they both have to be charged? Answer the same question if the bodies repel one another.

Exercise:

Problem:

How would you determine whether the charge on a particular rod is positive or negative?

Solution:

Take an object with a known charge, either positive or negative, and bring it close to the rod. If the known charged object is positive and it is repelled from the rod, the rod is charged positive. If the positively charged object is attracted to the rod, the rod is negatively charged.

Problems

Exercise:

Problem:

Common static electricity involves charges ranging from nanocoulombs to microcoulombs. (a) How many electrons are needed to form a charge of −2.00 nC? (b) How many electrons must be removed from a neutral object to leave a net charge of $0.500\ \mu\text{C}$?

Solution:

a. $2.00\ \times\ 10^{-9}\,\text{C}\left(\frac{1}{1.602\times10^{-19}}\ \text{e/C}\right)=1.248\ \times\ 10^{10}\,\text{electrons;}$
b. $0.500\ \times\ 10^{-6}\,\text{C}\left(\frac{1}{1.602\times10^{-19}}\ \text{e/C}\right)=3.121\ \times\ 10^{12}\,\text{electrons}$

Exercise:

Problem:

If $1.80\ \times\ 10^{20}$ electrons move through a pocket calculator during a full day’s operation, how many coulombs of charge moved through it?

Exercise:

Problem:

To start a car engine, the car battery moves 3.75×10^{21} electrons through the starter motor. How many coulombs of charge were moved?

Solution:

$\frac{3.750 \times 10^{21}\,\text{e}}{6.242 \times 10^{18}\,\text{e/C}} = –600.8\,\text{C}$

Exercise:

Problem:

A certain lightning bolt moves 40.0 C of charge. How many fundamental units of charge is this?

Exercise:

Problem:

A 2.5-g copper penny is given a charge of -2.0×10^{-9} C. (a) How many excess electrons are on the penny? (b) By what percent do the excess electrons change the mass of the penny?

Solution:

a. $2.0 \times 10^{-9}\,\text{C}\,(6.242 \times 10^{18}\,\text{e/C}) = 1.248 \times 10^{10}\,\text{e}$;

b. $9.109 \times 10^{-31}\,\text{kg}\,(1.248 \times 10^{10}\,\text{e}) = 1.137 \times 10^{-20}\,\text{kg}$,
$\frac{1.137 \times 10^{-20}\,\text{kg}}{2.5 \times 10^{-3}\,\text{kg}} = 4.548 \times 10^{-18}$ or $4.545 \times 10^{-16}\%$

Exercise:

Problem:

A 2.5-g copper penny is given a charge of 4.0×10^{-9} C. (a) How many electrons are removed from the penny? (b) If no more than one electron is removed from an atom, what percent of the atoms are ionized by this charging process?

Glossary

coulomb
: SI unit of electric charge

electric charge
: physical property of an object that causes it to be attracted toward or repelled from another charged object; each charged object generates and is influenced by a force called an electric force

electric force
: noncontact force observed between electrically charged objects

electron
: particle surrounding the nucleus of an atom and carrying the smallest unit of negative charge

electrostatic attraction
: phenomenon of two objects with opposite charges attracting each other

electrostatic repulsion
: phenomenon of two objects with like charges repelling each other

ion
: atom or molecule with more or fewer electrons than protons

law of conservation of charge
: net electric charge of a closed system is constant

neutron

neutral particle in the nucleus of an atom, with (nearly) the same mass as a proton

proton
: particle in the nucleus of an atom and carrying a positive charge equal in magnitude to the amount of negative charge carried by an electron

static electricity
: buildup of electric charge on the surface of an object; the arrangement of the charge remains constant (“static”)

Coulomb's Law

By the end of this section, you will be able to:

- Describe the electric force, both qualitatively and quantitatively
- Calculate the force that charges exert on each other
- Determine the direction of the electric force for different source charges
- Correctly describe and apply the superposition principle for multiple source charges

Experiments with electric charges have shown that if two objects each have electric charge, then they exert an electric force on each other. The magnitude of the force is linearly proportional to the net charge on each object and inversely proportional to the square of the distance between them. (Interestingly, the force does not depend on the mass of the objects.) The direction of the force vector is along the imaginary line joining the two objects and is dictated by the signs of the charges involved.

Let

- $q_1, q_2 =$ the net electric charges of the two objects;
- $\vec{\mathbf{r}}_{12} =$ the vector displacement from q_1 to q_2.

The electric force $\vec{\mathbf{F}}$ on one of the charges is proportional to the magnitude of its own charge and the magnitude of the other charge, and is inversely proportional to the square of the distance between them:

Equation:

$$F \propto \frac{q_1 q_2}{r_{12}^2}.$$

This proportionality becomes an equality with the introduction of a proportionality constant. For reasons that will become clear in a later chapter, the proportionality constant that we use is actually a collection of constants. (We discuss this constant shortly.)

Note:

Coulomb’s Law

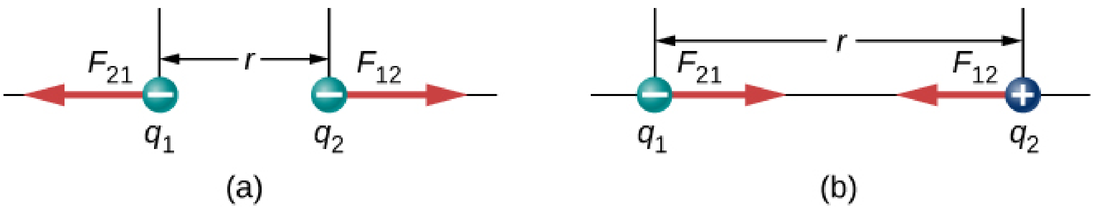
The magnitude of the electric force (or **Coulomb force**) between two electrically charged particles is equal to

Equation:

$$|\mathbf{F}_{12}| = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r_{12}^2}$$

The unit vector r has a magnitude of 1 and points along the axis as the charges. If the charges have the same sign, the force is in the same direction as r showing a repelling

force. If the charges have different signs, the force is in the opposite direction of r showing an attracting force. ([link]).



The electrostatic force $\vec{\mathbf{F}}$ between point charges q_1 and q_2 separated by a distance r is given by Coulomb's law. Note that Newton's third law (every force exerted creates an equal and opposite force) applies as usual—the force on q_1 is equal in magnitude and opposite in direction to the force it exerts on q_2. (a) Like charges; (b) unlike charges.

It is important to note that the electric force is not constant; it is a function of the separation distance between the two charges. If either the test charge or the source charge (or both) move, then $\vec{\mathbf{r}}$ changes, and therefore so does the force. An immediate consequence of this is that direct application of Newton's laws with this force can be mathematically difficult, depending on the specific problem at hand. It can (usually) be done, but we almost always look for easier methods of calculating whatever physical quantity we are interested in. (Conservation of energy is the most common choice.)

Finally, the new constant ε_0 in Coulomb's law is called the *permittivity of free space*, or (better) the **permittivity of vacuum**. It has a very important physical meaning that we will discuss in a later chapter; for now, it is simply an empirical proportionality constant. Its numerical value (to three significant figures) turns out to be

Equation:

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N} \cdot \mathrm{m}^2}.$$

These units are required to give the force in Coulomb's law the correct units of newtons. Note that in Coulomb's law, the permittivity of vacuum is only part of the proportionality constant. For convenience, we often define a Coulomb's constant:

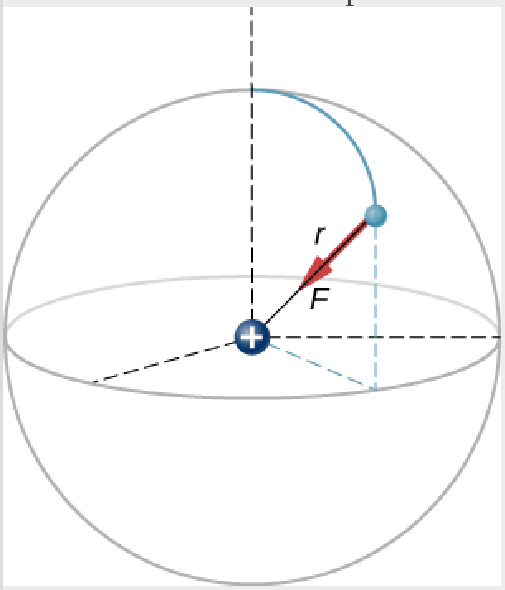
Equation:

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^{9} \frac{\mathrm{N} \cdot \mathrm{m}^2}{\mathrm{C}^2}.$$

Example:

The Force on the Electron in Hydrogen

A hydrogen atom consists of a single proton and a single electron. The proton has a charge of $+e$ and the electron has $-e$. In the "ground state" of the atom, the electron orbits the proton at most probable distance of 5.29×10^{-11} m ([link]). Calculate the electric force on the electron due to the proton.



A schematic depiction of a hydrogen atom, showing the force on the electron. This depiction is only to enable us to calculate the force; the hydrogen atom does not really look like this. Recall [link].

Strategy

For the purposes of this example, we are treating the electron and proton as two point particles, each with an electric charge, and we are told the distance between them; we are asked to calculate the force on the electron. We thus use Coulomb's law.

Solution

Our two charges and the distance between them are,

Equation:

$$\begin{array}{rcl} q_1 & = & +e = +1.602 \times 10^{-19}\ \text{C} \\ q_2 & = & -e = -1.602 \times 10^{-19}\ \text{C} \\ r & = & 5.29 \times 10^{-11}\ \text{m.} \end{array}$$

The magnitude of the force on the electron is

Equation:

$$F = \frac{1}{4\pi\epsilon_0} \frac{|e|^2}{r^2} = \frac{1}{4\pi\left(8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}\right)} \frac{\left(1.602 \times 10^{-19}\,\mathrm{C}\right)^2}{\left(5.29 \times 10^{-11}\,\mathrm{m}\right)^2} = 8.25 \times 10^{-8}\,\mathrm{N}.$$

As for the direction, since the charges on the two particles are opposite, the force is attractive; the force on the electron points radially directly toward the proton, everywhere in the electron's orbit. The force is thus expressed as

Equation:

$$\vec{\mathbf{F}} = \left(8.25 \times 10^{-8}\,\mathrm{N}\right)\hat{\mathbf{r}}.$$

Significance

This is a three-dimensional system, so the electron (and therefore the force on it) can be anywhere in an imaginary spherical shell around the proton. In this "classical" model of the hydrogen atom, the electrostatic force on the electron points in the inward centripetal direction, thus maintaining the electron's orbit. But note that the quantum mechanical model of hydrogen (discussed in Quantum Mechanics) is utterly different.

Note:

Exercise:

Problem:

Check Your Understanding What would be different if the electron also had a positive charge?

Solution:

The force would point outward.

Multiple Source Charges

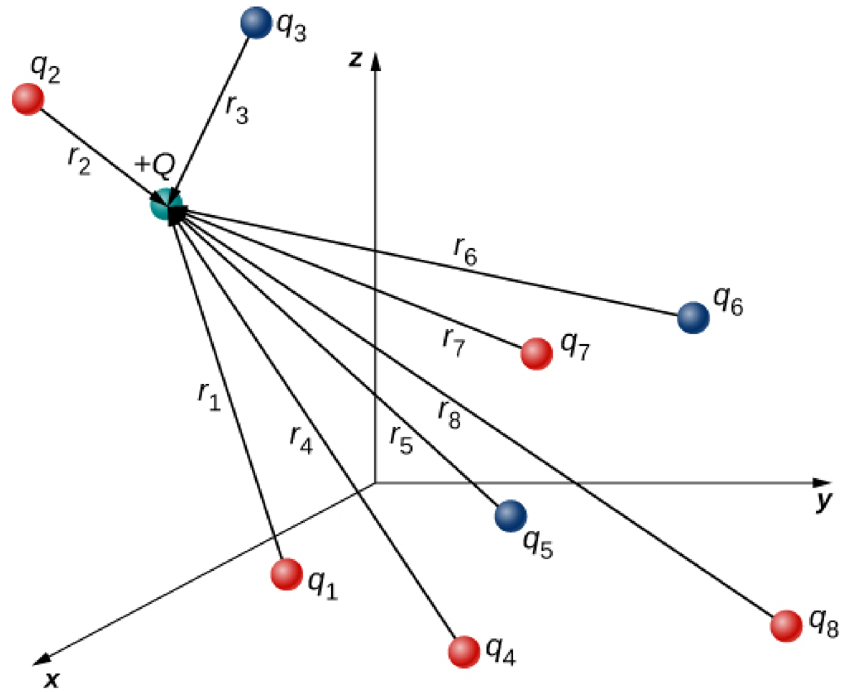
The analysis that we have done for two particles can be extended to an arbitrary number of particles; we simply repeat the analysis, two charges at a time. Specifically, we ask the question: Given N charges (which we refer to as source charge), what is the net electric force that they exert on some other point charge (which we call the test charge)? Note that we use these terms because we can think of the test charge being used to test the strength of the force provided by the source charges.

Like all forces that we have seen up to now, the net electric force on our test charge is simply the vector sum of each individual electric force exerted on it by each of the individual test charges. Thus, we can calculate the net force on the test charge Q by calculating the force on it from each source charge, taken one at a time, and then adding all those forces together (as vectors). This ability to simply add up individual forces in this way is referred to as the **principle of superposition**, and is one of the more important features of the electric force. In mathematical form, this becomes

Note:
Equation:

$$\vec{\mathbf{F}}(r) = \frac{1}{4\pi\varepsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i.$$

In this expression, Q represents the charge of the particle that is experiencing the electric force $\vec{\mathbf{F}}$, and is located at $\vec{\mathbf{r}}$ from the origin; the q_i's are the N source charges, and the vectors $\vec{\mathbf{r}}_i = r_i \hat{\mathbf{r}}_i$ are the displacements from the position of the ith charge to the position of Q. Each of the N unit vectors points directly from its associated source charge toward the test charge. All of this is depicted in [link]. Please note that there is no physical difference between Q and q_i; the difference in labels is merely to allow clear discussion, with Q being the charge we are determining the force on.



The eight source charges each apply a force on the single test charge *Q*. Each force can be calculated independently of the other seven forces. This is the essence of the superposition principle.

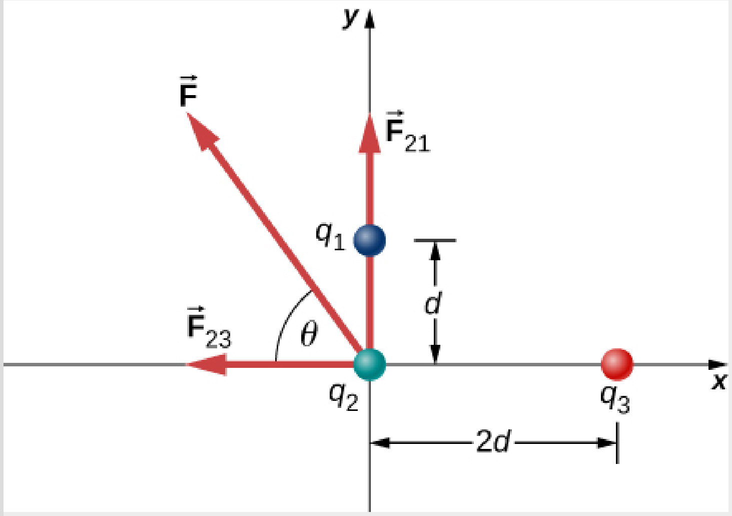
(Note that the force vector $\vec{\mathbf{F}}_i$ does not necessarily point in the same direction as the unit vector $\hat{\mathbf{r}}_i$; it may point in the opposite direction, $-\hat{\mathbf{r}}_i$. The signs of the source charge and test charge determine the direction of the force on the test charge.)

There is a complication, however. Just as the source charges each exert a force on the test charge, so too (by Newton's third law) does the test charge exert an equal and opposite force on each of the source charges. As a consequence, each source charge would change position. However, by [link], the force on the test charge is a function of position; thus, as the positions of the source charges change, the net force on the test charge necessarily changes, which changes the force, which again changes the positions. Thus, the entire mathematical analysis quickly becomes intractable. Later, we will learn techniques for handling this situation, but for now, we make the simplifying assumption that the source charges are fixed in place somehow, so that their positions are constant in time. (The test charge is allowed to move.) With this restriction in place, the analysis of charges is known as **electrostatics**, where "statics" refers to the constant (that is, static) positions of the source charges and the force is referred to as an **electrostatic force**.

Example:

The Net Force from Two Source Charges

Three different, small charged objects are placed as shown in [link]. The charges q_1 and q_3 are fixed in place; q_2 is free to move. Given $q_1 = 2e$, $q_2 = -3e$, and $q_3 = -5e$, and that $d = 2.0 \times 10^{-7}$ m, what is the net force on the middle charge q_2?



Source charges q_1 and q_3 each apply a force on q_2.

Strategy

We use Coulomb's law again. The way the question is phrased indicates that q_2 is our test charge, so that q_1 and q_3 are source charges. The principle of superposition says that the force on q_2 from each of the other charges is unaffected by the presence of the other charge. Therefore, we write down the force on q_2 from each and add them together as vectors.

Solution

We have two source charges $(q_1$ and $q_3)$, a test charge (q_2), distances $(r_{21}$ and $r_{23})$, and we are asked to find a force. This calls for Coulomb's law and superposition of forces. There are two forces:

Equation:

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{23} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_2 q_1}{r_{21}^2}\hat{\mathbf{j}} + \left(-\frac{q_2 q_3}{r_{23}^2}\hat{\mathbf{i}}\right)\right].$$

We can't add these forces directly because they don't point in the same direction: $\vec{\mathbf{F}}_{12}$ points only in the −*x*-direction, while $\vec{\mathbf{F}}_{13}$ points only in the +*y*-direction. The net force is obtained from applying the Pythagorean theorem to its *x*- and *y*-components:

Equation:

$$F = \sqrt{F_x^2 + F_y^2}$$

where
Equation:

$$\begin{aligned} F_x &= -F_{23} = -\frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}^2} \\ &= -\left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\frac{(4.806 \times 10^{-19}\ \text{C})(8.01 \times 10^{-19}\ \text{C})}{(4.00 \times 10^{-7}\ \text{m})^2} \\ &= -2.16 \times 10^{-14}\ \text{N} \end{aligned}$$

and
Equation:

$$\begin{aligned} F_y &= F_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{r_{21}^2} \\ &= \left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\frac{(4.806 \times 10^{-19}\ \text{C})(3.204 \times 10^{-19}\ \text{C})}{(2.00 \times 10^{-7}\ \text{m})^2} \\ &= 3.46 \times 10^{-14}\ \text{N}. \end{aligned}$$

We find that
Equation:

$$F = \sqrt{F_x^2 + F_y^2} = 4.08 \times 10^{-14}\ \text{N}$$

at an angle of
Equation:

$$\phi = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{3.46 \times 10^{-14}\ \text{N}}{-2.16 \times 10^{-14}\ \text{N}}\right) = -58^\circ,$$

that is, 58° above the −*x*-axis, as shown in the diagram.
Significance
Notice that when we substituted the numerical values of the charges, we did not include the negative sign of either q_2 or q_3. Recall that negative signs on vector quantities indicate a reversal of direction of the vector in question. But for electric forces, the direction of the force is determined by the types (signs) of both interacting charges; we determine the force directions by considering whether the signs of the two charges are the same or are opposite. If you also include negative signs from negative charges when you substitute numbers, you run the risk of mathematically reversing the direction of the force you are calculating. Thus, the safest thing to do is to calculate just the magnitude of the force, using the absolute values of the charges, and determine the directions physically.
It’s also worth noting that the only new concept in this example is how to calculate the electric forces; everything else (getting the net force from its components, breaking the

forces into their components, finding the direction of the net force) is the same as force problems you have done earlier.

Note:
Exercise:

Problem: **Check Your Understanding** What would be different if q_1 were negative?

Solution:

The net force would point 58° below the $-x$-axis.

Summary

- Coulomb's law gives the magnitude of the force vector between point charges. It is
  Equation:

$$\vec{\mathbf{F}}_{12}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

  where q_1 and q_2 are two point charges separated by a distance r. This Coulomb force is extremely basic, since most charges are due to point-like particles. It is responsible for all electrostatic effects and underlies most macroscopic forces.

Conceptual Questions

Exercise:

Problem:

Would defining the charge on an electron to be positive have any effect on Coulomb's law?

Exercise:

Problem:

An atomic nucleus contains positively charged protons and uncharged neutrons. Since nuclei do stay together, what must we conclude about the forces between these nuclear particles?

Solution:

The force holding the nucleus together must be greater than the electrostatic repulsive force on the protons.

Exercise:

Problem:

Is the force between two fixed charges influenced by the presence of other charges?

Problems

Exercise:

Problem:

Two point particles with charges $+3\ \mu\text{C}$ and $+5\ \mu\text{C}$ are held in place by 3-N forces on each charge in appropriate directions. (a) Draw a free-body diagram for each particle. (b) Find the distance between the charges.

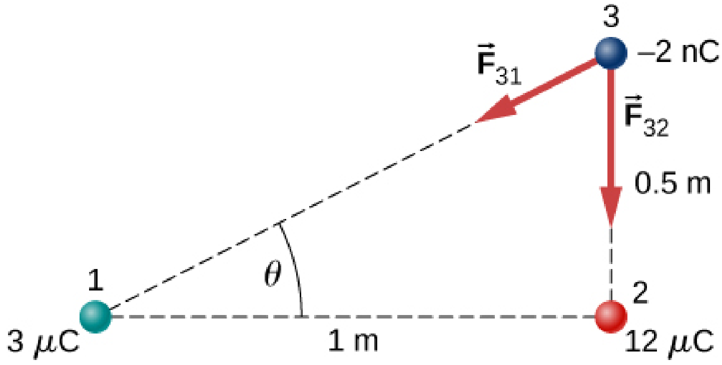
Exercise:

Problem:

Two charges $+3\ \mu\text{C}$ and $+12\ \mu\text{C}$ are fixed 1 m apart, with the second one to the right. Find the magnitude and direction of the net force on a −2-nC charge when placed at the following locations: (a) halfway between the two (b) half a meter to the left of the $+3\ \mu\text{C}$ charge (c) half a meter above the $+12\ \mu\text{C}$ charge in a direction perpendicular to the line joining the two fixed charges

Solution:

a. charge 1 is $3\ \mu\text{C}$; charge 2 is $12\ \mu\text{C}$, $\text{F}_{31} = 2.16 \times 10^{-4}$ N to the left,
$\text{F}_{32} = 8.63 \times 10^{-4}$ N to the right,
$\text{F}_{\text{net}} = 6.47 \times 10^{-4}$ N to the right;
b. $\text{F}_{31} = 2.16 \times 10^{-4}$ N to the right,
$\text{F}_{32} = 9.59 \times 10^{-5}$ N to the right,
$\text{F}_{\text{net}} = 3.12 \times 10^{-4}$ N to the right,



;

c. $\vec{\mathbf{F}}_{31x} = -2.76 \times 10^{-5}\,\text{N}\,\hat{\mathbf{i}}$,
$\vec{\mathbf{F}}_{31y} = -1.38 \times 10^{-5}\,\text{N}\,\hat{\mathbf{j}}$,
$\vec{\mathbf{F}}_{32y} = -8.63 \times 10^{-4}\,\text{N}\,\hat{\mathbf{j}}$
$\vec{\mathbf{F}}_{\text{net}} = -3.86 \times 10^{-5}\,\text{N}\,\hat{\mathbf{i}} - 8.83 \times 10^{-4}\,\text{N}\,\hat{\mathbf{j}}$

Exercise:

Problem:

In a salt crystal, the distance between adjacent sodium and chloride ions is $2.82 \times 10^{-10}\,\text{m}$. What is the force of attraction between the two singly charged ions?

Exercise:

Problem:

Protons in an atomic nucleus are typically $10^{-15}\,\text{m}$ apart. What is the electric force of repulsion between nuclear protons?

Solution:

$F = 230.7\,\text{N}$

Exercise:

Problem:

Suppose Earth and the Moon each carried a net negative charge $-Q$. Approximate both bodies as point masses and point charges.

(a) What value of Q is required to balance the gravitational attraction between Earth and the Moon?

(b) Does the distance between Earth and the Moon affect your answer? Explain.

(c) How many electrons would be needed to produce this charge?

Exercise:

Problem:

Point charges $q_1 = 50\ \mu\text{C}$ and $q_2 = -25\ \mu\text{C}$ are placed 1.0 m apart. What is the force on a third charge $q_3 = 20\ \mu\text{C}$ placed midway between q_1 and q_2?

Solution:

$F = 53.94\ \text{N}$

Exercise:

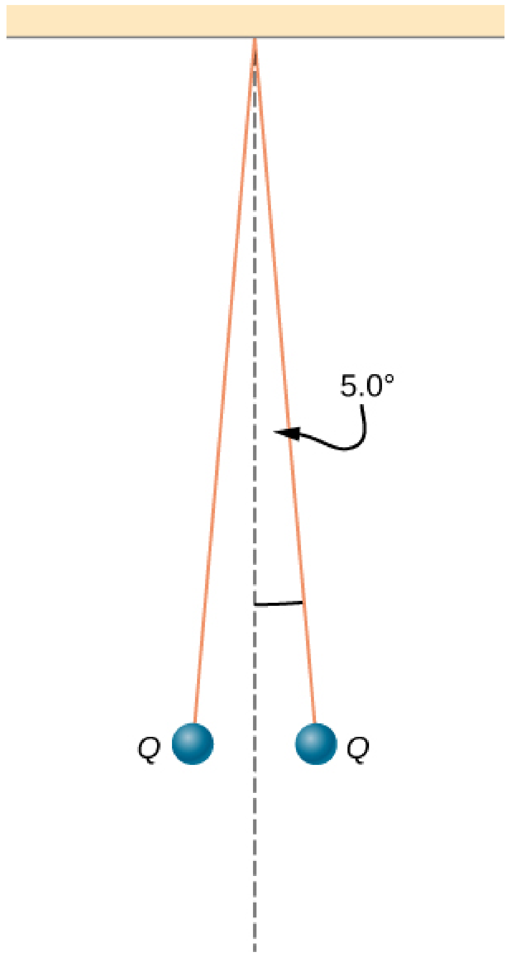
Problem:

Where must q_3 of the preceding problem be placed so that the net force on it is zero?

Exercise:

Problem:

Two small balls, each of mass 5.0 g, are attached to silk threads 50 cm long, which are in turn tied to the same point on the ceiling, as shown below. When the balls are given the same charge Q, the threads hang at 5.0° to the vertical, as shown below. What is the magnitude of Q? What are the signs of the two charges?



Solution:

The tension is $T = 0.049\ \text{N}$. The horizontal component of the tension is $0.0043\ \text{N}$
$d = 0.088\ \text{m}, \quad q = 6.1\ \times\ 10^{-8}\ \text{C}.$
The charges can be positive or negative, but both have to be the same sign.

Exercise:

Problem:

Point charges $Q_1 = 2.0\ \mu\text{C}$ and $Q_2 = 4.0\ \mu\text{C}$ are located at $\vec{\mathbf{r}}_1 = (4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 5.0\widehat{\mathbf{k}})\text{m}$ and $\vec{\mathbf{r}}_2 = (8.0\hat{\mathbf{i}} + 5.0\hat{\mathbf{j}} - 9.0\widehat{\mathbf{k}})\text{m}$. What is the force of Q_2 on Q_1?

Exercise:

Problem:

The net excess charge on two small spheres (small enough to be treated as point charges) is Q. Show that the force of repulsion between the spheres is greatest when each sphere has an excess charge $Q/2$. Assume that the distance between the spheres is so large compared with their radii that the spheres can be treated as point charges.

Solution:

Let the charge on one of the spheres be nQ, where n is a fraction between 0 and 1. In the numerator of Coulomb's law, the term involving the charges is $nQ(1-n)Q$. This is equal to $(n-n^2)Q^2$. Finding the maximum of this term gives $1-2n=0 \Rightarrow n=\frac{1}{2}$

Exercise:

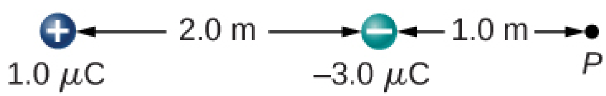
Problem:

Two small, identical conducting spheres repel each other with a force of 0.050 N when they are 0.25 m apart. After a conducting wire is connected between the spheres and then removed, they repel each other with a force of 0.060 N. What is the original charge on each sphere?

Exercise:

Problem:

A charge $q=2.0\ \mu\text{C}$ is placed at the point P shown below. What is the force on q?



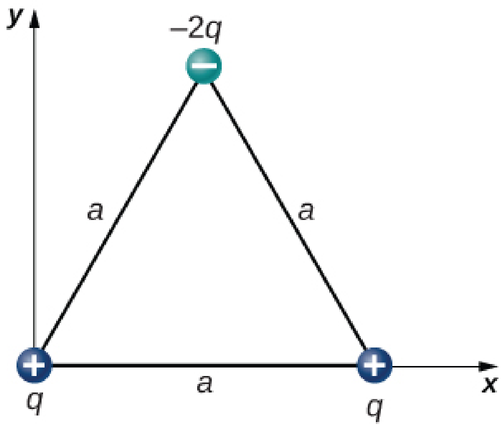
Solution:

Define right to be the positive direction and hence left is the negative direction, then $F=-0.05\ \text{N}$

Exercise:

Problem:

What is the net electric force on the charge located at the lower right-hand corner of the triangle shown here?



Exercise:

Problem:

Two fixed particles, each of charge 5.0×10^{-6} C, are 24 cm apart. What force do they exert on a third particle of charge -2.5×10^{-6} C that is 13 cm from each of them?

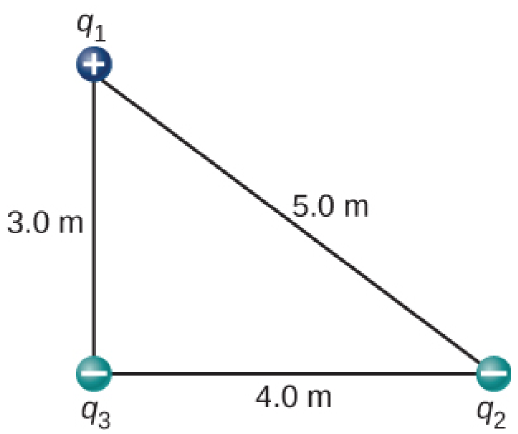
Solution:

The particles form triangle of sides 13, 13, and 24 cm. The *x*-components cancel, whereas there is a contribution to the *y*-component from both charges 24 cm apart. The *y*-axis passing through the third charge bisects the 24-cm line, creating two right triangles of sides 5, 12, and 13 cm.
$F_y = 2.56\,\text{N}$ in the negative *y*-direction since the force is attractive. The net force from both charges is $\vec{\mathbf{F}}_{\text{net}} = -5.12\,\text{N}\hat{\mathbf{j}}$.

Exercise:

Problem:

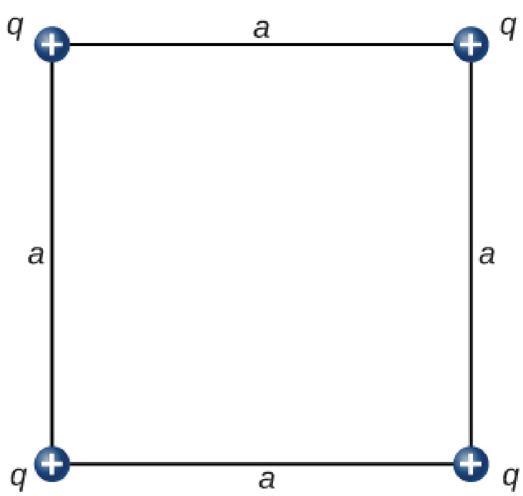
The charges $q_1 = 2.0 \times 10^{-7}$ C, $q_2 = -4.0 \times 10^{-7}$ C, and $q_3 = -1.0 \times 10^{-7}$ C are placed at the corners of the triangle shown below. What is the force on q_1?



Exercise:

Problem:

What is the force on the charge q at the lower-right-hand corner of the square shown here?



Solution:

The diagonal is $\sqrt{2}a$ and the components of the force due to the diagonal charge has a factor $\cos\theta = \frac{1}{\sqrt{2}}$;

$\vec{\mathbf{F}}_{\text{net}} = \left[k\frac{q^2}{a^2} + k\frac{q^2}{2a^2}\ \frac{1}{\sqrt{2}}\right]\hat{\mathbf{i}} - \left[k\frac{q^2}{a^2} + k\frac{q^2}{2a^2}\ \frac{1}{\sqrt{2}}\right]\hat{\mathbf{j}}$

Exercise:

Problem:

Point charges $q_1 = 10\ \mu\text{C}$ and $q_2 = -30\ \mu\text{C}$ are fixed at $r_1 = \left(3.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}}\right)\text{m}$ and $r_2 = \left(9.0\hat{\mathbf{i}} + 6.0\hat{\mathbf{j}}\right)\text{m}$. What is the force of q_2 on q_1?

Glossary

Coulomb force
: another term for the electrostatic force

Coulomb’s law
: mathematical equation calculating the electrostatic force vector between two charged particles

electrostatic force
: amount and direction of attraction or repulsion between two charged bodies; the assumption is that the source charges have no acceleration

electrostatics
: study of charged objects which are not in motion

permittivity of vacuum
: also called the permittivity of free space, and constant describing the strength of the electric force in a vacuum

principle of superposition
: useful fact that we can simply add up all of the forces due to charges acting on an object

Electric Field
By the end of this section, you will be able to:

- Explain the purpose of the electric field concept
- Describe the properties of the electric field
- Calculate the field of a collection of source charges of either sign

As we showed in the preceding section, the net electric force on a test charge is the vector sum of all the electric forces acting on it, from all of the various source charges, located at their various positions. But what if we use a different test charge, one with a different magnitude, or sign, or both? Or suppose we have a dozen different test charges we wish to try at the same location? We would have to calculate the sum of the forces from scratch. Fortunately, it is possible to define a quantity, called the **electric field**, which is independent of the test charge. It only depends on the configuration of the source charges, and once found, allows us to calculate the force on any test charge.

Defining a Field

Suppose we have N source charges $q_1, q_2, q_3, \ldots, q_N$ located at positions $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \ldots, \vec{\mathbf{r}}_N$, applying N electrostatic forces on a test charge Q. The net force on Q is (see [link])

Equation:

$$\begin{aligned}
\vec{\mathbf{F}} &= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \cdots + \vec{\mathbf{F}}_N \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{Qq_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{Qq_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{Qq_3}{r_3^2}\hat{\mathbf{r}}_3 + \cdots + \frac{Qq_N}{r_1^2}\hat{\mathbf{r}}_N\right) \\
&= Q\left[\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{q_3}{r_3^2}\hat{\mathbf{r}}_3 + \cdots + \frac{q_N}{r_1^2}\hat{\mathbf{r}}_N\right)\right].
\end{aligned}$$

We can rewrite this as

Note:

Equation:

$$\vec{\mathbf{F}} = Q\vec{\mathbf{E}}$$

where

Equation:

$$\vec{\mathbf{E}} \equiv \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{q_3}{r_3^2}\hat{\mathbf{r}}_3 + \cdots + \frac{q_N}{r_1^2}\hat{\mathbf{r}}_N\right)$$

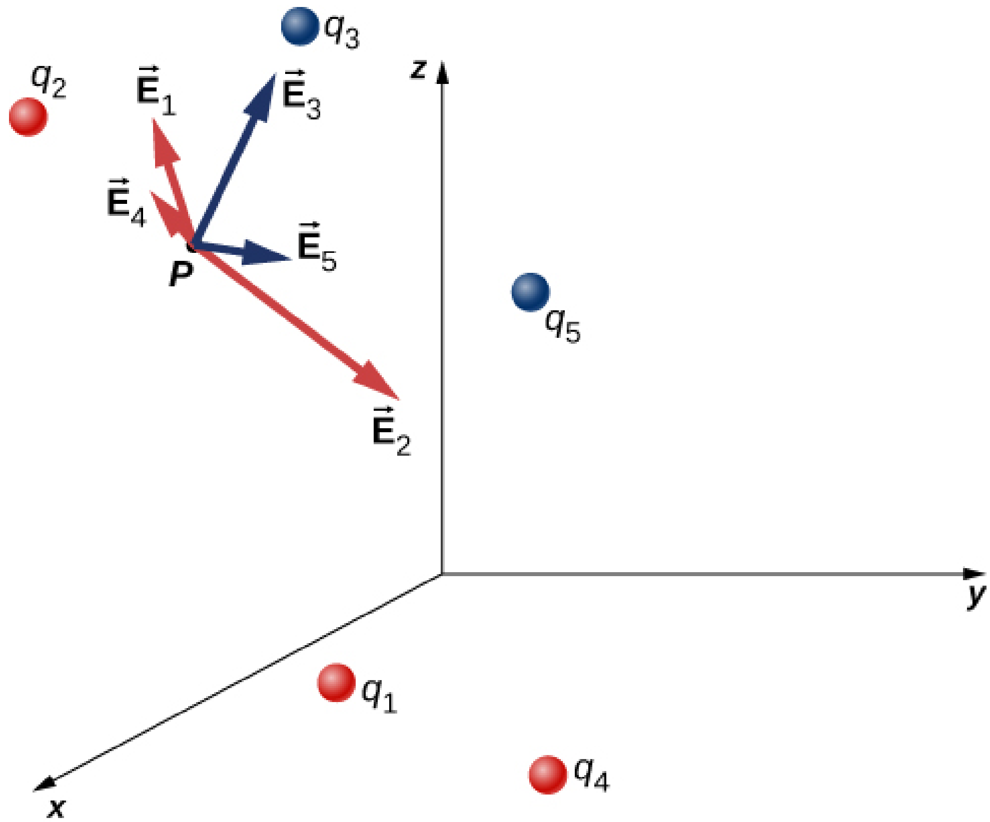
or, more compactly,

Note:

Equation:

$$\vec{\mathbf{E}}(P) \equiv \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i.$$

This expression is called the electric field at position $P = P\left(x, y, z\right)$ of the N source charges. Here, P is the location of the point in space where you are calculating the field and is relative to the positions $\vec{\mathbf{r}}_i$ of the source charges ([link]). Note that we have to impose a coordinate system to solve actual problems.



Each of these eight source charges creates its own electric field at every point in space; shown here are the field vectors at an arbitrary point *P*. Like the electric force, the net electric field obeys the superposition principle.

Notice that the calculation of the electric field makes no reference to the test charge. Thus, the physically useful approach is to calculate the electric field and then use it to calculate the force on some test charge later, if needed. Different test charges experience different forces [link], but it is the same electric field [link]. That being said, recall that there is no fundamental difference between a test charge and a source charge; these are merely convenient labels for the system of interest. Any charge produces an electric field; however, just as Earth's orbit is not affected by Earth's own gravity, a charge is not subject to a force due to the electric field it generates. Charges are only subject to forces from the electric fields of other charges.

In this respect, the electric field $\vec{\mathbf{E}}$ of a point charge is similar to the gravitational field $\vec{\mathbf{g}}$ of Earth; once we have calculated the gravitational field at some point in space, we can use it any time we want to calculate the resulting force on any mass we choose to place at that point. In fact, this is exactly what we do when we say the gravitational field of Earth (near Earth's surface) has a value of $9.81\ \mathrm{m/s}^2$, and then we calculate the resulting force (i.e., weight) on different masses. Also, the general expression for calculating $\vec{\mathbf{g}}$ at arbitrary distances from the center of Earth (i.e., not just near Earth's surface) is very similar to the expression for $\vec{\mathbf{E}}$: $\vec{\mathbf{g}} = G\frac{M}{r^2}\hat{\mathbf{r}}$, where G is a proportionality constant, playing the same role for $\vec{\mathbf{g}}$ as $\frac{1}{4\pi\varepsilon_0}$ does for $\vec{\mathbf{E}}$. The value of $\vec{\mathbf{g}}$ is calculated once and is then used in an endless number of problems.

To push the analogy further, notice the units of the electric field: From $F = QE$, the units of E are newtons per coulomb, N/C, that is, the electric field applies a force on each unit charge. Now notice the units of g: From $w = mg$, the units of g are newtons per kilogram, N/kg, that is, the gravitational field applies a force on each unit mass. We could say that the gravitational field of Earth, near Earth's surface, has a value of 9.81 N/kg.

The Meaning of "Field"

Recall from your studies of gravity that the word "field" in this context has a precise meaning. A field, in physics, is a physical quantity whose value depends on (is a function of) position, relative to the source of the field. In the case of the electric field, [link] shows that the value of $\vec{\mathbf{E}}$ (both the magnitude and the direction) depends on where in space the point P is located, measured from the locations $\vec{\mathbf{r}}_i$ of the source charges q_i.

In addition, since the electric field is a vector quantity, the electric field is referred to as a *vector field*. (The gravitational field is also a vector field.) In contrast, a field that has only a magnitude at every point is a *scalar field*. The temperature in a room is an example of a scalar field. It is a field because the temperature, in general, is different at different locations in the room, and it is a scalar field because temperature is a scalar quantity.

Also, as you did with the gravitational field of an object with mass, you should picture the electric field of a charge-bearing object (the source charge) as a continuous, immaterial substance that surrounds the source charge, filling all of space—in principle, to $\pm\infty$ in all directions. The field exists at every physical point in space. To put it another way, the electric charge on an object alters the space around the charged object in such a way that all other electrically charged objects in space experience an electric force as a result of being in that field. The electric field, then, is the mechanism by which the electric properties of the source charge are transmitted to and through the rest of the universe. (Again, the range of the electric force is infinite.)

We will see in subsequent chapters that the speed at which electrical phenomena travel is the same as the speed of light. There is a deep connection between the electric field and light.

Superposition

Yet another experimental fact about the field is that it obeys the superposition principle. In this context, that means that we can (in principle) calculate the total electric field of many source charges by calculating the electric field of only q_1 at position P, then calculate the field of q_2 at P, while—and this is the crucial idea—ignoring the field of, and indeed even the existence of, q_1. We can repeat this process, calculating the field of each individual source charge, independently of the existence of any of the other charges. The total electric field, then, is the vector sum of all these fields. That, in essence, is what [link] says.

In the next section, we describe how to determine the shape of an electric field of a source charge distribution and how to sketch it.

The Direction of the Field

[link] enables us to determine the magnitude of the electric field, but we need the direction also. We use the convention that the direction of any electric field vector is the same as the direction of the electric force vector that the field would apply to a positive test charge placed in that field. Such a charge would be repelled by positive source charges (the force on it would point

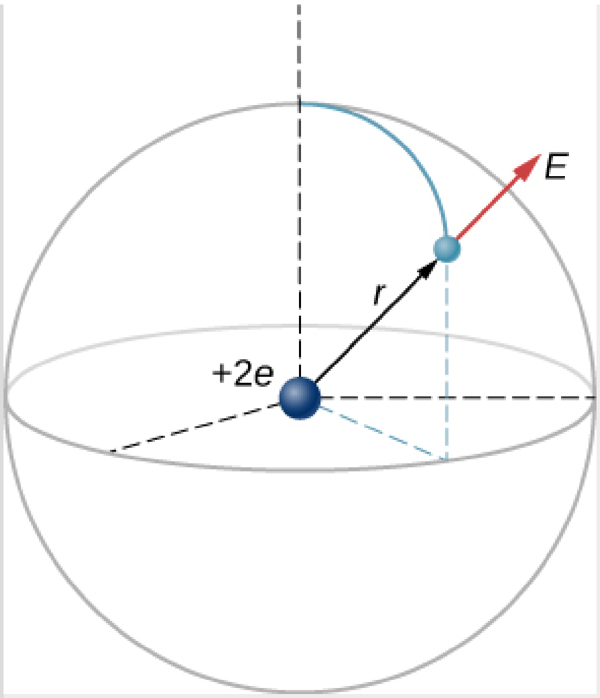
away from the positive source charge) but attracted to negative charges (the force points toward the negative source).

Note:
Direction of the Electric Field

By convention, all electric fields $\vec{\mathbf{E}}$ point away from positive source charges and point toward negative source charges.

Note:
Add charges to the Electric Field of Dreams and see how they react to the electric field. Turn on a background electric field and adjust the direction and magnitude.

Example:
The *E*-field of an Atom
In an ionized helium atom, the most probable distance between the nucleus and the electron is $r = 26.5 \times 10^{-12}$ m. What is the electric field due to the nucleus at the location of the electron?
Strategy
Note that although the electron is mentioned, it is not used in any calculation. The problem asks for an electric field, not a force; hence, there is only one charge involved, and the problem specifically asks for the field due to the nucleus. Thus, the electron is a red herring; only its distance matters. Also, since the distance between the two protons in the nucleus is much, much smaller than the distance of the electron from the nucleus, we can treat the two protons as a single charge +2*e* ([link]).



A schematic representation of a helium atom. Again, helium physically looks nothing like this, but this sort of diagram is helpful for calculating the electric field of the nucleus.

Solution
The electric field is calculated by
Equation:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i.$$

Since there is only one source charge (the nucleus), this expression simplifies to
Equation:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}.$$

Here $q = 2e = 2\left(1.6 \times 10^{-19}\text{ C}\right)$ (since there are two protons) and *r* is given; substituting gives

Equation:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\left(8.85 \times 10^{-12}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\frac{2\left(1.6 \times 10^{-19}\text{ C}\right)}{\left(26.5 \times 10^{-12}\text{ m}\right)^2}\hat{\mathbf{r}} = 4.1 \times 10^{12}\frac{\text{N}}{\text{C}}\hat{\mathbf{r}}.$$

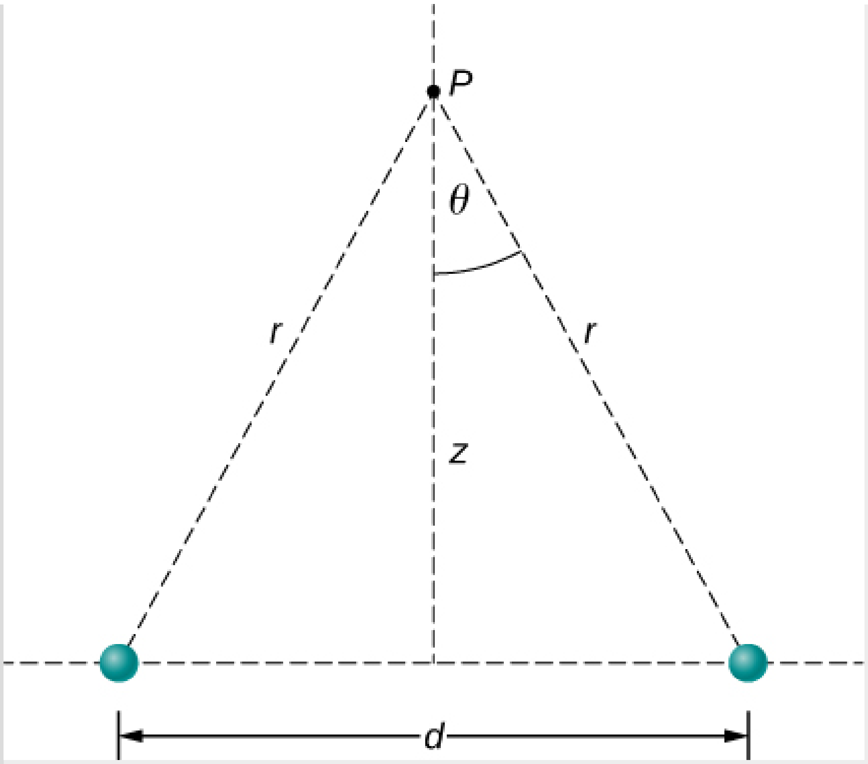
The direction of $\vec{\mathbf{E}}$ is radially away from the nucleus in all directions. Why? Because a positive test charge placed in this field would accelerate radially away from the nucleus (since it is also positively charged), and again, the convention is that the direction of the electric field vector is defined in terms of the direction of the force it would apply to positive test charges.

Example:

The *E*-Field above Two Equal Charges

(a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges $+q$ that are a distance d apart ([link]). Check that your result is consistent with what you'd expect when $z \gg d$.

(b) The same as part (a), only this time make the right-hand charge $-q$ instead of $+q$.



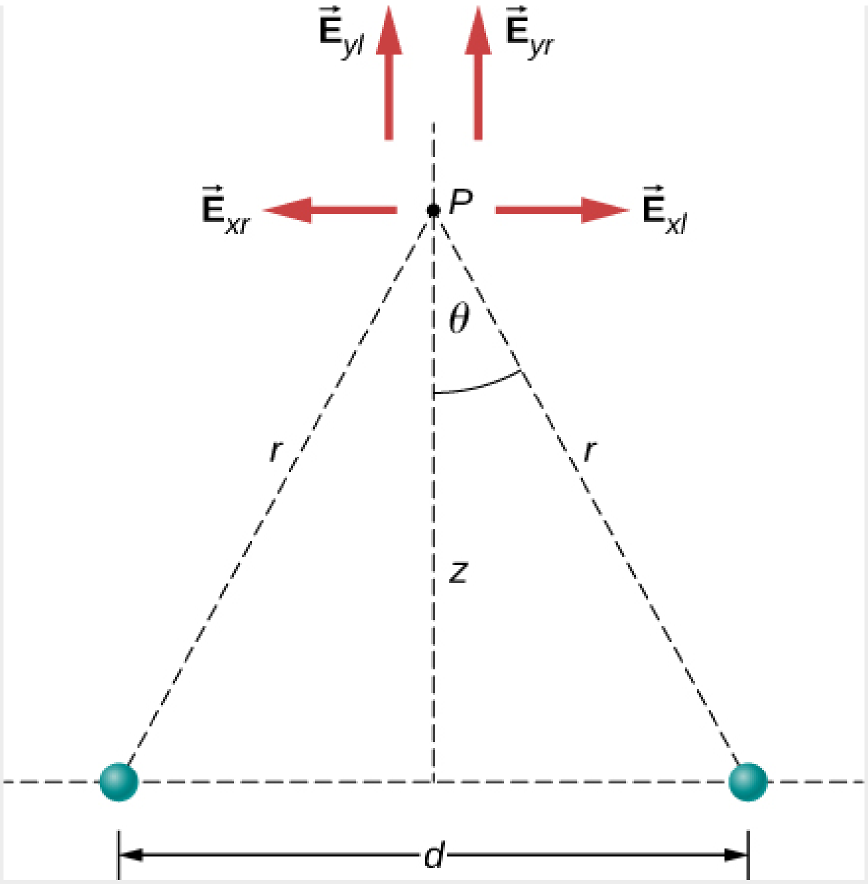
Finding the field of two identical source charges at the point *P*. Due to the symmetry, the net field at *P* is entirely vertical. (Notice that this is *not* true away from the midline between the charges.)

Strategy

We add the two fields as vectors, per [link]. Notice that the system (and therefore the field) is symmetrical about the vertical axis; as a result, the horizontal components of the field vectors cancel. This simplifies the math. Also, we take care to express our final answer in terms of only quantities that are given in the original statement of the problem: *q*, *z*, *d*, and constants (π, ε_0).

Solution

a. By symmetry, the horizontal (*x*)-components of $\vec{\mathbf{E}}$ cancel ([link]);
$$E_x = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\sin\theta - \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\sin\theta = 0.$$



Note that the horizontal components of the electric fields from the two charges cancel each other out, while the vertical components add together.

The vertical (z)-component is given by

Equation:

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r^2}\cos\theta.$$

Since none of the other components survive, this is the entire electric field, and it points in the $\hat{\mathbf{k}}$ direction. Notice that this calculation uses the principle of **superposition**; we calculate the fields of the two

charges independently and then add them together.

What we want to do now is replace the quantities in this expression that we don't know (such as *r*), or can't easily measure (such as $\cos\theta$) with quantities that we do know, or can measure. In this case, by geometry,

Equation:

$$r^2 = z^2 + \left(\frac{d}{2}\right)^2$$

and

Equation:

$$\cos\theta = \frac{z}{r} = \frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}.$$

Thus, substituting,

Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]}\frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}\hat{\mathbf{k}}.$$

Simplifying, the desired answer is

Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{\mathbf{k}}.$$

b. If the source charges are equal and opposite, the vertical components cancel because $E_z = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta - \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta = 0$

and we get, for the horizontal component of $\vec{\mathbf{E}}$,

Equation:

$$\begin{aligned}\vec{\mathbf{E}}(z) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\sin\theta\hat{\mathbf{i}} - \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}\sin\theta\hat{\mathbf{i}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2q}{r^2}\sin\theta\hat{\mathbf{i}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2q}{\left[z^2+\left(\frac{d}{2}\right)^2\right]}\frac{\left(\frac{d}{2}\right)}{\left[z^2+\left(\frac{d}{2}\right)^2\right]^{1/2}}\hat{\mathbf{i}}.\end{aligned}$$

This becomes
Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[z^2+\left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{\mathbf{i}}.$$

Significance
It is a very common and very useful technique in physics to check whether your answer is reasonable by evaluating it at extreme cases. In this example, we should evaluate the field expressions for the cases $d = 0$, $z \gg d$, and $z \to \infty$, and confirm that the resulting expressions match our physical expectations. Let's do so:
Let's start with [link], the field of two identical charges. From far away (i.e., $z \gg d$), the two source charges should "merge" and we should then "see" the field of just one charge, of size $2q$. So, let $z \gg d$; then we can neglect d^2 in [link] to obtain
Equation:

$$\begin{aligned}\lim_{d\to 0}\vec{\mathbf{E}} &= \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2\right]^{3/2}}\widehat{\mathbf{k}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2qz}{z^3}\widehat{\mathbf{k}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{(2q)}{z^2}\widehat{\mathbf{k}},\end{aligned}$$

which is the correct expression for a field at a distance *z* away from a charge $2q$.
Next, we consider the field of equal and opposite charges, [link]. It can be shown (via a Taylor expansion) that for $d \ll z \ll \infty$, this becomes

Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\,\frac{qd}{z^3}\hat{\mathbf{i}},$$

which is the field of a dipole, a system that we will study in more detail later. (Note that the units of $\vec{\mathbf{E}}$ are still correct in this expression, since the units of *d* in the numerator cancel the unit of the "extra" *z* in the denominator.) If *z* is *very* large $(z \rightarrow \infty)$, then $E \rightarrow 0$, as it should; the two charges "merge" and so cancel out.

Note:
Exercise:

Problem:

Check Your Understanding What is the electric field due to a single point particle?

Solution:

$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\hat{r}$

Note:
Try this simulation of electric field hockey to get the charge in the goal by placing other charges on the field.

Summary

- The electric field is an alteration of space caused by the presence of an electric charge. The electric field mediates the electric force between a source charge and a test charge.

- The electric field, like the electric force, obeys the superposition principle
- The field is a vector; by definition, it points away from positive charges and toward negative charges.

Conceptual Questions

Exercise:

Problem:

When measuring an electric field, could we use a negative rather than a positive test charge?

Solution:

Either sign of the test charge could be used, but the convention is to use a positive test charge.

Exercise:

Problem:

During fair weather, the electric field due to the net charge on Earth points downward. Is Earth charged positively or negatively?

Exercise:

Problem:

If the electric field at a point on the line between two charges is zero, what do you know about the charges?

Solution:

The charges are of the same sign.

Exercise:

Problem:

Two charges lie along the *x*-axis. Is it true that the net electric field always vanishes at some point (other than infinity) along the *x*-axis?

Problems

Exercise:

Problem:

A particle of charge $2.0\ \times\ 10^{-8}$ C experiences an upward force of magnitude $4.0\ \times\ 10^{-6}$ N when it is placed in a particular point in an electric field. (a) What is the electric field at that point? (b) If a charge $q=-1.0\ \times\ 10^{-8}$ C is placed there, what is the force on it?

Solution:

a. $E=2.0\ \times\ 10^{-2}\frac{\text{N}}{\text{C}}$ up;
b. $F=2.0\ \times\ 10^{-6}$ N down

Exercise:

Problem:

On a typical clear day, the atmospheric electric field points downward and has a magnitude of approximately 100 N/C. Compare the gravitational and electric forces on a small dust particle of mass $2.0\ \times\ 10^{-15}$ g that carries a single electron charge. What is the acceleration (both magnitude and direction) of the dust particle?

Exercise:

Problem:

Consider an electron that is 10^{-10} m from an alpha particle ($q = 3.2 \times 10^{-19}$ C). (a) What is the electric field due to the alpha particle at the location of the electron? (b) What is the electric field due to the electron at the location of the alpha particle? (c) What is the electric force on the alpha particle? On the electron?

Solution:

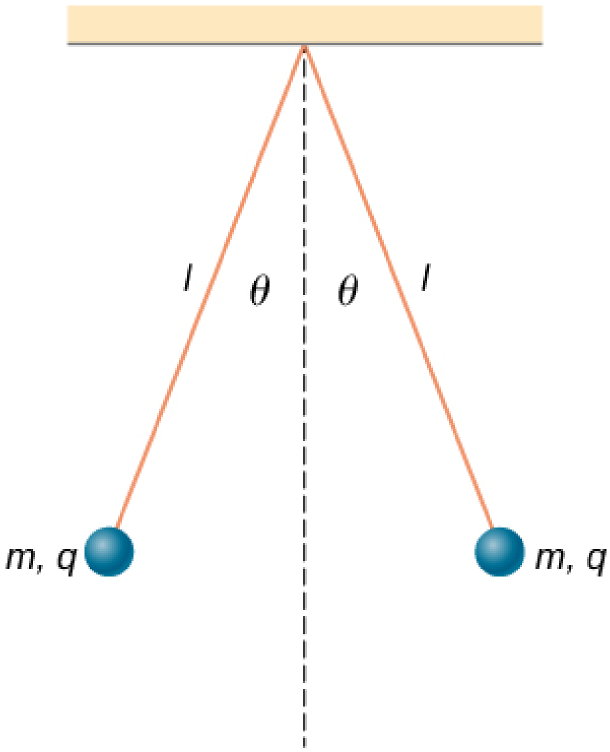
a. $E = 2.88 \times 10^{11}$ N/C;
b. $E = 1.44 \times 10^{11}$ N/C;
c. $F = 4.61 \times 10^{-8}$ N on alpha particle;
$F = 4.61 \times 10^{-8}$ N on electron

Exercise:

Problem:

Each the balls shown below carries a charge q and has a mass m. The length of each thread is l, and at equilibrium, the balls are separated by an angle 2θ. How does θ vary with q and l? Show that θ satisfies

$$\sin(\theta)^2 \tan(\theta) = \frac{q^2}{16\pi\varepsilon_0 g l^2 m}.$$



Exercise:

Problem:

What is the electric field at a point where the force on a -2.0×10^{-6} $-$C charge is $\left(4.0\hat{\mathbf{i}} - 6.0\hat{\mathbf{j}}\right) \times 10^{-6}$ N?

Solution:

$$E = \left(-2.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}}\right) \text{ N}$$

Exercise:

Problem:

A proton is suspended in the air by an electric field at the surface of Earth. What is the strength of this electric field?

Exercise:

Problem:

The electric field in a particular thundercloud is $2.0 \times 10^5\ \mathrm{N/C}$. What is the acceleration of an electron in this field?

Solution:

$F = 3.204 \times 10^{-14}\ \mathrm{N}$,
$a = 3.517 \times 10^{16}\ \mathrm{m/s^2}$

Exercise:

Problem:

A small piece of cork whose mass is 2.0 g is given a charge of $5.0 \times 10^{-7}\ \mathrm{C}$. What electric field is needed to place the cork in equilibrium under the combined electric and gravitational forces?

Exercise:

Problem:

If the electric field is $100\ \mathrm{N/C}$ at a distance of 50 cm from a point charge q, what is the value of q?

Solution:

$q = 2.78 \times 10^{-9}\ \mathrm{C}$

Exercise:

Problem:

What is the electric field of a proton at the first Bohr orbit for hydrogen $(r = 5.29 \times 10^{-11}\ \mathrm{m})$? What is the force on the electron in that orbit?

Exercise:

Problem:

(a) What is the electric field of an oxygen nucleus at a point that is 10^{-10} m from the nucleus? (b) What is the force this electric field exerts on a second oxygen nucleus placed at that point?

Solution:

a. $E = 1.15 \times 10^{12}\,\text{N/C}$;
b. $F = 1.47 \times 10^{-6}\,\text{N}$

Exercise:

Problem:

Two point charges, $q_1 = 2.0 \times 10^{-7}$ C and $q_2 = -6.0 \times 10^{-8}$ C, are held 25.0 cm apart. (a) What is the electric field at a point 5.0 cm from the negative charge and along the line between the two charges? (b)What is the force on an electron placed at that point?

Exercise:

Problem:

Point charges $q_1 = 50\,\mu\text{C}$ and $q_2 = -25\,\mu\text{C}$ are placed 1.0 m apart. (a) What is the electric field at a point midway between them? (b) What is the force on a charge $q_3 = 20\,\mu\text{C}$ situated there?

Solution:

If the q_2 is to the right of q_1, the electric field vector from both charges point to the right. a. $E = 2.70 \times 10^6\,\text{N/C}$;
b. $F = 54.0\,\text{N}$

Exercise:

Problem:

Can you arrange the two point charges $q_1 = -2.0 \times 10^{-6}$ C and $q_2 = 4.0 \times 10^{-6}$ C along the x-axis so that $E = 0$ at the origin?

Exercise:

Problem:

Point charges $q_1 = q_2 = 4.0 \times 10^{-6}$ C are fixed on the *x*-axis at $x = -3.0\,\mathrm{m}$ and $x = 3.0\,\mathrm{m}$. What charge *q* must be placed at the origin so that the electric field vanishes at $x = 0, y = 3.0\,\mathrm{m}$?

Solution:

There is 45° right triangle geometry. The *x*-components of the electric field at $y = 3\,\mathrm{m}$ cancel. The *y*-components give
$E(y = 3\,\mathrm{m}) = 2.83 \times 10^3\,\mathrm{N/C}$.
At the origin we have a a negative charge of magnitide
$q = -2.83 \times 10^{-6}$ C.

Glossary

electric field
: physical phenomenon created by a charge; it "transmits" a force between a two charges

superposition
: concept that states that the net electric field of multiple source charges is the vector sum of the field of each source charge calculated individually

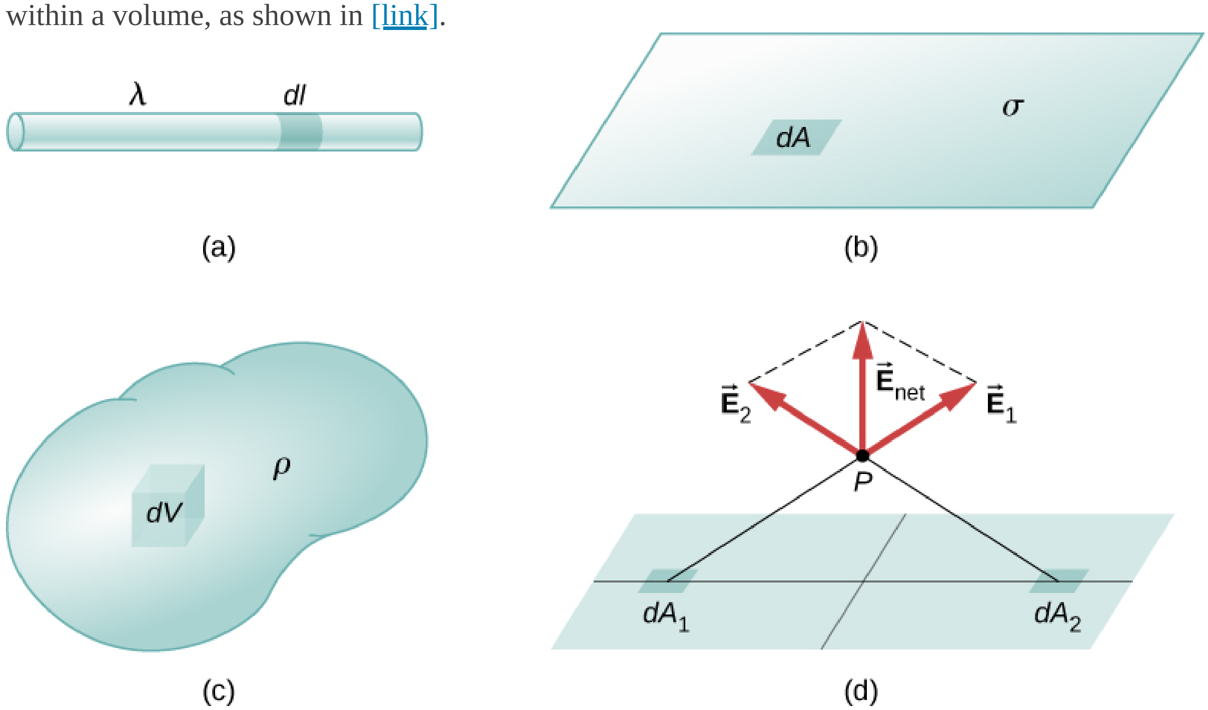
Calculating Electric Fields of Charge Distributions
By the end of this section, you will be able to:

- Explain what a continuous source charge distribution is and how it is related to the concept of quantization of charge
- Describe line charges, surface charges, and volume charges
- Calculate the field of a continuous source charge distribution of either sign

The charge distributions we have seen so far have been discrete: made up of individual point particles. This is in contrast with a **continuous charge distribution**, which has at least one nonzero dimension. If a charge distribution is continuous rather than discrete, we can generalize the definition of the electric field. We simply divide the charge into infinitesimal pieces and treat each piece as a point charge.

Note that because charge is quantized, there is no such thing as a "truly" continuous charge distribution. However, in most practical cases, the total charge creating the field involves such a huge number of discrete charges that we can safely ignore the discrete nature of the charge and consider it to be continuous. This is exactly the kind of approximation we make when we deal with a bucket of water as a continuous fluid, rather than a collection of H_2O molecules.

Our first step is to define a charge density for a charge distribution along a line, across a surface, or within a volume, as shown in [link].



The configuration of charge differential elements for a (a) line charge, (b) sheet of charge, and (c) a volume of charge. Also note that (d) some of the components of the total electric field cancel out, with the remainder resulting in a net electric field.

Definitions of charge density:

- $\lambda \equiv$ charge per unit length (**linear charge density**); units are coulombs per meter (C/m)
- $\sigma \equiv$ charge per unit area (**surface charge density**); units are coulombs per square meter $(\mathrm{C/m^2})$
- $\rho \equiv$ charge per unit volume (**volume charge density**); units are coulombs per cubic meter $(\mathrm{C/m^3})$

Then, for a line charge, a surface charge, and a volume charge, the summation in [link] becomes an integral and q_i is replaced by $dq = \lambda dl$, σdA, or ρdV, respectively:
Equation:

$$\text{Point charges:} \qquad \vec{\mathbf{E}}\,(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\left(\frac{q_i}{r^2}\right)\hat{\mathbf{r}}$$

Equation:

$$\text{Line charge:} \qquad \vec{\mathbf{E}}\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)\hat{\mathbf{r}}$$

Equation:

$$\text{Surface charge:} \qquad \vec{\mathbf{E}}\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{surface}}\left(\frac{\sigma dA}{r^2}\right)\hat{\mathbf{r}}$$

Equation:

$$\text{Volume charge:} \qquad \vec{\mathbf{E}}\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}}\left(\frac{\rho dV}{r^2}\right)\hat{\mathbf{r}}$$

The integrals are generalizations of the expression for the field of a point charge. They implicitly include and assume the principle of superposition. The “trick” to using them is almost always in coming up with correct expressions for *dl*, *dA*, or *dV*, as the case may be, expressed in terms of *r*, and also expressing the charge density function appropriately. It may be constant; it might be dependent on location.

Note carefully the meaning of *r* in these equations: It is the distance from the charge element $(q_i, \lambda dl, \sigma dA, \rho dV)$ to the location of interest, $P\,(x, y, z)$ (the point in space where you want to determine the field). However, don’t confuse this with the meaning of $\hat{\mathbf{r}}$; we are using it and the vector notation $\vec{\mathbf{E}}$ to write three integrals at once. That is, [link] is actually
Equation:

$$E_x\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_x, \quad E_y\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_y, \quad E_z\,(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_z.$$

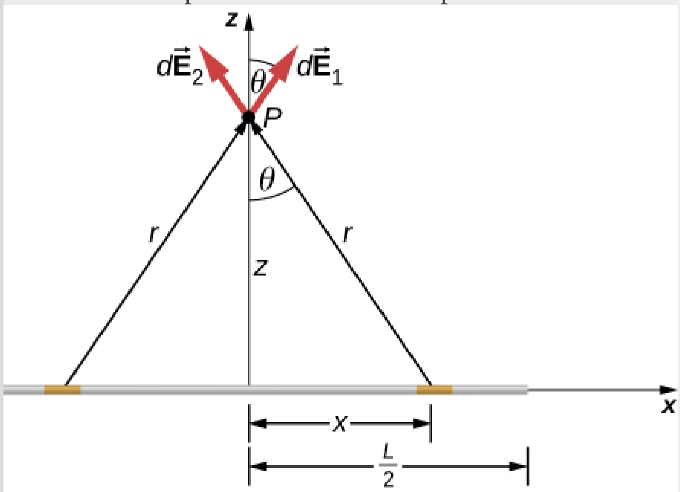
Example:

Electric Field of a Line Segment

Find the electric field a distance z above the midpoint of a straight line segment of length L that carries a uniform line charge density λ.

Strategy

Since this is a continuous charge distribution, we conceptually break the wire segment into differential pieces of length dl, each of which carries a differential amount of charge $dq = \lambda dl$. Then, we calculate the differential field created by two symmetrically placed pieces of the wire, using the symmetry of the setup to simplify the calculation ([link]). Finally, we integrate this differential field expression over the length of the wire (half of it, actually, as we explain below) to obtain the complete electric field expression.



A uniformly charged segment of wire. The electric field at point P can be found by applying the superposition principle to symmetrically placed charge elements and integrating.

Solution

Before we jump into it, what do we expect the field to "look like" from far away? Since it is a finite line segment, from far away, it should look like a point charge. We will check the expression we get to see if it meets this expectation.

The electric field for a line charge is given by the general expression

Equation:

$$\vec{\mathbf{E}}\left(P\right) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{\mathbf{r}}.$$

The symmetry of the situation (our choice of the two identical differential pieces of charge) implies the horizontal (x)-components of the field cancel, so that the net field points in the z-direction. Let's check this formally.

The total field $\vec{\mathbf{E}}\left(P\right)$ is the vector sum of the fields from each of the two charge elements (call them $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$, for now):

Equation:

$$\vec{\mathbf{E}}(P) = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = E_{1x}\hat{\mathbf{i}} + E_{1z}\widehat{\mathbf{k}} + E_{2x}\left(-\hat{\mathbf{i}}\right) + E_{2z}\widehat{\mathbf{k}}.$$

Because the two charge elements are identical and are the same distance away from the point *P* where we want to calculate the field, $E_{1x} = E_{2x}$, so those components cancel. This leaves
Equation:

$$\vec{\mathbf{E}}(P) = E_{1z}\widehat{\mathbf{k}} + E_{2z}\widehat{\mathbf{k}} = E_1 \cos\theta\widehat{\mathbf{k}} + E_2 \cos\theta\widehat{\mathbf{k}}.$$

These components are also equal, so we have
Equation:

$$\begin{aligned}\vec{\mathbf{E}}(P) &= \tfrac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r^2}\cos\theta\widehat{\mathbf{k}} + \tfrac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r^2}\cos\theta\widehat{\mathbf{k}} \\ &= \tfrac{1}{4\pi\varepsilon_0}\int_0^{L/2} \frac{2\lambda dx}{r^2}\cos\theta\widehat{\mathbf{k}}\end{aligned}$$

where our differential line element *dl* is *dx*, in this example, since we are integrating along a line of charge that lies on the *x*-axis. (The limits of integration are 0 to $\frac{L}{2}$, not $-\frac{L}{2}$ to $+\frac{L}{2}$, because we have constructed the net field from two differential pieces of charge *dq*. If we integrated along the entire length, we would pick up an erroneous factor of 2.)
In principle, this is complete. However, to actually calculate this integral, we need to eliminate all the variables that are not given. In this case, both *r* and *θ* change as we integrate outward to the end of the line charge, so those are the variables to get rid of. We can do that the same way we did for the two point charges: by noticing that
Equation:

$$r = \left(z^2 + x^2\right)^{1/2}$$

and
Equation:

$$\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2 + x^2\right)^{1/2}}.$$

Substituting, we obtain
Equation:

$$\begin{aligned}\vec{\mathbf{E}}(P) &= \tfrac{1}{4\pi\varepsilon_0}\int_0^{L/2} \frac{2\lambda dx}{\left(z^2 + x^2\right)}\,\tfrac{z}{\left(z^2+x^2\right)^{1/2}}\widehat{\mathbf{k}} \\ &= \tfrac{1}{4\pi\varepsilon_0}\int_0^{L/2} \frac{2\lambda z}{\left(z^2 + x^2\right)^{3/2}}dx\widehat{\mathbf{k}} \\ &= \tfrac{2\lambda z}{4\pi\varepsilon_0}\left[\tfrac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^{L/2}\widehat{\mathbf{k}}\end{aligned}$$

which simplifies to
Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0} \frac{\lambda L}{z\sqrt{z^2 + \frac{L^2}{4}}}\widehat{\mathbf{k}}.$$

Significance

Notice, once again, the use of symmetry to simplify the problem. This is a very common strategy for calculating electric fields. The fields of nonsymmetrical charge distributions have to be handled with multiple integrals and may need to be calculated numerically by a computer.

Note:

Exercise:

Problem:

Check Your Understanding How would the strategy used above change to calculate the electric field at a point a distance *z* above one end of the finite line segment?

Solution:

We will no longer be able to take advantage of symmetry. Instead, we will need to calculate each of the two components of the electric field with their own integral.

Example:

Electric Field of an Infinite Line of Charge

Find the electric field a distance *z* above the midpoint of an infinite line of charge that carries a uniform line charge density λ.

Strategy

This is exactly like the preceding example, except the limits of integration will be $-\infty$ to $+\infty$.

Solution

Again, the horizontal components cancel out, so we wind up with

Equation:

$$\vec{\mathbf{E}}\,(P) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{\lambda dx}{r^2} \cos\theta\widehat{\mathbf{k}}$$

where our differential line element *dl* is *dx*, in this example, since we are integrating along a line of charge that lies on the *x*-axis. Again,

Equation:

$$\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2 + x^2\right)^{1/2}}.$$

Substituting, we obtain

Equation:

$$\begin{aligned}\vec{\mathbf{E}}(P) &= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda dx}{(z^2+x^2)}\frac{z}{(z^2+x^2)^{1/2}}\hat{\mathbf{k}}\\ &= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda z}{(z^2+x^2)^{3/2}}dx\hat{\mathbf{k}}\\ &= \frac{\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Big|_{-\infty}^{\infty}\hat{\mathbf{k}},\end{aligned}$$

which simplifies to

Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\hat{\mathbf{k}}.$$

Significance

Our strategy for working with continuous charge distributions also gives useful results for charges with infinite dimension.

In the case of a finite line of charge, note that for $z \gg L$, z^2 dominates the L in the denominator, so that [link] simplifies to

Equation:

$$\vec{\mathbf{E}} \approx \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{k}}.$$

If you recall that $\lambda L = q$, the total charge on the wire, we have retrieved the expression for the field of a point charge, as expected.

In the limit $L \to \infty$, on the other hand, we get the field of an **infinite straight wire**, which is a straight wire whose length is much, much greater than either of its other dimensions, and also much, much greater than the distance at which the field is to be calculated:

Note:

Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\hat{\mathbf{k}}.$$

An interesting artifact of this infinite limit is that we have lost the usual $1/r^2$ dependence that we are used to. This will become even more intriguing in the case of an infinite plane.

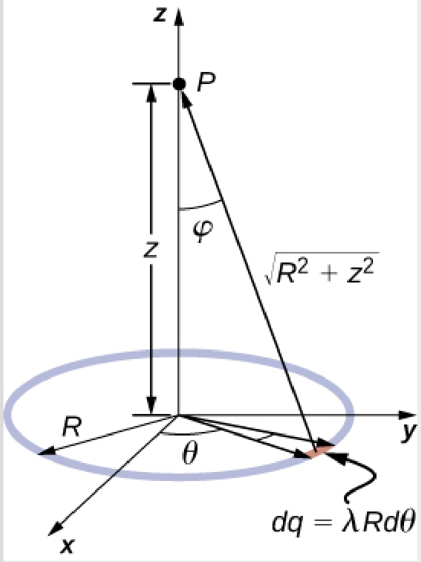
Example:

Electric Field due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric field at a point on the axis passing through the center of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use polar coordinates shown in [link].



The system and variable for calculating the electric field due to a ring of charge.

Solution

The electric field for a line charge is given by the general expression

Equation:

$$\vec{\mathbf{E}}\left(P\right) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\frac{\lambda dl}{r^2}\hat{\mathbf{r}}.$$

A general element of the arc between θ and $\theta + d\theta$ is of length $Rd\theta$ and therefore contains a charge equal to $\lambda Rd\theta$. The element is at a distance of $r = \sqrt{z^2 + R^2}$ from P, the angle is $\cos\phi = \frac{z}{\sqrt{z^2+R^2}}$, and therefore the electric field is

Equation:

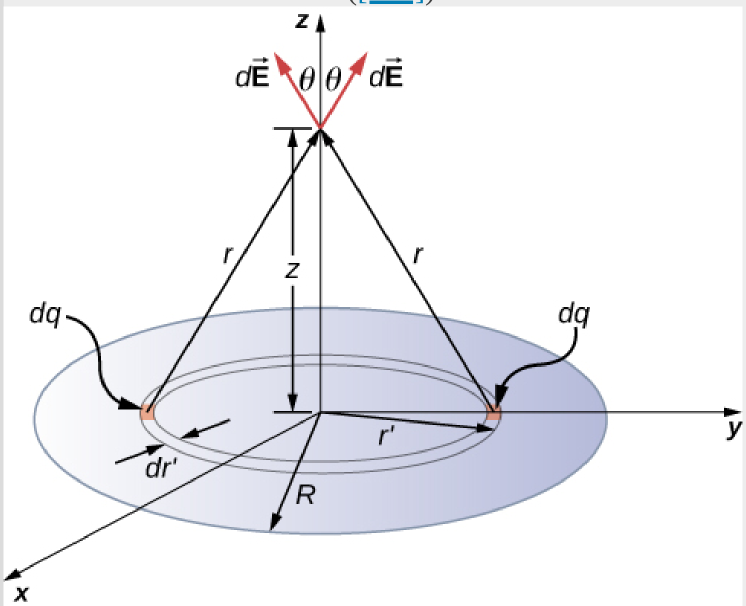
$$\begin{aligned}\vec{\mathbf{E}}(P) &= \frac{1}{4\pi\varepsilon_0}\int_{\mathbf{line}}\frac{\lambda dl}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{\lambda R d\theta}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}}\widehat{\mathbf{z}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{\lambda Rz}{(z^2+R^2)^{3/2}}\widehat{\mathbf{z}}\int_0^{2\pi}d\theta = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda Rz}{(z^2+R^2)^{3/2}}\widehat{\mathbf{z}} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q_{\mathbf{tot}}z}{(z^2+R^2)^{3/2}}\widehat{\mathbf{z}}.\end{aligned}$$

Significance
As usual, symmetry simplified this problem, in this particular case resulting in a trivial integral. Also, when we take the limit of $z >> R$, we find that
Equation:

$$\vec{\mathbf{E}} \approx \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{tot}}}{z^2}\widehat{\mathbf{z}},$$

as we expect.

Example:
The Field of a Disk
Find the electric field of a circular thin disk of radius *R* and uniform charge density at a distance *z* above the center of the disk ([link])



A uniformly charged disk. As in the line charge example, the field above the center of this disk can be calculated by taking advantage of the symmetry of the charge distribution.

Strategy

The electric field for a surface charge is given by
Equation:

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{surface}}\frac{\sigma dA}{r^2}\hat{\mathbf{r}}.$$

To solve surface charge problems, we break the surface into symmetrical differential “stripes” that match the shape of the surface; here, we’ll use rings, as shown in the figure. Again, by symmetry, the horizontal components cancel and the field is entirely in the vertical $(\widehat{\mathbf{k}})$ direction. The vertical component of the electric field is extracted by multiplying by $\cos\theta$, so
Equation:

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\varepsilon_0}\int_{\text{surface}}\frac{\sigma dA}{r^2}\cos\theta\,\widehat{\mathbf{k}}.$$

As before, we need to rewrite the unknown factors in the integrand in terms of the given quantities. In this case,
Equation:

$$\begin{aligned} dA &= 2\pi r' dr' \\ r^2 &= {r'}^2 + z^2 \\ \cos\theta &= \tfrac{z}{\left({r'}^2+z^2\right)^{1/2}}. \end{aligned}$$

(Please take note of the two different “r’s” here; r is the distance from the differential ring of charge to the point P where we wish to determine the field, whereas r' is the distance from the center of the disk to the differential ring of charge.) Also, we already performed the polar angle integral in writing down dA.
Solution
Substituting all this in, we get
Equation:

$$\begin{aligned} \vec{\mathbf{E}}(P) &= \vec{\mathbf{E}}\left(z\right) = \tfrac{1}{4\pi\varepsilon_0}\int_0^R\frac{\sigma\left(2\pi r' dr'\right)z}{\left({r'}^2+z^2\right)^{3/2}}\widehat{\mathbf{k}} \\ &= \tfrac{1}{4\pi\varepsilon_0}(2\pi\sigma z)\left(\tfrac{1}{z} - \tfrac{1}{\sqrt{R^2+z^2}}\right)\widehat{\mathbf{k}} \end{aligned}$$

or, more simply,
Equation:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\left(2\pi\sigma - \frac{2\pi\sigma z}{\sqrt{R^2+z^2}}\right)\widehat{\mathbf{k}}.$$

Significance
Again, it can be shown (via a Taylor expansion) that when $z \gg R$, this reduces to
Equation:

$$\vec{\mathbf{E}}(z) \approx \frac{1}{4\pi\varepsilon_0}\frac{\sigma\pi R^2}{z^2}\widehat{\mathbf{k}},$$

which is the expression for a point charge $Q = \sigma\pi R^2$.

Note:
Exercise:

Problem:

Check Your Understanding How would the above limit change with a uniformly charged rectangle instead of a disk?

Solution:

The point charge would be $Q = \sigma ab$ where a and b are the sides of the rectangle but otherwise identical.

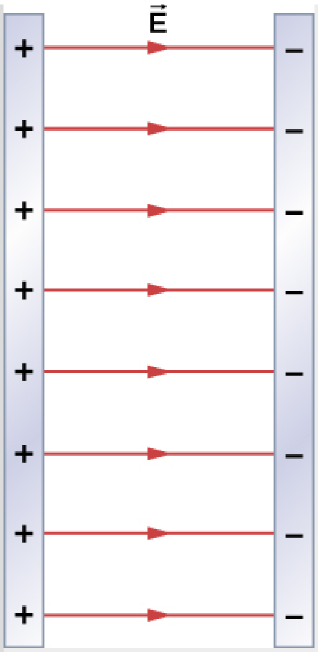
As $R \to \infty$, [link] reduces to the field of an **infinite plane**, which is a flat sheet whose area is much, much greater than its thickness, and also much, much greater than the distance at which the field is to be calculated:

Note:
Equation:

$$\vec{\mathbf{E}} = \frac{\sigma}{2\varepsilon_0}\widehat{\mathbf{k}}.$$

Note that this field is constant. This surprising result is, again, an artifact of our limit, although one that we will make use of repeatedly in the future. To understand why this happens, imagine being placed above an infinite plane of constant charge. Does the plane look any different if you vary your altitude? No—you still see the plane going off to infinity, no matter how far you are from it. It is important to note that [link] is because we are above the plane. If we were below, the field would point in the $-\widehat{\mathbf{k}}$ direction.

Example:
The Field of Two Infinite Planes
Find the electric field everywhere resulting from two infinite planes with equal but opposite charge densities ([link]).



Two charged infinite planes. Note the direction of the electric field.

Strategy
We already know the electric field resulting from a single infinite plane, so we may use the principle of superposition to find the field from two.
Solution
The electric field points away from the positively charged plane and toward the negatively charged plane. Since the σ are equal and opposite, this means that in the region outside of the two planes, the electric fields cancel each other out to zero.
However, in the region between the planes, the electric fields add, and we get
Equation:

$$\vec{\mathbf{E}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}$$

for the electric field. The $\hat{\mathbf{i}}$ is because in the figure, the field is pointing in the $+x$-direction.
Significance
Systems that may be approximated as two infinite planes of this sort provide a useful means of creating uniform electric fields.

Note:
Exercise:

Problem:

Check Your Understanding What would the electric field look like in a system with two parallel positively charged planes with equal charge densities?

Solution:

The electric field would be zero in between, and have magnitude $\frac{\sigma}{\varepsilon_0}$ everywhere else.

Summary

- A very large number of charges can be treated as a continuous charge distribution, where the calculation of the field requires integration. Common cases are:
  - one-dimensional (like a wire); uses a line charge density λ
  - two-dimensional (metal plate); uses surface charge density σ
  - three-dimensional (metal sphere); uses volume charge density ρ
- The “source charge” is a differential amount of charge dq. Calculating dq depends on the type of source charge distribution:
  Equation:

$$dq = \lambda dl; \ \ dq = \sigma dA; \ \ dq = \rho dV.$$

- Symmetry of the charge distribution is usually key.
- Important special cases are the field of an “infinite” wire and the field of an “infinite” plane.

Conceptual Questions

Exercise:

Problem:

Give a plausible argument as to why the electric field outside an infinite charged sheet is constant.

Solution:

At infinity, we would expect the field to go to zero, but because the sheet is infinite in extent, this is not the case. Everywhere you are, you see an infinite plane in all directions.

Exercise:

Problem:

Compare the electric fields of an infinite sheet of charge, an infinite, charged conducting plate, and infinite, oppositely charged parallel plates.

Exercise:

Problem:

Describe the electric fields of an infinite charged plate and of two infinite, charged parallel plates in terms of the electric field of an infinite sheet of charge.

Solution:

The infinite charged plate would have $E = \frac{\sigma}{2\varepsilon_0}$ everywhere. The field would point toward the plate if it were negatively charged and point away from the plate if it were positively charged. The electric field of the parallel plates would be zero between them if they had the same charge, and E would be $E = \frac{\sigma}{\varepsilon_0}$ everywhere else. If the charges were opposite, the situation is reversed, zero outside the plates and $E = \frac{\sigma}{\varepsilon_0}$ between them.

Exercise:

Problem:

A negative charge is placed at the center of a ring of uniform positive charge. What is the motion (if any) of the charge? What if the charge were placed at a point on the axis of the ring other than the center?

Problems

Exercise:

Problem:

A thin conducting plate 1.0 m on the side is given a charge of -2.0×10^{-6} C. An electron is placed 1.0 cm above the center of the plate. What is the acceleration of the electron?

Exercise:

Problem:

Calculate the magnitude and direction of the electric field 2.0 m from a long wire that is charged uniformly at $\lambda = 4.0 \times 10^{-6}\ \text{C/m}$.

Solution:

$\vec{\mathbf{E}}(z) = 3.6 \times 10^4\ \text{N/C}\widehat{\mathbf{k}}$

Exercise:

Problem:

Two thin conducting plates, each 25.0 cm on a side, are situated parallel to one another and 5.0 mm apart. If 10^{11} electrons are moved from one plate to the other, what is the electric field between the plates?

Exercise:

Problem:

The charge per unit length on the thin rod shown below is λ. What is the electric field at the point *P*? (*Hint*: Solve this problem by first considering the electric field $d\vec{\mathbf{E}}$ at *P* due to a small segment *dx* of the rod, which contains charge $dq = \lambda dx$. Then find the net field by integrating $d\vec{\mathbf{E}}$ over the length of the rod.)

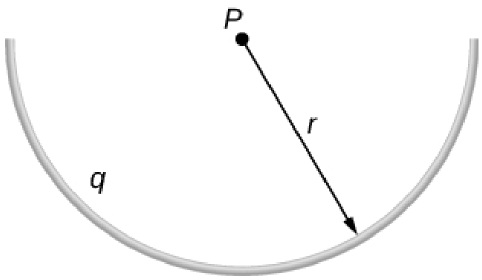
Solution:

$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(x+a)^2}, \quad E = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{l+a} - \frac{1}{a}\right]$

Exercise:

Problem:

The charge per unit length on the thin semicircular wire shown below is λ. What is the electric field at the point *P*?



Exercise:

Problem:

Two thin parallel conducting plates are placed 2.0 cm apart. Each plate is 2.0 cm on a side; one plate carries a net charge of $8.0\,\mu\text{C}$, and the other plate carries a net charge of $-8.0\,\mu\text{C}$. What is the charge density on the inside surface of each plate? What is the electric field between the plates?

Solution:

$\sigma = 0.02\,\text{C/m}^2 \quad E = 2.26\,\times\,10^9\,\text{N/C}$

Exercise:

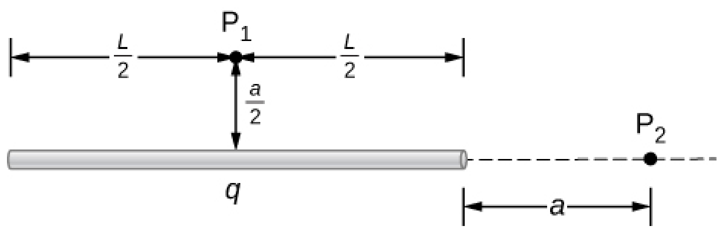
Problem:

A thin conducting plate 2.0 m on a side is given a total charge of $-10.0\ \mu\text{C}$. (a) What is the electric field 1.0 cm above the plate? (b) What is the force on an electron at this point? (c) Repeat these calculations for a point 2.0 cm above the plate. (d) When the electron moves from 1.0 to 2,0 cm above the plate, how much work is done on it by the electric field?

Exercise:

Problem:

A total charge *q* is distributed uniformly along a thin, straight rod of length *L* (see below). What is the electric field at P_1? At P_2?



Solution:

At P_1: $\vec{\mathbf{E}}(y) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{y\sqrt{y^2+\frac{L^2}{4}}}\hat{\mathbf{j}} \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q}{\frac{a}{2}\sqrt{(\frac{a}{2})^2+\frac{L^2}{4}}}\hat{\mathbf{j}} = \frac{1}{\pi\varepsilon_0}\frac{q}{a\sqrt{a^2+L^2}}\hat{\mathbf{j}}$

At P_2: Put the origin at the end of L.

$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(x+a)^2}, \quad \vec{\mathbf{E}} = -\frac{q}{4\pi\varepsilon_0 l}\left[\frac{1}{l+a} - \frac{1}{a}\right]\hat{\mathbf{i}}$

Exercise:

Problem:

Charge is distributed along the entire *x*-axis with uniform density λ. How much work does the electric field of this charge distribution do on an electron that moves along the *y*-axis from $y = a$ to $y = b$?

Exercise:

Problem:

Charge is distributed along the entire *x*-axis with uniform density λ_x and along the entire *y*-axis with uniform density λ_y. Calculate the resulting electric field at (a) $\vec{\mathbf{r}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ and (b) $\vec{\mathbf{r}} = c\widehat{\mathbf{k}}$.

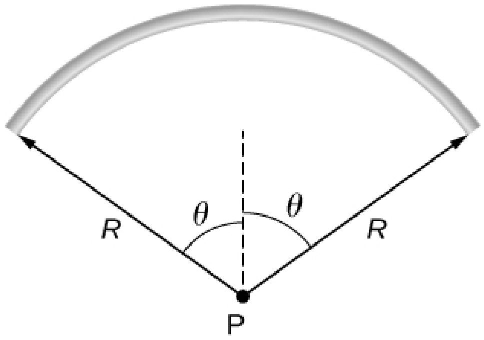
Solution:

a. $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda_x}{b}\hat{\mathbf{i}} + \frac{1}{4\pi\varepsilon_0}\frac{2\lambda_y}{a}\hat{\mathbf{j}}$; b. $\frac{1}{4\pi\varepsilon_0}\frac{2(\lambda_x+\lambda_y)}{c}\widehat{\mathbf{k}}$

Exercise:

Problem:

A rod bent into the arc of a circle subtends an angle 2θ at the center P of the circle (see below). If the rod is charged uniformly with a total charge Q, what is the electric field at P?



Exercise:

Problem:

A proton moves in the electric field $\vec{\mathbf{E}} = 200\hat{\mathbf{i}}\,\text{N/C}$. (a) What are the force on and the acceleration of the proton? (b) Do the same calculation for an electron moving in this field.

Solution:

a. $\vec{\mathbf{F}} = 3.2 \times 10^{-17}\,\text{N}\hat{\mathbf{i}}$,
$\vec{\mathbf{a}} = 1.92 \times 10^{10}\,\text{m/s}^2\hat{\mathbf{i}}$;
b. $\vec{\mathbf{F}} = -3.2 \times 10^{-17}\,\text{N}\hat{\mathbf{i}}$,
$\vec{\mathbf{a}} = -3.51 \times 10^{13}\,\text{m/s}^2\hat{\mathbf{i}}$

Exercise:

Problem:

An electron and a proton, each starting from rest, are accelerated by the same uniform electric field of 200 N/C. Determine the distance and time for each particle to acquire a kinetic energy of 3.2×10^{-16} J.

Exercise:

Problem:

A spherical water droplet of radius $25\,\mu\text{m}$ carries an excess 250 electrons. What vertical electric field is needed to balance the gravitational force on the droplet at the surface of the earth?

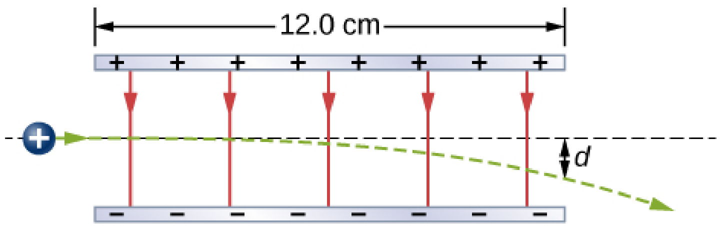
Solution:

$m = 6.5 \times 10^{-11}\,\text{kg}$,
$E = 1.6 \times 10^{7}\,\text{N/C}$

Exercise:

Problem:

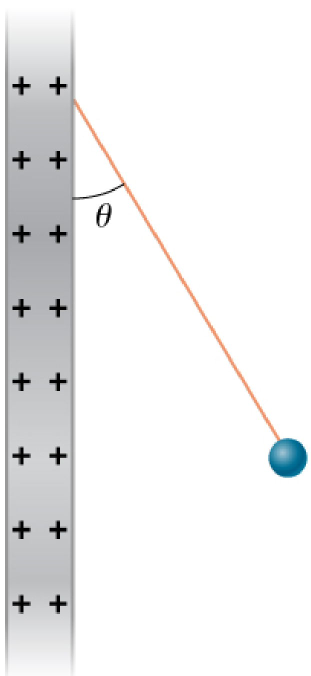
A proton enters the uniform electric field produced by the two charged plates shown below. The magnitude of the electric field is $4.0 \times 10^5\ \mathrm{N/C}$, and the speed of the proton when it enters is $1.5 \times 10^7\ \mathrm{m/s}$. What distance *d* has the proton been deflected downward when it leaves the plates?



Exercise:

Problem:

Shown below is a small sphere of mass 0.25 g that carries a charge of $9.0 \times 10^{-10}\ \mathrm{C}$. The sphere is attached to one end of a very thin silk string 5.0 cm long. The other end of the string is attached to a large vertical conducting plate that has a charge density of $30 \times 10^{-6}\ \mathrm{C/m^2}$. What is the angle that the string makes with the vertical?



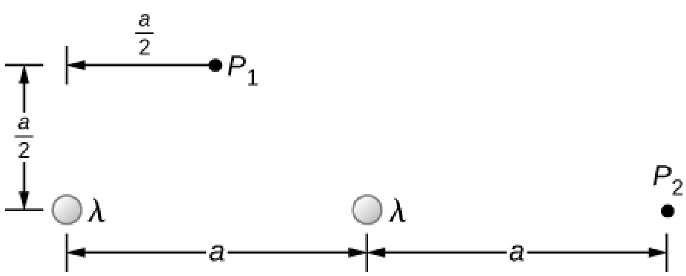
Solution:

$E = 1.70 \times 10^6 \text{ N/C},$
$F = 1.53 \times 10^{-3} \text{ N} \;\; T\cos\theta = mg \;\; T\sin\theta = qE,$
$\tan\theta = 0.62 \Rightarrow \theta = 32.0^\circ,$
This is independent of the length of the string.

Exercise:

Problem:

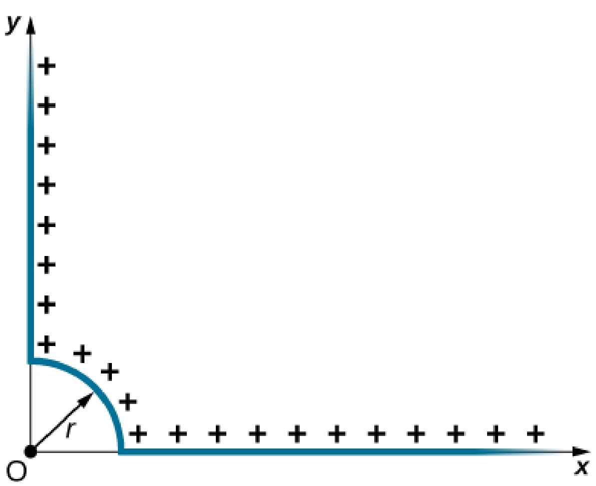
Two infinite rods, each carrying a uniform charge density λ, are parallel to one another and perpendicular to the plane of the page. (See below.) What is the electrical field at P_1? At P_2?



Exercise:

Problem:

Positive charge is distributed with a uniform density λ along the positive *x*-axis from r to ∞, along the positive *y*-axis from r to ∞, and along a 90° arc of a circle of radius *r*, as shown below. What is the electric field at *O*?



Solution:

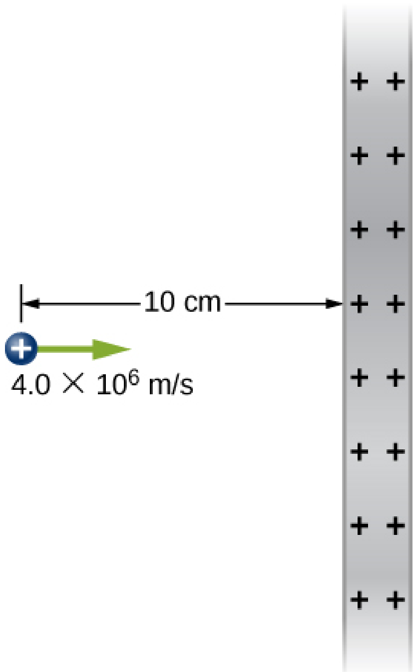
circular arc $dE_x(-\hat{\mathbf{i}}) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2}\cos\theta(-\hat{\mathbf{i}}),$
$\vec{\mathbf{E}}_x = \frac{\lambda}{4\pi\varepsilon_0 r}(-\hat{\mathbf{i}}),$
$dE_y(-\hat{\mathbf{i}}) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda ds}{r^2}\sin\theta(-\hat{\mathbf{j}}),$

$\vec{\mathbf{E}}_y = \frac{\lambda}{4\pi\varepsilon_0 r}(-\hat{\mathbf{j}})$;
y-axis: $\vec{\mathbf{E}}_x = \frac{\lambda}{4\pi\varepsilon_0 r}(-\hat{\mathbf{i}})$;
x-axis: $\vec{\mathbf{E}}_y = \frac{\lambda}{4\pi\varepsilon_0 r}(-\hat{\mathbf{j}})$,
$\vec{\mathbf{E}} = \frac{\lambda}{2\pi\varepsilon_0 r}(-\hat{\mathbf{i}}) + \frac{\lambda}{2\pi\varepsilon_0 r}(-\hat{\mathbf{j}})$

Exercise:

Problem:

From a distance of 10 cm, a proton is projected with a speed of $v = 4.0 \times 10^6$ m/s directly at a large, positively charged plate whose charge density is $\sigma = 2.0 \times 10^{-5}\ \mathrm{C/m^2}$. (See below.) (a) Does the proton reach the plate? (b) If not, how far from the plate does it turn around?



Exercise:

Problem:

A particle of mass m and charge $-q$ moves along a straight line away from a fixed particle of charge Q. When the distance between the two particles is r_0, $-q$ is moving with a speed v_0. (a) Use the work-energy theorem to calculate the maximum separation of the charges. (b) What do you have to assume about v_0 to make this calculation? (c) What is the minimum value of v_0 such that $-q$ escapes from Q?

Solution:

a. $W = \frac{1}{2}m(v^2 - v_0^2)$, $\frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{1}{2}m(v^2 - v_0^2) \Rightarrow r_0 - r = \frac{4\pi\varepsilon_0}{Qq}\ \frac{1}{2}rr_0 m(v^2 - v_0^2)$
; b. $r_0 - r$ is negative; therefore, $v_0 > v$,

$r \to \infty$, and $v \to 0$: $\frac{Qq}{4\pi\varepsilon_0}\left(-\frac{1}{r_0}\right) = -\frac{1}{2}mv_0^2 \Rightarrow v_0 = \sqrt{\frac{Qq}{2\pi\varepsilon_0 m r_0}}$

Glossary

continuous charge distribution
: total source charge composed of so large a number of elementary charges that it must be treated as continuous, rather than discrete

infinite plane
: flat sheet in which the dimensions making up the area are much, much greater than its thickness, and also much, much greater than the distance at which the field is to be calculated; its field is constant

infinite straight wire
: straight wire whose length is much, much greater than either of its other dimensions, and also much, much greater than the distance at which the field is to be calculated

linear charge density
: amount of charge in an element of a charge distribution that is essentially one-dimensional (the width and height are much, much smaller than its length); its units are C/m

surface charge density
: amount of charge in an element of a two-dimensional charge distribution (the thickness is small); its units are $\mathrm{C/m^2}$

volume charge density
: amount of charge in an element of a three-dimensional charge distribution; its units are $\mathrm{C/m^3}$

Electric Field Lines

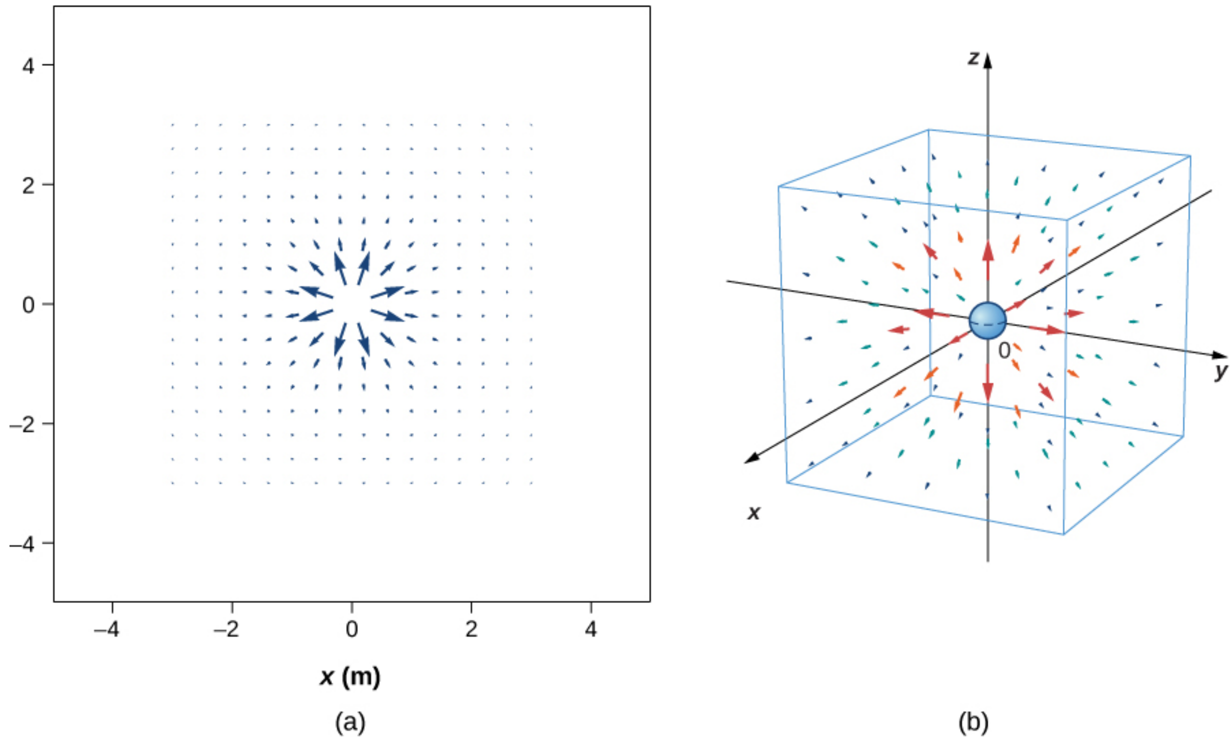
By the end of this section, you will be able to:

- Explain the purpose of an electric field diagram
- Describe the relationship between a vector diagram and a field line diagram
- Explain the rules for creating a field diagram and why these rules make physical sense
- Sketch the field of an arbitrary source charge

Now that we have some experience calculating electric fields, let’s try to gain some insight into the geometry of electric fields. As mentioned earlier, our model is that the charge on an object (the source charge) alters space in the region around it in such a way that when another charged object (the test charge) is placed in that region of space, that test charge experiences an electric force. The concept of electric **field lines**, and of electric field line diagrams, enables us to visualize the way in which the space is altered, allowing us to visualize the field. The purpose of this section is to enable you to create sketches of this geometry, so we will list the specific steps and rules involved in creating an accurate and useful sketch of an electric field.

It is important to remember that electric fields are three-dimensional. Although in this book we include some pseudo-three-dimensional images, several of the diagrams that you’ll see (both here, and in subsequent chapters) will be two-dimensional projections, or cross-sections. Always keep in mind that in fact, you’re looking at a three-dimensional phenomenon.

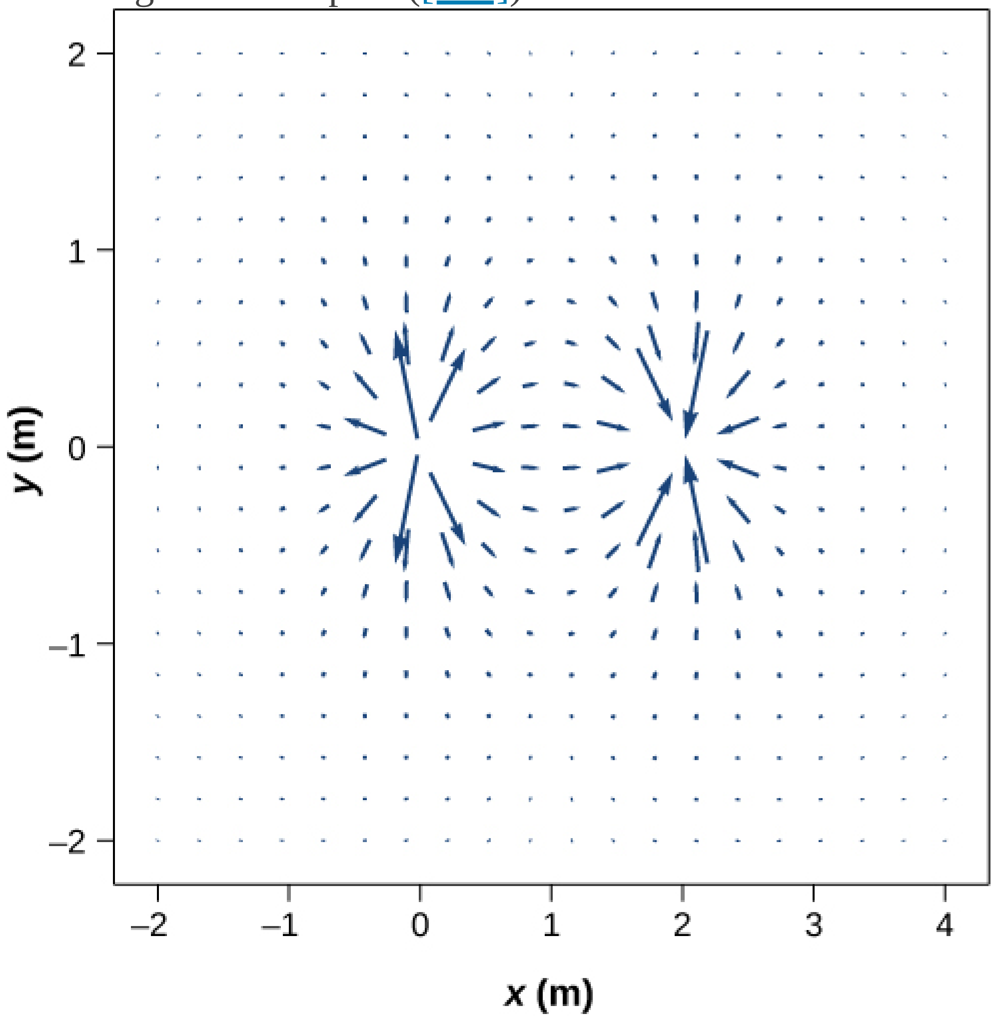
Our starting point is the physical fact that the electric field of the source charge causes a test charge in that field to experience a force. By definition, electric field vectors point in the same direction as the electric force that a (hypothetical) positive test charge would experience, if placed in the field ([link])



The electric field of a positive point charge. A large number of field vectors are shown. Like all vector arrows, the length of each vector is proportional to the magnitude of the field at each point. (a) Field in two dimensions; (b) field in three dimensions.

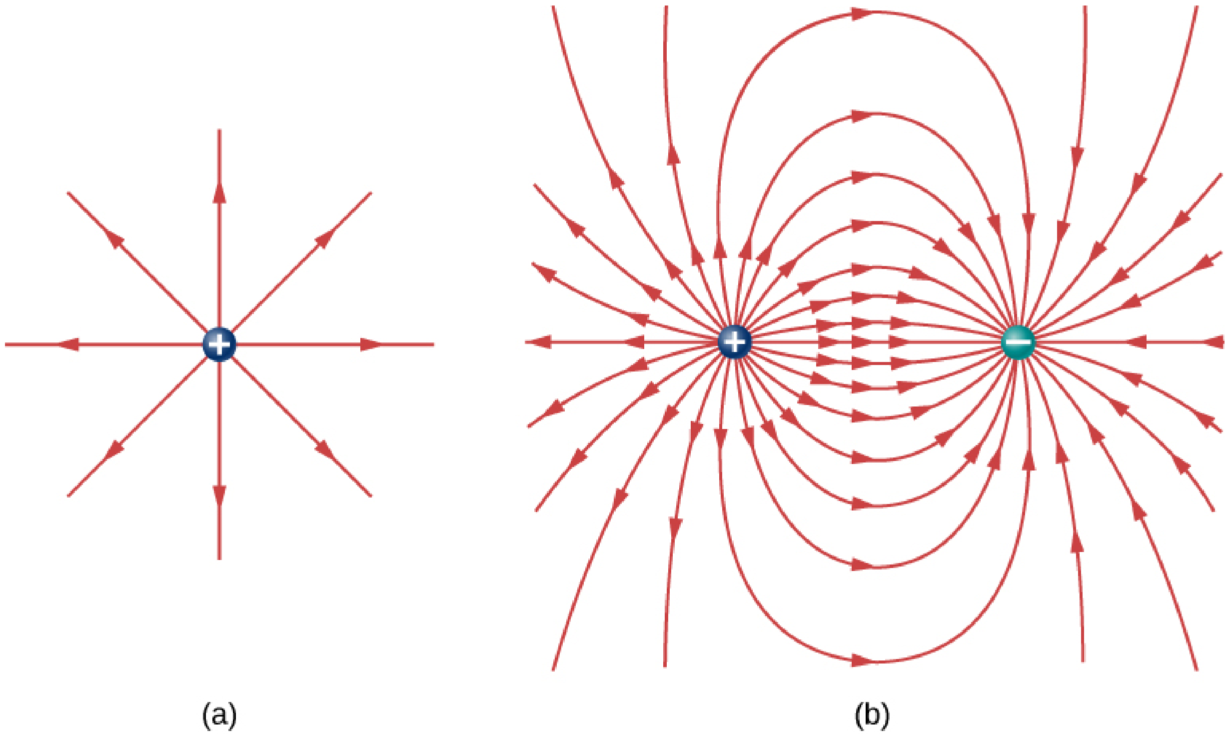
We’ve plotted many field vectors in the figure, which are distributed uniformly around the source charge. Since the electric field is a vector, the arrows that we draw correspond at every point in space to both the magnitude and the direction of the field at that point. As always, the length of the arrow that we draw corresponds to the magnitude of the field vector at that point. For a point source charge, the length decreases by the square of the distance from the source charge. In addition, the direction of the field vector is radially away from the source charge, because the direction of the electric field is defined by the direction of the force that a positive test charge would experience in that field. (Again, keep in mind that the actual field is three-dimensional; there are also field lines pointing out of and into the page.)

This diagram is correct, but it becomes less useful as the source charge distribution becomes more complicated. For example, consider the vector field diagram of a dipole ([link]).



The vector field of a dipole. Even with just two identical charges, the vector field diagram becomes difficult to understand.

There is a more useful way to present the same information. Rather than drawing a large number of increasingly smaller vector arrows, we instead connect all of them together, forming continuous lines and curves, as shown in [link].

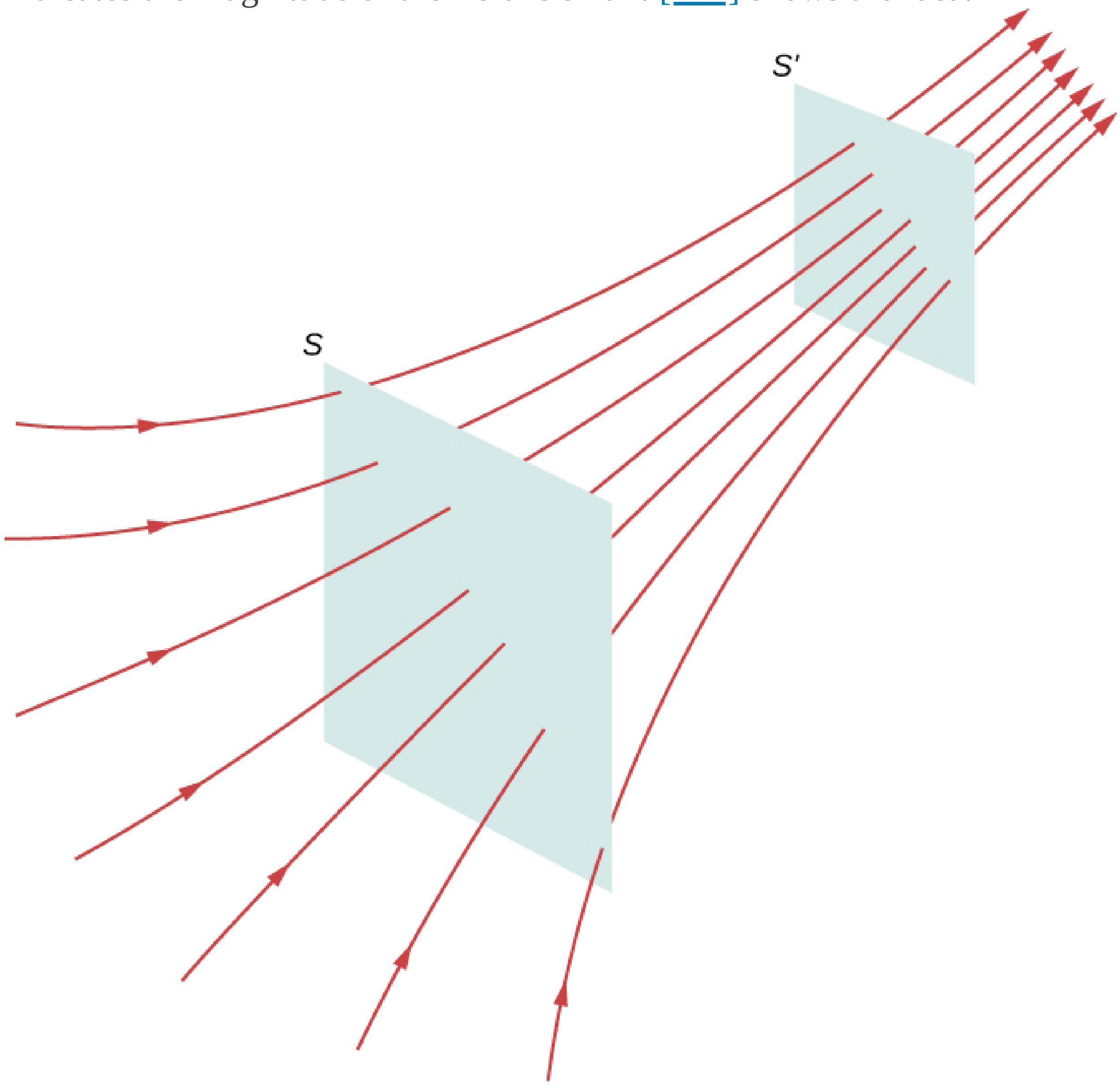


(a) The electric field line diagram of a positive point charge. (b) The field line diagram of a dipole. In both diagrams, the magnitude of the field is indicated by the field line density. The field *vectors* (not shown here) are everywhere tangent to the field lines.

Although it may not be obvious at first glance, these field diagrams convey the same information about the electric field as do the vector diagrams. First, the direction of the field at every point is simply the direction of the field vector at that same point. In other words, at any point in space, the field vector at each point is tangent to the field line at that same point. The arrowhead placed on a field line indicates its direction.

As for the magnitude of the field, that is indicated by the **field line density** —that is, the number of field lines per unit area passing through a small cross-sectional area perpendicular to the electric field. This field line density is drawn to be proportional to the magnitude of the field at that cross-section. As a result, if the field lines are close together (that is, the field line density is greater), this indicates that the magnitude of the field is

large at that point. If the field lines are far apart at the cross-section, this indicates the magnitude of the field is small. [link] shows the idea.



Electric field lines passing through imaginary areas. Since the number of lines passing through each area is the same, but the areas themselves are different, the field line density is different. This indicates different magnitudes of the electric field at these points.

In [link], the same number of field lines passes through both surfaces (S and $S\prime$), but the surface S is larger than surface $S\prime$. Therefore, the density of field lines (number of lines per unit area) is larger at the location of $S\prime$, indicating that the electric field is stronger at the location of $S\prime$ than at S. The rules for creating an electric field diagram are as follows.

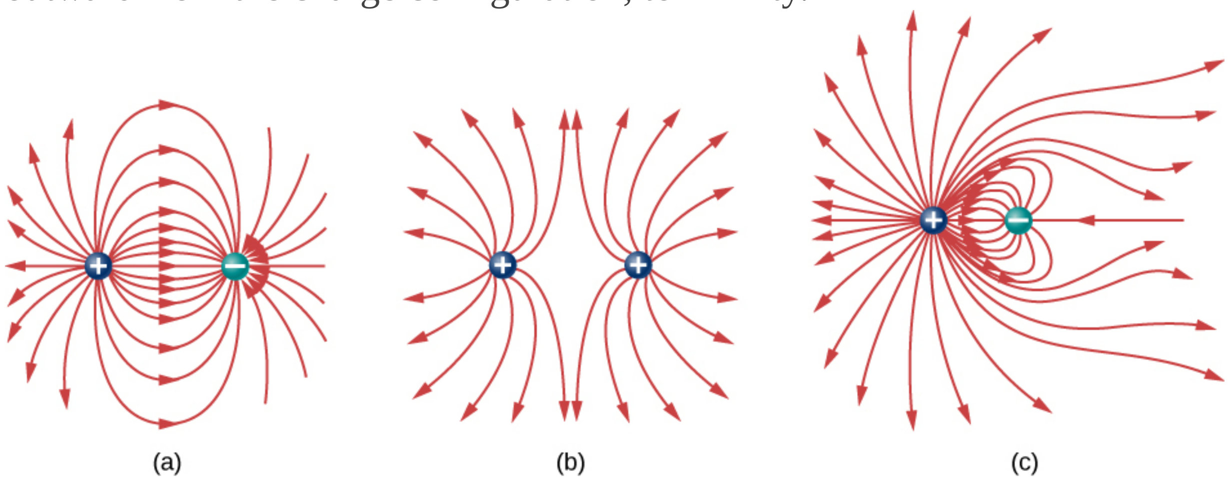
Note:
Problem-Solving Strategy: Drawing Electric Field Lines

1. Electric field lines either originate on positive charges or come in from infinity, and either terminate on negative charges or extend out to infinity.
2. The number of field lines originating or terminating at a charge is proportional to the magnitude of that charge. A charge of $2q$ will have twice as many lines as a charge of q.
3. At every point in space, the field vector at that point is tangent to the field line at that same point.
4. The field line density at any point in space is proportional to (and therefore is representative of) the magnitude of the field at that point in space.
5. Field lines can never cross. Since a field line represents the direction of the field at a given point, if two field lines crossed at some point, that would imply that the electric field was pointing in two different directions at a single point. This in turn would suggest that the (net) force on a test charge placed at that point would point in two different directions. Since this is obviously impossible, it follows that field lines must never cross.

Always keep in mind that field lines serve only as a convenient way to visualize the electric field; they are not physical entities. Although the direction and relative intensity of the electric field can be deduced from a set of field lines, the lines can also be misleading. For example, the field lines drawn to represent the electric field in a region must, by necessity, be

discrete. However, the actual electric field in that region exists at every point in space.

Field lines for three groups of discrete charges are shown in [link]. Since the charges in parts (a) and (b) have the same magnitude, the same number of field lines are shown starting from or terminating on each charge. In (c), however, we draw three times as many field lines leaving the $+3q$ charge as entering the $-q$. The field lines that do not terminate at $-q$ emanate outward from the charge configuration, to infinity.



Three typical electric field diagrams. (a) A dipole. (b) Two identical charges. (c) Two charges with opposite signs and different magnitudes. Can you tell from the diagram which charge has the larger magnitude?

The ability to construct an accurate electric field diagram is an important, useful skill; it makes it much easier to estimate, predict, and therefore calculate the electric field of a source charge. The best way to develop this skill is with software that allows you to place source charges and then will draw the net field upon request. We strongly urge you to search the Internet for a program. Once you've found one you like, run several simulations to get the essential ideas of field diagram construction. Then practice drawing field diagrams, and checking your predictions with the computer-drawn diagrams.

Note:
One example of a field-line drawing program is from the PhET “Charges and Fields” simulation.

Summary

- Electric field diagrams assist in visualizing the field of a source charge.
- The magnitude of the field is proportional to the field line density.
- Field vectors are everywhere tangent to field lines.

Conceptual Questions

Exercise:

Problem:

If a point charge is released from rest in a uniform electric field, will it follow a field line? Will it do so if the electric field is not uniform?

Solution:

yes; no

Exercise:

Problem:

Under what conditions, if any, will the trajectory of a charged particle not follow a field line?

Exercise:

Problem:

How would you experimentally distinguish an electric field from a gravitational field?

Solution:

At the surface of Earth, the gravitational field is always directed in toward Earth's center. An electric field could move a charged particle in a different direction than toward the center of Earth. This would indicate an electric field is present.

Exercise:

Problem:

A representation of an electric field shows 10 field lines perpendicular to a square plate. How many field lines should pass perpendicularly through the plate to depict a field with twice the magnitude?

Exercise:

Problem:

What is the ratio of the number of electric field lines leaving a charge $10q$ and a charge q?

Solution:

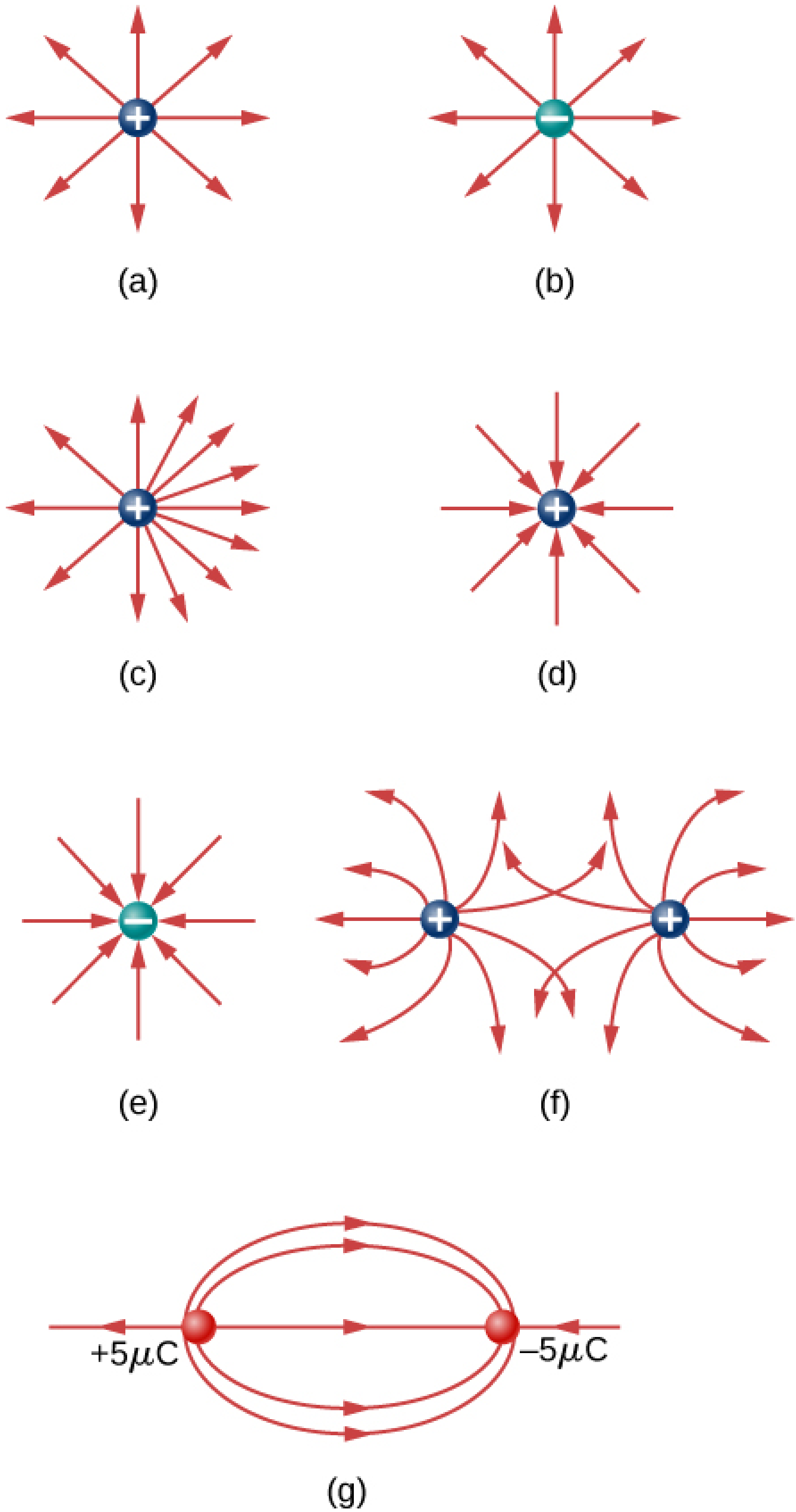
10

Problems

Exercise:

Problem:

Which of the following electric field lines are incorrect for point charges? Explain why.



(a)
(b)
(c)
(d)
(e)
(f)
+5μC
–5μC
(g)

Exercise:

Problem:

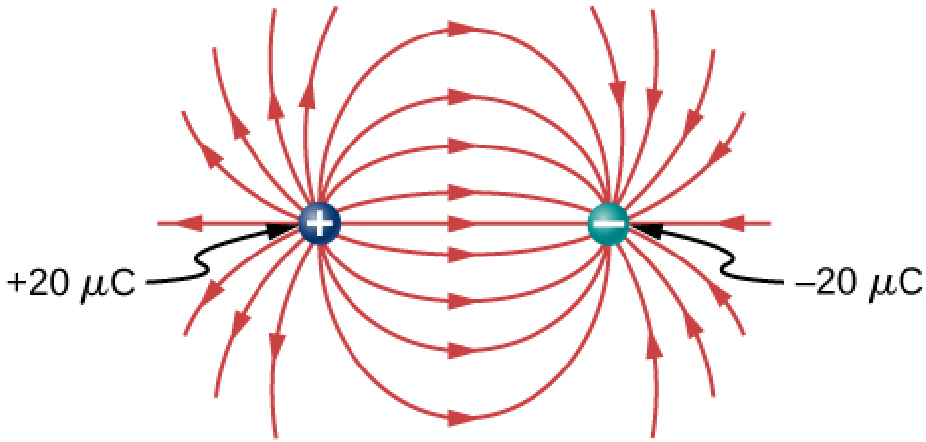
In this exercise, you will practice drawing electric field lines. Make sure you represent both the magnitude and direction of the electric field adequately. Note that the number of lines into or out of charges is proportional to the charges.

(a) Draw the electric field lines map for two charges $+20\ \mu\mathrm{C}$ and $-20\ \mu\mathrm{C}$ situated 5 cm from each other.

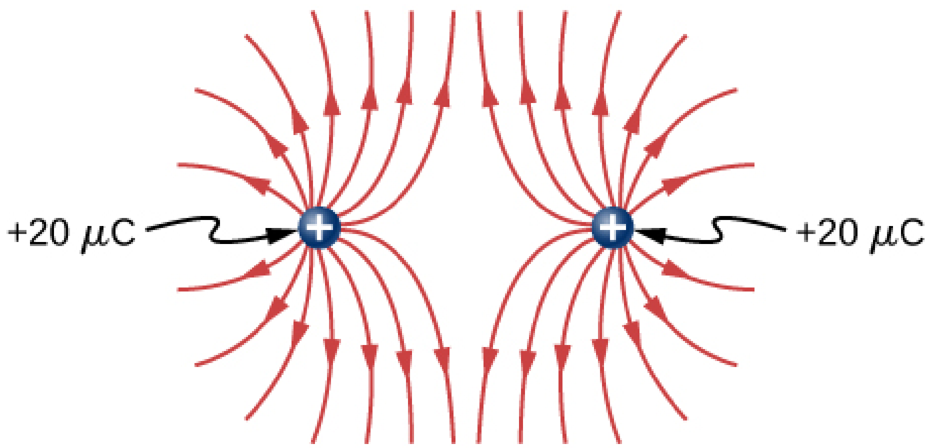
(b) Draw the electric field lines map for two charges $+20\ \mu\mathrm{C}$ and $+20\ \mu\mathrm{C}$ situated 5 cm from each other.

(c) Draw the electric field lines map for two charges $+20\ \mu\mathrm{C}$ and $-30\ \mu\mathrm{C}$ situated 5 cm from each other.

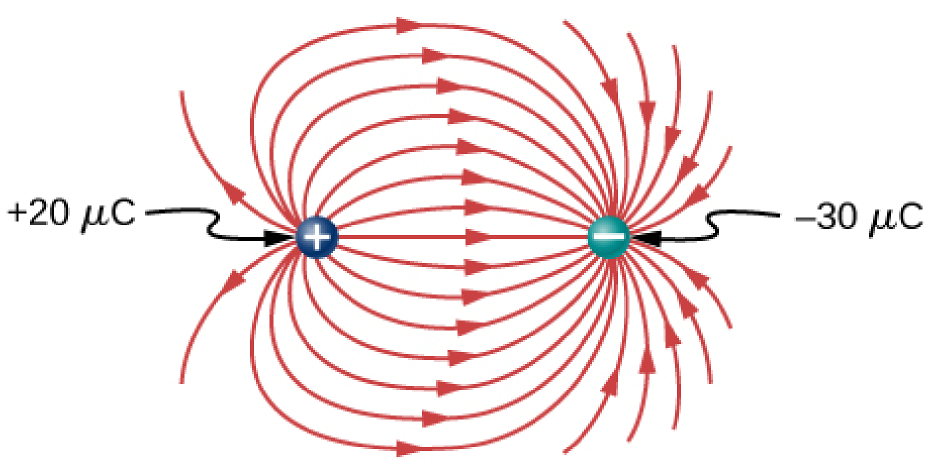
Solution:



(a)



(b)



(c)

Exercise:

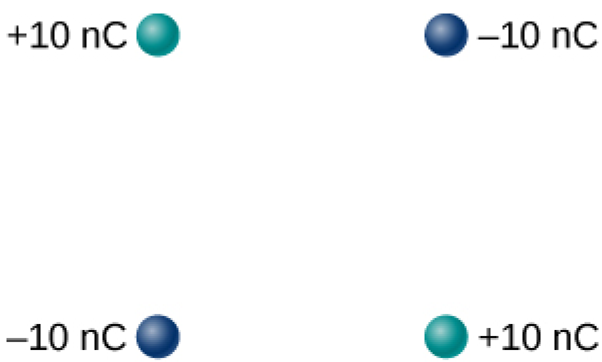
Problem:

Draw the electric field for a system of three particles of charges $+1\ \mu\text{C}$, $+2\ \mu\text{C}$, and $-3\ \mu\text{C}$ fixed at the corners of an equilateral triangle of side 2 cm.

Exercise:

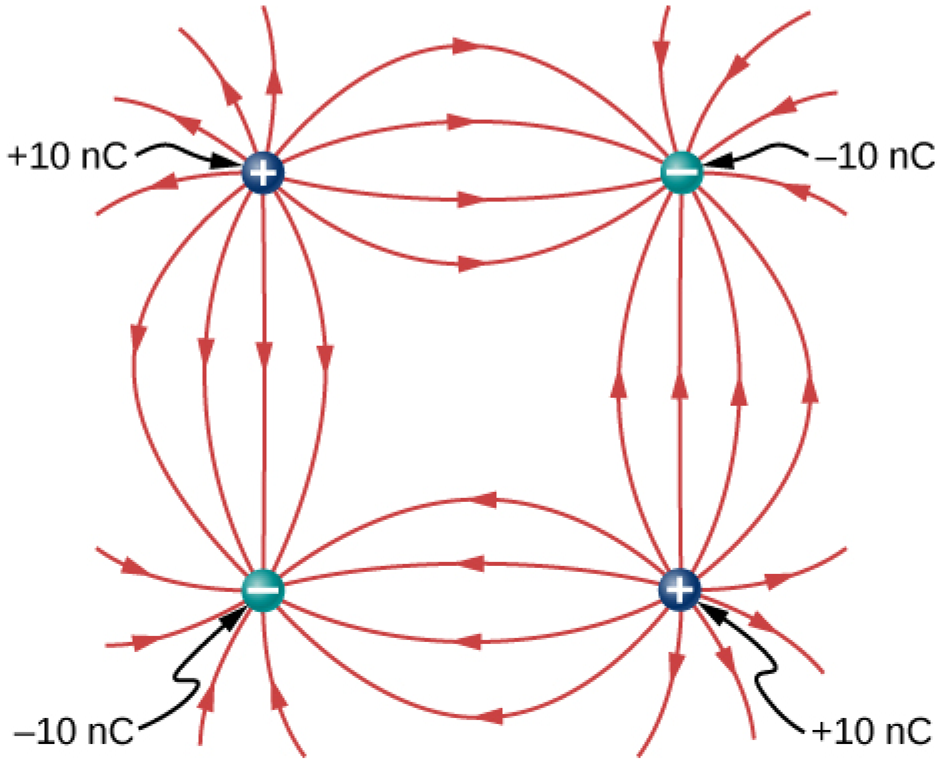
Problem:

Two charges of equal magnitude but opposite sign make up an electric dipole. A quadrupole consists of two electric dipoles that are placed anti-parallel at two edges of a square as shown.



Draw the electric field of the charge distribution.

Solution:



Exercise:

Problem:

Suppose the electric field of an isolated point charge decreased with distance as $1/r^{2+\delta}$ rather than as $1/r^2$. Show that it is then impossible to draw continous field lines so that their number per unit area is proportional to E.

Glossary

field line
: smooth, usually curved line that indicates the direction of the electric field

field line density
: number of field lines per square meter passing through an imaginary area; its purpose is to indicate the field strength at different points in space

Introduction
class="introduction"

This chapter introduces the concept of flux, which relates a physical quantity and the area through which it is flowing. Although we introduce this concept with the electric field, the concept may be used for many other quantities, such as fluid flow. (credit: modification of work by “Alessandro”/Flickr)

Flux is a general and broadly applicable concept in physics. However, in this chapter, we concentrate on the flux of the electric field. This allows us to introduce Gauss's law, which is particularly useful for finding the electric fields of charge distributions exhibiting spatial symmetry. The main topics discussed here are

1. **Electric flux.** We define electric flux for both open and closed surfaces.
2. **Gauss's law.** We derive Gauss's law for an arbitrary charge distribution and examine the role of electric flux in Gauss's law.
3. **Calculating electric fields with Gauss's law.** The main focus of this chapter is to explain how to use Gauss's law to find the electric fields of spatially symmetrical charge distributions. We discuss the importance of choosing a Gaussian surface and provide examples involving the applications of Gauss's law.
4. **Electric fields in conductors.** Gauss's law provides useful insight into the absence of electric fields in conducting materials.

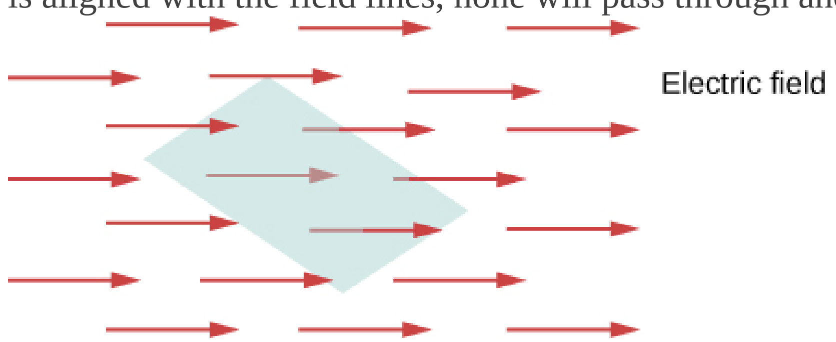
So far, we have found that the electrostatic field begins and ends at point charges and that the field of a point charge varies inversely with the square of the distance from that charge. These characteristics of the electrostatic field lead to an important mathematical relationship known as Gauss's law. This law is named in honor of the extraordinary German mathematician and scientist Karl Friedrich Gauss ([link]). Gauss's law gives us an elegantly simple way of finding the electric field, and, as you will see, it can be much easier to use than the integration method described in the previous chapter. However, there is a catch—Gauss's law has a limitation in that, while always true, it can be readily applied only for charge distributions with certain symmetries.

Karl Friedrich Gauss (1777–1855) was a legendary mathematician of the nineteenth century. Although his major contributions were to the field of mathematics, he also did important work in physics and astronomy.

Electric Flux
By the end of this section, you will be able to:

- Define the concept of flux
- Describe electric flux
- Calculate electric flux for a given situation

The concept of **flux** describes how much of something goes through a given area. More formally, it is the dot product of a vector field (in this chapter, the electric field) with an area. You may conceptualize the flux of an electric field as a measure of the number of electric field lines passing through an area ([link]). The larger the area, the more field lines go through it and, hence, the greater the flux; similarly, the stronger the electric field is (represented by a greater density of lines), the greater the flux. On the other hand, if the area rotated so that the plane is aligned with the field lines, none will pass through and there will be no flux.

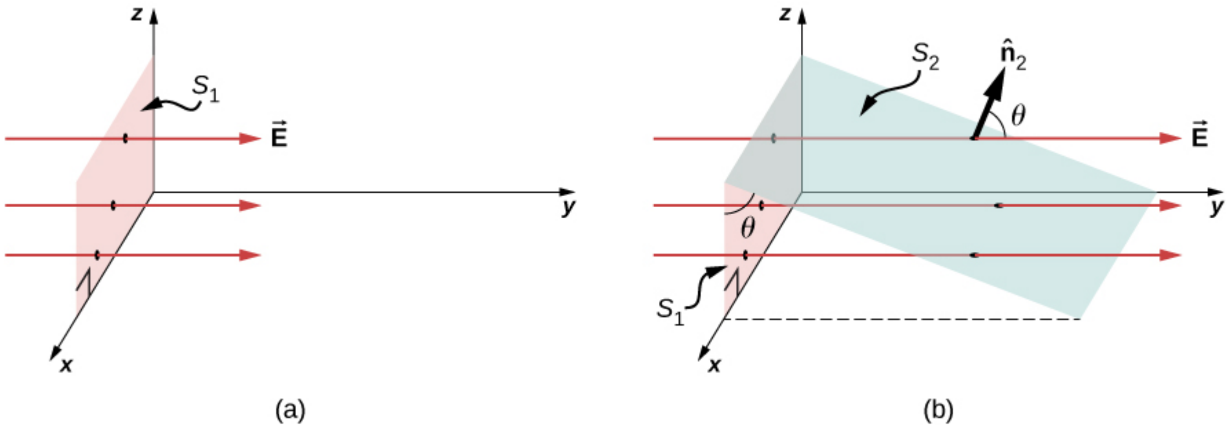


The flux of an electric field through the shaded area captures information about the “number” of electric field lines passing through the area. The numerical value of the electric flux depends on the magnitudes of the electric field and the area, as well as the relative orientation of the area with respect to the direction of the electric field.

A macroscopic analogy that might help you imagine this is to put a hula hoop in a flowing river. As you change the angle of the hoop relative to the direction of the current, more or less of the flow will go through the hoop. Similarly, the amount of flow through the hoop depends on the strength of the current and the size of the hoop. Again, flux is a general concept; we can also use it to describe the amount of sunlight hitting a solar panel or the amount of energy a telescope receives from a distant star, for example.

To quantify this idea, [link](a) shows a planar surface S_1 of area A_1 that is perpendicular to the uniform electric field $\vec{\mathbf{E}} = E\hat{\mathbf{y}}$. If N field lines pass through S_1, then we know from the definition of electric field lines (Electric Charges and Fields) that $N/A_1 \propto E$, or $N \propto EA_1$.

The quantity EA_1 is the **electric flux** through S_1. We represent the electric flux through an open surface like S_1 by the symbol Φ. Electric flux is a scalar quantity and has an SI unit of newton-meters squared per coulomb ($\mathrm{N \cdot m^2/C}$). Notice that $N \propto EA_1$ may also be written as $N \propto \Phi$, demonstrating that *electric flux is a measure of the number of field lines crossing a surface*.



(a) A planar surface S_1 of area A_1 is perpendicular to the electric field $E\hat{\mathbf{j}}$. *N* field lines cross surface S_1. (b) A surface S_2 of area A_2 whose projection onto the *xz*-plane is S_1.The same number of field lines cross each surface.

Now consider a planar surface that is not perpendicular to the field. How would we represent the electric flux? [link](b) shows a surface S_2 of area A_2 that is inclined at an angle θ to the *xz*-plane and whose projection in that plane is S_1 (area A_1). The areas are related by $A_2 \cos\theta = A_1$. Because the same number of field lines crosses both S_1 and S_2, the fluxes through both surfaces must be the same. The flux through S_2 is therefore $\Phi = EA_1 = EA_2 \cos\theta$. Designating $\hat{\mathbf{n}}_2$ as a unit vector normal to S_2 (see [link](b)), we obtain

Equation:

$$\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}_2 A_2.$$

Note:
Check out this video to observe what happens to the flux as the area changes in size and angle, or the electric field changes in strength.

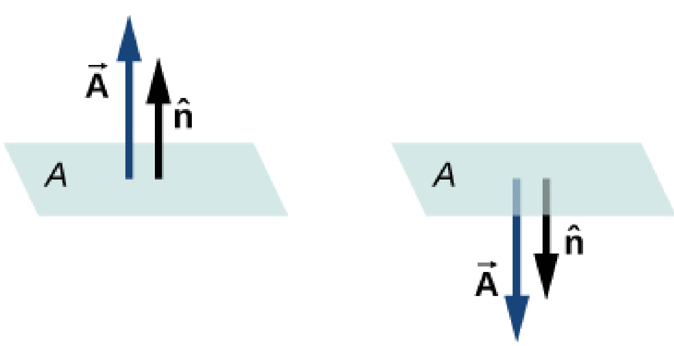
Area Vector

For discussing the flux of a vector field, it is helpful to introduce an area vector $\vec{\mathbf{A}}$. This allows us to write the last equation in a more compact form. What should the magnitude of the area vector be? What should the direction of the area vector be? What are the implications of how you answer the previous question?

The **area vector** of a flat surface of area A has the following magnitude and direction:

- Magnitude is equal to area (A)
- Direction is along the normal to the surface ($\hat{\mathbf{n}}$); that is, perpendicular to the surface.

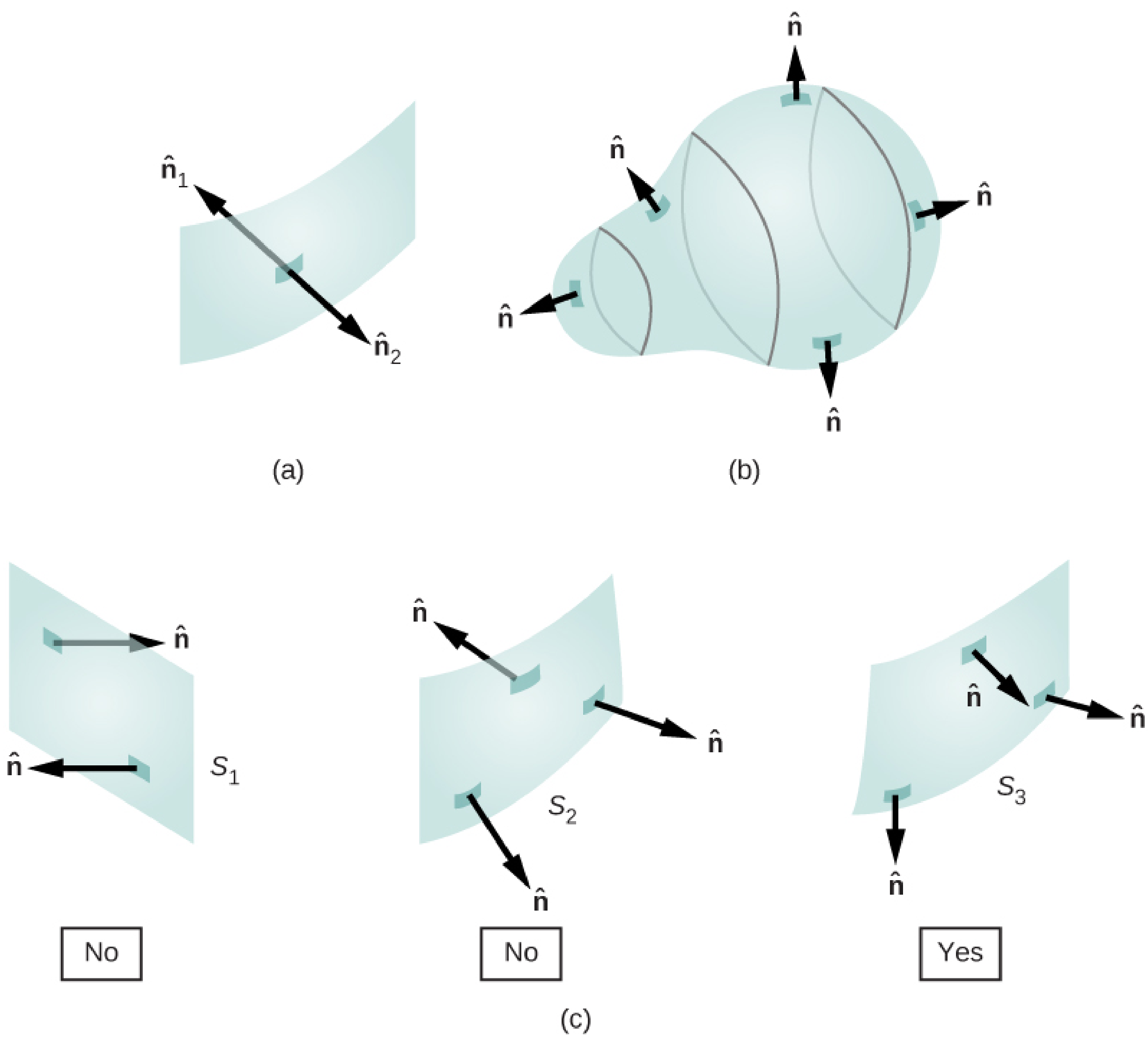
Since the normal to a flat surface can point in either direction from the surface, the direction of the area vector of an open surface needs to be chosen, as shown in [link].



The direction of the area vector of an open surface needs to be chosen; it could be either of the two cases displayed here. The area vector of a part of a closed surface is defined to point from the inside of the closed space to

the outside. This rule gives a unique direction.

Since $\hat{\mathbf{n}}$ is a unit normal to a surface, it has two possible directions at every point on that surface ([link](a)). For an open surface, we can use either direction, as long as we are consistent over the entire surface. Part (c) of the figure shows several cases.



(a) Two potential normal vectors arise at every point on a surface. (b) The outward normal is used to calculate the flux through a closed surface. (c)

Only S_3 has been given a consistent set of normal vectors that allows us to define the flux through the surface.

However, if a surface is closed, then the surface encloses a volume. In that case, the direction of the normal vector at any point on the surface points from the inside to the outside. On a *closed surface* such as that of [link](b), $\hat{\mathbf{n}}$ is chosen to be the *outward normal* at every point, to be consistent with the sign convention for electric charge.

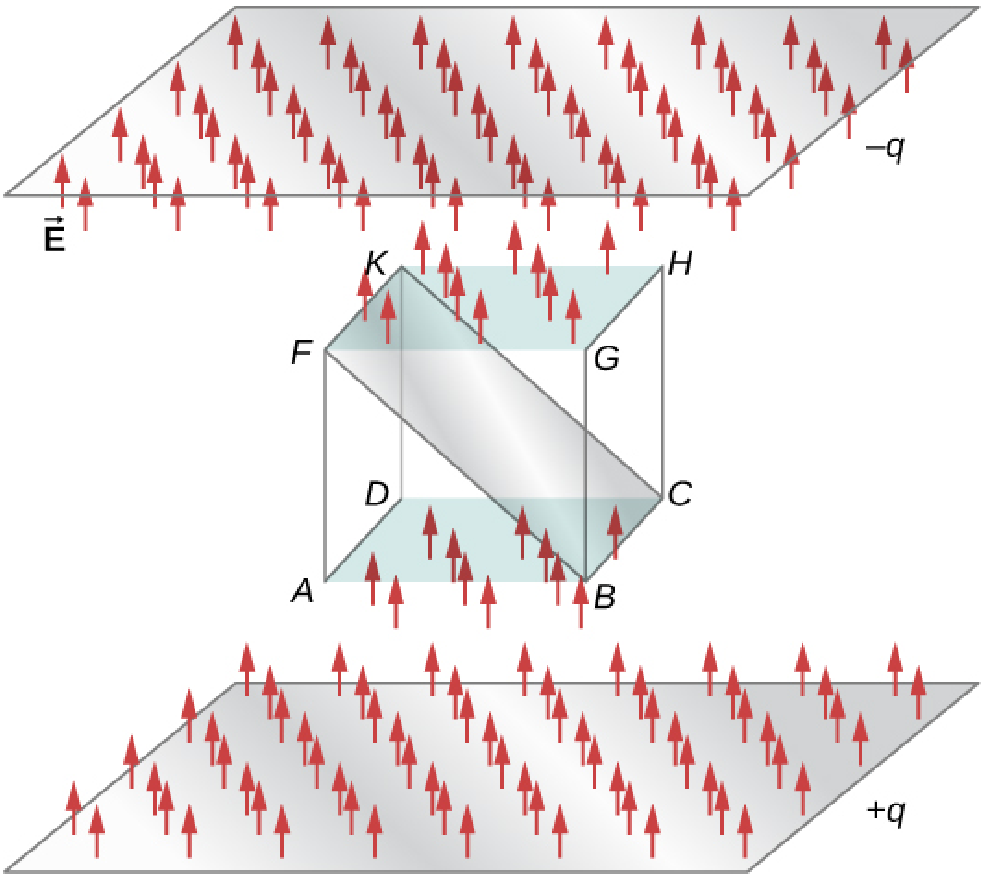
Electric Flux

Now that we have defined the area vector of a surface, we can define the electric flux of a uniform electric field through a flat area as the scalar product of the electric field and the area vector, as defined in Products of Vectors:

Note:
Equation:

$$\Phi = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} \,(\text{uniform}\, \overrightarrow{\mathbf{E}},\ \text{flat surface}).$$

[link] shows the electric field of an oppositely charged, parallel-plate system and an imaginary box between the plates. The electric field between the plates is uniform and points from the positive plate toward the negative plate. A calculation of the flux of this field through various faces of the box shows that the net flux through the box is zero. Why does the flux cancel out here?

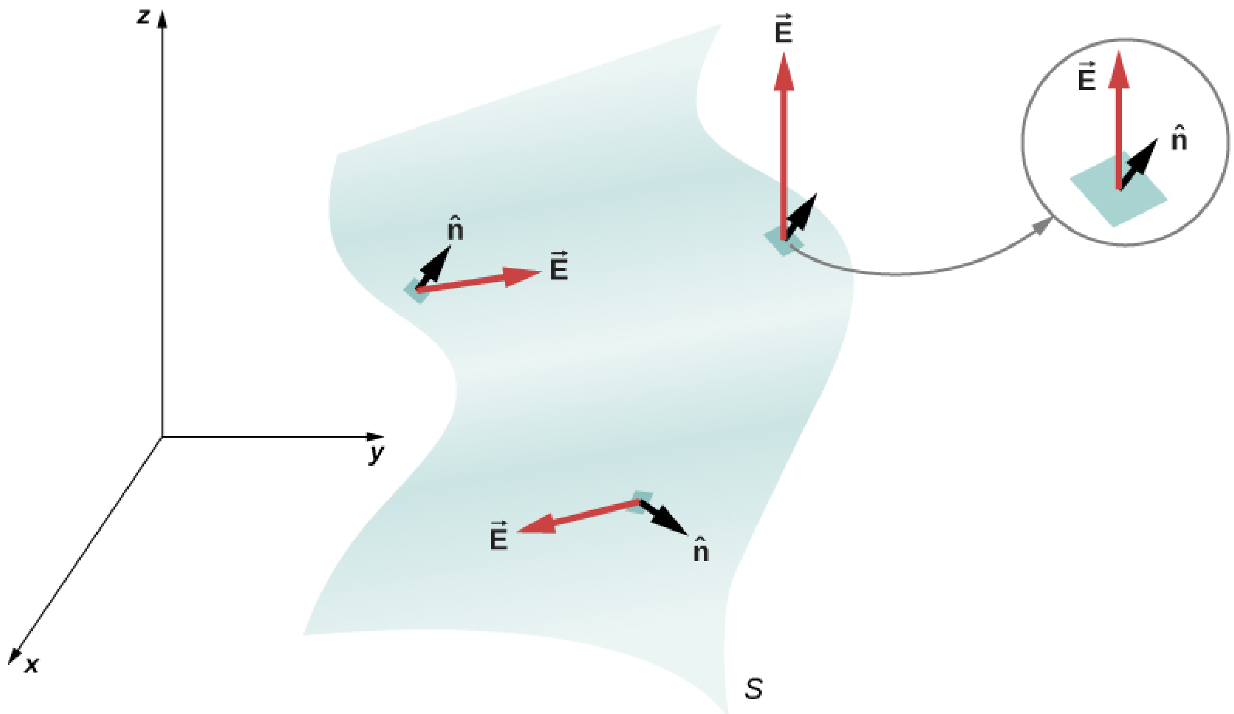


Electric flux through a cube, placed between two charged plates. Electric flux through the bottom face (*ABCD*) is negative, because $\vec{\mathbf{E}}$ is in the opposite direction to the normal to the surface. The electric flux through the top face (*FGHK*) is positive, because the electric field and the normal are in the same direction. The electric flux through the other faces is zero, since the electric field is perpendicular to the normal vectors of those faces. The net electric flux through the cube is the sum of fluxes through the six faces. Here, the net flux through the cube is equal to zero. The magnitude of the flux through rectangle *BCKF* is equal to the magnitudes of the flux through both the top and bottom faces.

The reason is that the sources of the electric field are outside the box. Therefore, if any electric field line enters the volume of the box, it must also exit somewhere on the surface because there is no charge inside for the lines to land on. Therefore,

quite generally, electric flux through a closed surface is zero if there are no sources of electric field, whether positive or negative charges, inside the enclosed volume. In general, when field lines leave (or “flow out of”) a closed surface, Φ is positive; when they enter (or “flow into”) the surface, Φ is negative.

Any smooth, non-flat surface can be replaced by a collection of tiny, approximately flat surfaces, as shown in [link]. If we divide a surface S into small patches, then we notice that, as the patches become smaller, they can be approximated by flat surfaces. This is similar to the way we treat the surface of Earth as locally flat, even though we know that globally, it is approximately spherical.



A surface is divided into patches to find the flux.

To keep track of the patches, we can number them from 1 through N . Now, we define the area vector for each patch as the area of the patch pointed in the direction of the normal. Let us denote the area vector for the ith patch by $\delta \vec{\mathbf{A}}_i$. (We have used the symbol δ to remind us that the area is of an arbitrarily small

patch.) With sufficiently small patches, we may approximate the electric field over any given patch as uniform. Let us denote the average electric field at the location of the *i*th patch by $\vec{\mathbf{E}}_i$.

Equation:

$$\vec{\mathbf{E}}_i = \text{average electric field over the } i\text{th patch.}$$

Therefore, we can write the electric flux Φ_i through the area of the *i*th patch as

Equation:

$$\Phi_i = \overrightarrow{\mathbf{E}}_i \cdot \delta\overrightarrow{\mathbf{A}}_i \ (i\text{th patch}).$$

The flux through each of the individual patches can be constructed in this manner and then added to give us an estimate of the net flux through the entire surface *S*, which we denote simply as Φ.

Equation:

$$\Phi = \sum_{i=1}^{N} \Phi_i = \sum_{i=1}^{N} \overrightarrow{\mathbf{E}}_i \cdot \delta\overrightarrow{\mathbf{A}}_i \ (N \text{ patch estimate}).$$

This estimate of the flux gets better as we decrease the size of the patches. However, when you use smaller patches, you need more of them to cover the same surface. In the limit of infinitesimally small patches, they may be considered to have area *dA* and unit normal $\widehat{\mathbf{n}}$. Since the elements are infinitesimal, they may be assumed to be planar, and $\overrightarrow{\mathbf{E}}_i$ may be taken as constant over any element. Then the flux $d\Phi$ through an area *dA* is given by $d\Phi = \overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA$. It is positive when the angle between $\overrightarrow{\mathbf{E}}_i$ and $\widehat{\mathbf{n}}$ is less than 90° and negative when the angle is greater than 90°. The net flux is the sum of the infinitesimal flux elements over the entire surface. With infinitesimally small patches, you need infinitely many patches, and the limit of the sum becomes a surface integral. With $\int_S$ representing the integral over *S*,

Note:
Equation:

$$\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \text{ (open surface).}$$

In practical terms, surface integrals are computed by taking the antiderivatives of both dimensions defining the area, with the edges of the surface in question being the bounds of the integral.

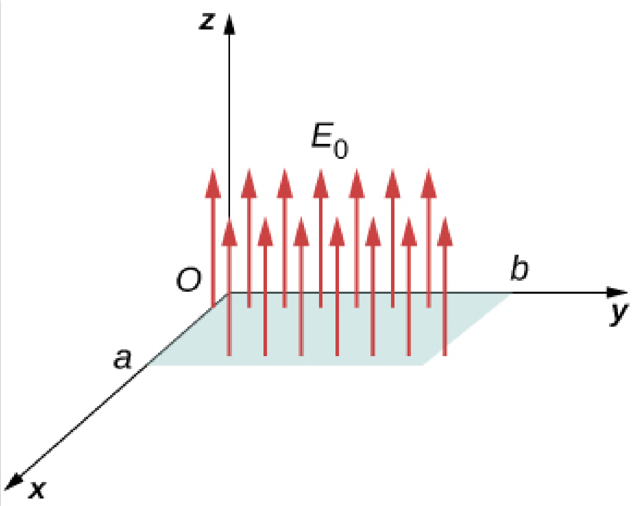
To distinguish between the flux through an open surface like that of [link] and the flux through a closed surface (one that completely bounds some volume), we represent flux through a closed surface by

Note:
Equation:

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \text{ (closed surface)}$$

where the circle through the integral symbol simply means that the surface is closed, and we are integrating over the entire thing. If you only integrate over a portion of a closed surface, that means you are treating a subset of it as an open surface.

Example:
Flux of a Uniform Electric Field
A constant electric field of magnitude E_0 points in the direction of the positive *z*-axis ([link]). What is the electric flux through a rectangle with sides *a* and *b* in the (a) *xy*-plane and in the (b) *xz*-plane?



Calculating the flux of E_0 through a rectangular surface.

Strategy

Apply the definition of flux: $\Phi = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}}$ (uniform $\vec{\mathbf{E}}$), where the definition of dot product is crucial.

Solution

a. In this case, $\Phi = \vec{\mathbf{E}}_0 \cdot \vec{\mathbf{A}} = E_0 A = E_0 ab$.
b. Here, the direction of the area vector is either along the positive *y*-axis or toward the negative *y*-axis. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

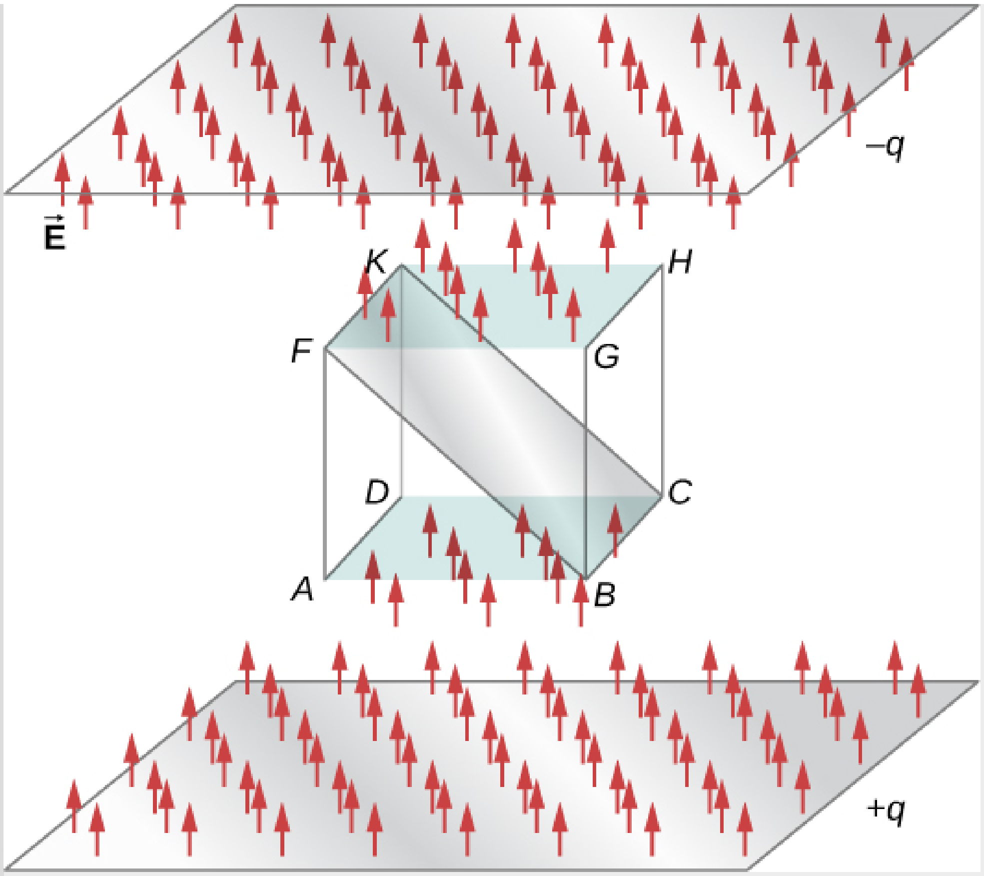
Significance

The relative directions of the electric field and area can cause the flux through the area to be zero.

Example:

Flux of a Uniform Electric Field through a Closed Surface

A constant electric field of magnitude E_0 points in the direction of the positive *z*-axis ([link]). What is the net electric flux through a cube?



Calculating the flux of E_0 through a closed cubic surface.

Strategy

Apply the definition of flux: $\Phi = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}}$ (uniform $\vec{\mathbf{E}}$), noting that a closed surface eliminates the ambiguity in the direction of the area vector.

Solution

Through the top face of the cube, $\Phi = \vec{\mathbf{E}}_0 \cdot \vec{\mathbf{A}} = E_0 A$.

Through the bottom face of the cube, $\Phi = \vec{\mathbf{E}}_0 \cdot \vec{\mathbf{A}} = -E_0 A$, because the area vector here points downward.

Along the other four sides, the direction of the area vector is perpendicular to the direction of the electric field. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

The net flux is $\Phi_{\text{net}} = E_0 A - E_0 A + 0 + 0 + 0 + 0 = 0$.

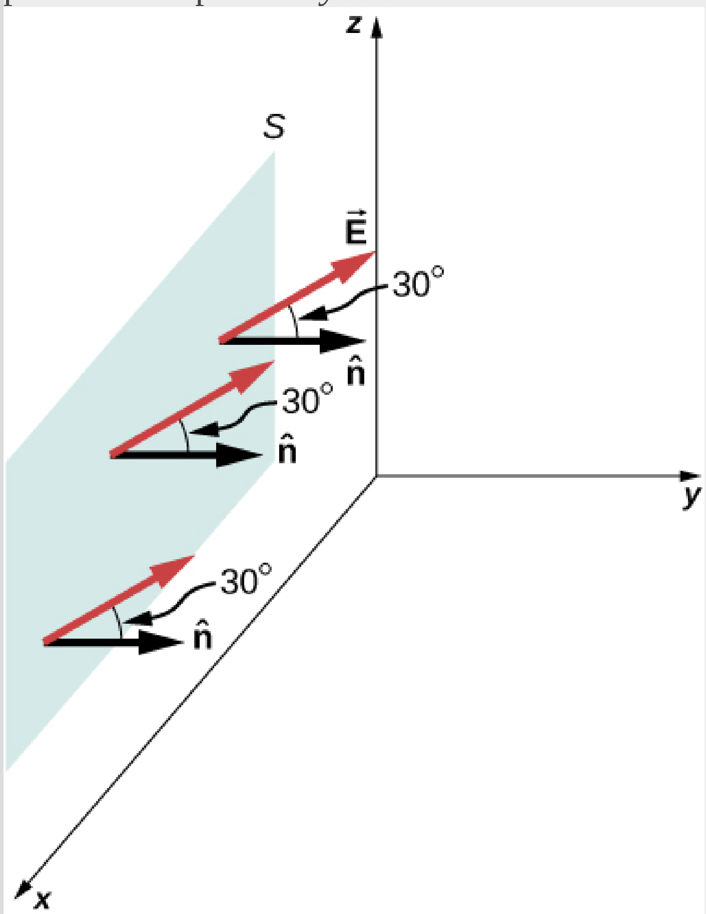
Significance

The net flux of a uniform electric field through a closed surface is zero.

Example:

Electric Flux through a Plane, Integral Method

A uniform electric field $\vec{\mathbf{E}}$ of magnitude 10 N/C is directed parallel to the *yz*-plane at 30° above the *xy*-plane, as shown in [link]. What is the electric flux through the plane surface of area $6.0\ \mathrm{m}^2$ located in the *xz*-plane? Assume that $\hat{\mathbf{n}}$ points in the positive *y*-direction.



The electric field produces a net electric flux through the surface *S*.

Strategy

Apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$, where the direction and magnitude of the electric field are constant.

Solution

The angle between the uniform electric field $\vec{\mathbf{E}}$ and the unit normal $\hat{\mathbf{n}}$ to the planar surface is 30°. Since both the direction and magnitude are constant, E comes outside the integral. All that is left is a surface integral over dA, which is A. Therefore, using the open-surface equation, we find that the electric flux through the surface is

Equation:

$$\begin{aligned}\Phi &= \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = EA \cos\theta \\ &= (10\,\mathrm{N/C})(6.0\,\mathrm{m^2})(\cos 30^\circ) = 52\,\mathrm{N \cdot m^2/C}.\end{aligned}$$

Significance

Again, the relative directions of the field and the area matter, and the general equation with the integral will simplify to the simple dot product of area and electric field.

Note:

Exercise:

Problem:

Check Your Understanding What angle should there be between the electric field and the surface shown in [link] in the previous example so that no electric flux passes through the surface?

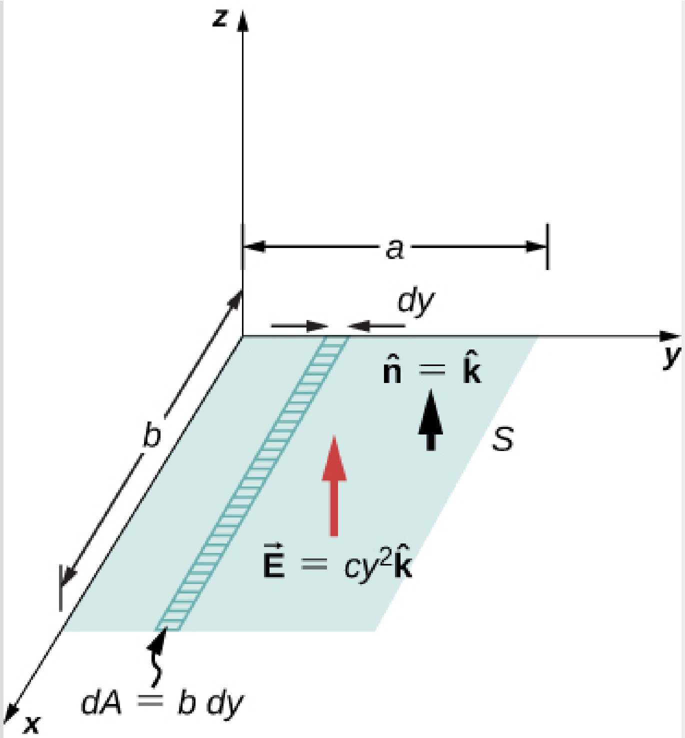
Solution:

Place it so that its unit normal is perpendicular to $\vec{\mathbf{E}}$.

Example:

Inhomogeneous Electric Field

What is the total flux of the electric field $\vec{\mathbf{E}} = cy^2\hat{\mathbf{k}}$ through the rectangular surface shown in [link]?



Since the electric field is not constant over the surface, an integration is necessary to determine the flux.

Strategy

Apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$. We assume that the unit normal $\hat{\mathbf{n}}$ to the given surface points in the positive *z*-direction, so $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. Since the electric field is not uniform over the surface, it is necessary to divide the surface into infinitesimal strips along which $\vec{\mathbf{E}}$ is essentially constant. As shown in [link], these strips are parallel to the *x*-axis, and each strip has an area $dA = b\,dy$.

Solution

From the open surface integral, we find that the net flux through the rectangular surface is

Equation:

$$\begin{aligned}\Phi &= \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \int_0^a (cy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}(b\, dy) \\ &= cb \int_0^a y^2\, dy = \frac{1}{3}\, a^3bc.\end{aligned}$$

Significance

For a non-constant electric field, the integral method is required.

Note:

Exercise:

Problem:

Check Your Understanding If the electric field in [link] is $\vec{\mathbf{E}} = mx\hat{\mathbf{k}}$, what is the flux through the rectangular area?

Solution:

$mab^2/2$

Summary

- The electric flux through a surface is proportional to the number of field lines crossing that surface. Note that this means the magnitude is proportional to the portion of the field perpendicular to the area.
- The electric flux is obtained by evaluating the surface integral

  Equation:

  $$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}},$$

  where the notation used here is for a closed surface *S*.

Conceptual Questions

Exercise:

Problem:

Discuss how to orient a planar surface of area A in a uniform electric field of magnitude E_0 to obtain (a) the maximum flux and (b) the minimum flux through the area.

Solution:

a. If the planar surface is perpendicular to the electric field vector, the maximum flux would be obtained. b. If the planar surface were parallel to the electric field vector, the minimum flux would be obtained.

Exercise:

Problem:

What are the maximum and minimum values of the flux in the preceding question?

Exercise:

Problem:

The net electric flux crossing a closed surface is always zero. True or false?

Solution:

False. The net electric flux crossing a closed surface is always zero if and only if the net charge enclosed is zero.

Exercise:

Problem:

The net electric flux crossing an open surface is never zero. True or false?

Problems

Exercise:

Problem:

A uniform electric field of magnitude 1.1×10^4 N/C is perpendicular to a square sheet with sides 2.0 m long. What is the electric flux through the sheet?

Exercise:

Problem:

Calculate the flux through the sheet of the previous problem if the plane of the sheet is at an angle of 60° to the field. Find the flux for both directions of the unit normal to the sheet.

Solution:

$\Phi = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} \rightarrow EA \cos\theta = 2.2 \times 10^4 \text{N} \cdot \text{m}^2/\text{C}$ electric field in direction of unit normal; $\Phi = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} \rightarrow EA \cos\theta = -2.2 \times 10^4 \text{N} \cdot \text{m}^2/\text{C}$ electric field opposite to unit normal

Exercise:

Problem:

Find the electric flux through a rectangular area $3 \text{ cm} \times 2 \text{ cm}$ between two parallel plates where there is a constant electric field of 30 N/C for the following orientations of the area: (a) parallel to the plates, (b) perpendicular to the plates, and (c) the normal to the area making a 30° angle with the direction of the electric field. Note that this angle can also be given as $180^\circ + 30^\circ$.

Exercise:

Problem:

The electric flux through a square-shaped area of side 5 cm near a large charged sheet is found to be $3 \times 10^{-5} \text{ N} \cdot \text{m}^2/\text{C}$ when the area is parallel to the plate. Find the charge density on the sheet.

Solution:

$\frac{3 \times 10^{-5} \text{ N} \cdot \text{m}^2/\text{C}}{(0.05 \text{ m})^2} = E \Rightarrow \sigma = 2.12 \times 10^{-13} \text{ C/m}^2$

Exercise:

Problem:

Two large rectangular aluminum plates of area $150\ \mathrm{cm}^2$ face each other with a separation of 3 mm between them. The plates are charged with equal amount of opposite charges, $\pm 20\ \mu\mathrm{C}$. The charges on the plates face each other. Find the flux through a circle of radius 3 cm between the plates when the normal to the circle makes an angle of 5° with a line perpendicular to the plates. Note that this angle can also be given as $180^\circ + 5^\circ$.

Exercise:

Problem:

A square surface of area $2\ \mathrm{cm}^2$ is in a space of uniform electric field of magnitude $10^3\ \mathrm{N/C}$. The amount of flux through it depends on how the square is oriented relative to the direction of the electric field. Find the electric flux through the square, when the normal to it makes the following angles with electric field: (a) 30°, (b) 90°, and (c) 0°. Note that these angles can also be given as $180^\circ + \theta$.

Solution:

a. $\Phi = 0.17\ \mathrm{N \cdot m^2/C}$;
b. $\Phi = 0$; c.
$\Phi = EA \cos 0^\circ = 1.0 \times 10^3\ \mathrm{N/C}(2.0 \times 10^{-4}\ \mathrm{m})^2 \cos 0^\circ = 0.20\ \mathrm{N \cdot m^2/C}$

Exercise:

Problem:

A vector field is pointed along the *z*-axis, $\vec{\mathbf{v}} = \frac{\alpha}{x^2+y^2}\ \widehat{\mathbf{z}}$. (a) Find the flux of the vector field through a rectangle in the *xy*-plane between $a < x < b$ and $c < y < d$. (b) Do the same through a rectangle in the *yz*-plane between $a < z < b$ and $c < y < d$. (Leave your answer as an integral.)

Exercise:

Problem:

Consider the uniform electric field $\vec{\mathbf{E}} = (4.0\,\hat{\mathbf{j}} + 3.0\,\widehat{\mathbf{k}}) \times 10^3\ \mathrm{N/C}$. What is its electric flux through a circular area of radius 2.0 m that lies in the *xy*-plane?

Solution:

$\Phi = 3.8 \times 10^4\,\text{N} \cdot \text{m}^2/\text{C}$

Exercise:

Problem:

Repeat the previous problem, given that the circular area is (a) in the *yz*-plane and (b) 45° above the *xy*-plane.

Exercise:

Problem:

An infinite charged wire with charge per unit length λ lies along the central axis of a cylindrical surface of radius *r* and length *l*. What is the flux through the surface due to the electric field of the charged wire?

Solution:

$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\varepsilon_0}\,\frac{2\lambda}{z}\widehat{\mathbf{k}},\ \int \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\,dA = \frac{\lambda}{\varepsilon_0}l$

Glossary

area vector
: vector with magnitude equal to the area of a surface and direction perpendicular to the surface

electric flux
: dot product of the electric field and the area through which it is passing

flux
: quantity of something passing through a given area

Explaining Gauss's Law
By the end of this section, you will be able to:

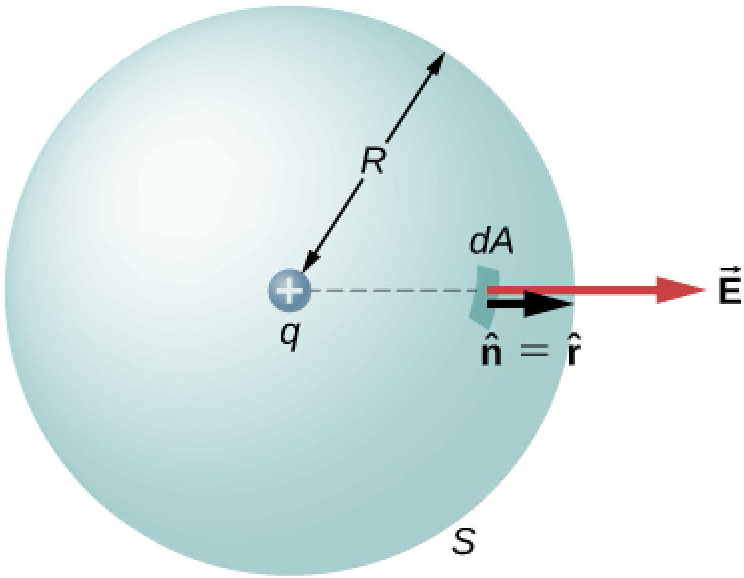
- State Gauss's law
- Explain the conditions under which Gauss's law may be used
- Apply Gauss's law in appropriate systems

We can now determine the electric flux through an arbitrary closed surface due to an arbitrary charge distribution. We found that if a closed surface does not have any charge inside where an electric field line can terminate, then any electric field line entering the surface at one point must necessarily exit at some other point of the surface. Therefore, if a closed surface does not have any charges inside the enclosed volume, then the electric flux through the surface is zero. Now, what happens to the electric flux if there are some charges inside the enclosed volume? Gauss's law gives a quantitative answer to this question.

To get a feel for what to expect, let's calculate the electric flux through a spherical surface around a positive point charge *q*, since we already know the electric field in such a situation. Recall that when we place the point charge at the origin of a coordinate system, the electric field at a point *P* that is at a distance *r* from the charge at the origin is given by
Equation:

$$\vec{\mathbf{E}}_P = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the radial vector from the charge at the origin to the point *P*. We can use this electric field to find the flux through the spherical surface of radius *r*, as shown in [link].



A closed spherical surface surrounding a point charge *q*.

Then we apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA$ to this system and substitute known values. On the sphere, $\widehat{\mathbf{n}} = \hat{\mathbf{r}}$ and $r = R$, so for an infinitesimal area *dA*,

Equation:

$$d\Phi = \overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dA = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2}\, dA.$$

We now find the net flux by integrating this flux over the surface of the sphere:

Equation:

$$\Phi = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \oint_S dA = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} (4\pi R^2) = \frac{q}{\varepsilon_0}.$$

where the total surface area of the spherical surface is $4\pi R^2$. This gives the flux through the closed spherical surface at radius *r* as

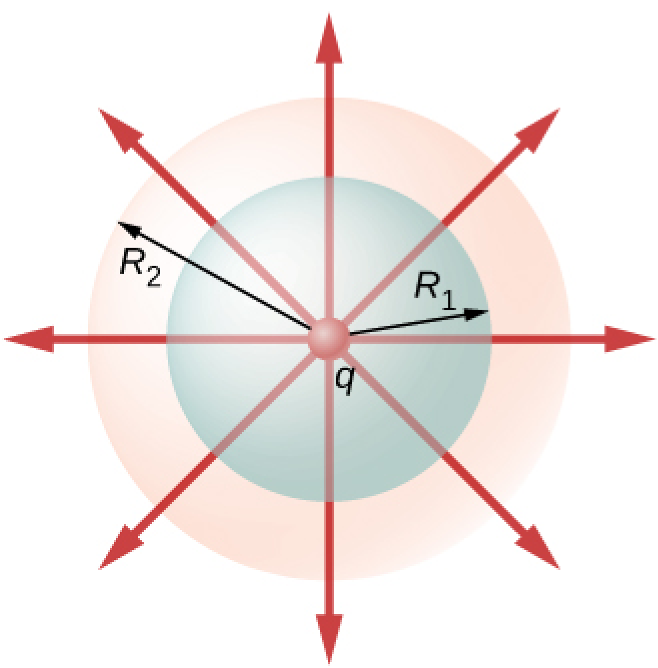
Equation:

$$\Phi = \frac{q}{\varepsilon_0}.$$

A remarkable fact about this equation is that the flux is independent of the size of the spherical surface. This can be directly attributed to the fact that the electric field of a point charge decreases as $1/r^2$ with distance, which just cancels the r^2 rate of increase of the surface area.

Electric Field Lines Picture

An alternative way to see why the flux through a closed spherical surface is independent of the radius of the surface is to look at the electric field lines. Note that every field line from q that pierces the surface at radius R_1 also pierces the surface at R_2 ([link]).

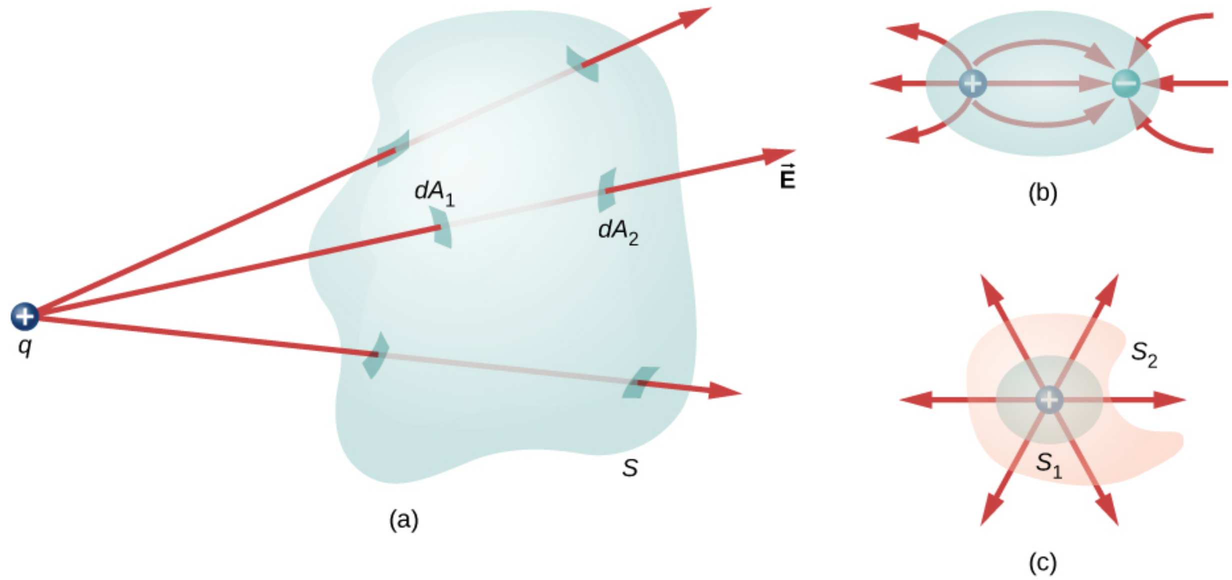


Flux through spherical surfaces of radii R_1 and R_2 enclosing a charge

q are equal, independent of the size of the surface, since all E-field lines that pierce one surface from the inside to outside direction also pierce the other surface in the same direction.

Therefore, the net number of electric field lines passing through the two surfaces from the inside to outside direction is equal. This net number of electric field lines, which is obtained by subtracting the number of lines in the direction from outside to inside from the number of lines in the direction from inside to outside gives a visual measure of the electric flux through the surfaces.

You can see that if no charges are included within a closed surface, then the electric flux through it must be zero. A typical field line enters the surface at dA_1 and leaves at dA_2. Every line that enters the surface must also leave that surface. Hence the net "flow" of the field lines into or out of the surface is zero ([link](a)). The same thing happens if charges of equal and opposite sign are included inside the closed surface, so that the total charge included is zero (part (b)). A surface that includes the same amount of charge has the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surface encloses the same amount of charge (part (c)).



Understanding the flux in terms of field lines. (a) The electric flux through a closed surface due to a charge outside that surface is zero. (b) Charges are enclosed, but because the net charge included is zero, the net flux through the closed surface is also zero. (c) The shape and size of the surfaces that enclose a charge does not matter because all surfaces enclosing the same charge have the same flux.

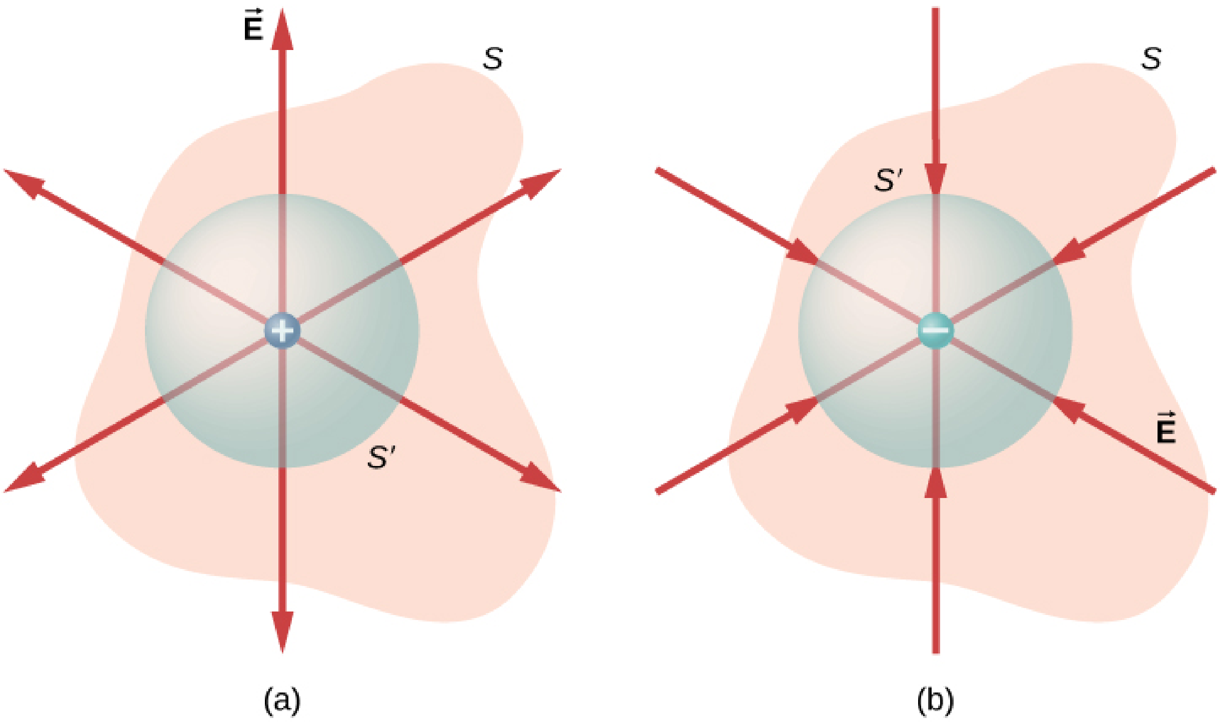
Statement of Gauss's Law

Gauss's law generalizes this result to the case of any number of charges and any location of the charges in the space inside the closed surface. According to Gauss's law, the flux of the electric field $\overrightarrow{\mathbf{E}}$ through any closed surface, also called a **Gaussian surface**, is equal to the net charge enclosed $\left(q_{\text{enc}}\right)$ divided by the permittivity of free space $\left(\varepsilon_0\right)$:

Equation:

$$\Phi_{\text{Closed Surface}} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

This equation holds for *charges of either sign*, because we define the area vector of a closed surface to point outward. If the enclosed charge is negative (see [link](b)), then the flux through either S or $S\prime$ is negative.

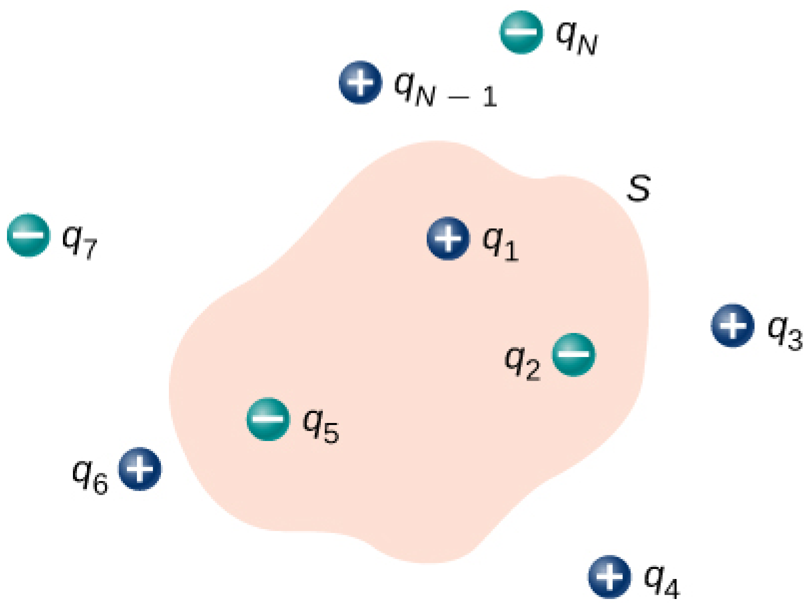


The electric flux through any closed surface surrounding a point charge q is given by Gauss's law. (a) Enclosed charge is positive. (b) Enclosed charge is negative.

The Gaussian surface does not need to correspond to a real, physical object; indeed, it rarely will. It is a mathematical construct that may be of any shape, provided that it is closed. However, since our goal is to integrate the flux over it, we tend to choose shapes that are highly symmetrical.

If the charges are discrete point charges, then we just add them. If the charge is described by a continuous distribution, then we need to integrate appropriately to find the total charge that resides inside the enclosed volume. For example, the flux through the Gaussian surface S of [link] is

$\Phi = (q_1 + q_2 + q_5)/\varepsilon_0$. Note that q_{enc} is simply the sum of the point charges. If the charge distribution were continuous, we would need to integrate appropriately to compute the total charge within the Gaussian surface.



The flux through the Gaussian surface shown, due to the charge distribution, is $\Phi = |q_1| + |q_2| + |q_5|/\varepsilon_0$.

Recall that the principle of superposition holds for the electric field. Therefore, the total electric field at any point, including those on the chosen Gaussian surface, is the sum of all the electric fields present at this point. This allows us to write Gauss's law in terms of the total electric field.

Note:
Gauss's Law

The flux Φ of the electric field $\vec{\mathbf{E}}$ through any closed surface S (a Gaussian surface) is equal to the net charge enclosed $\left(q_{\text{enc}}\right)$ divided by the permittivity of free space $\left(\varepsilon_0\right)$:

Equation:

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

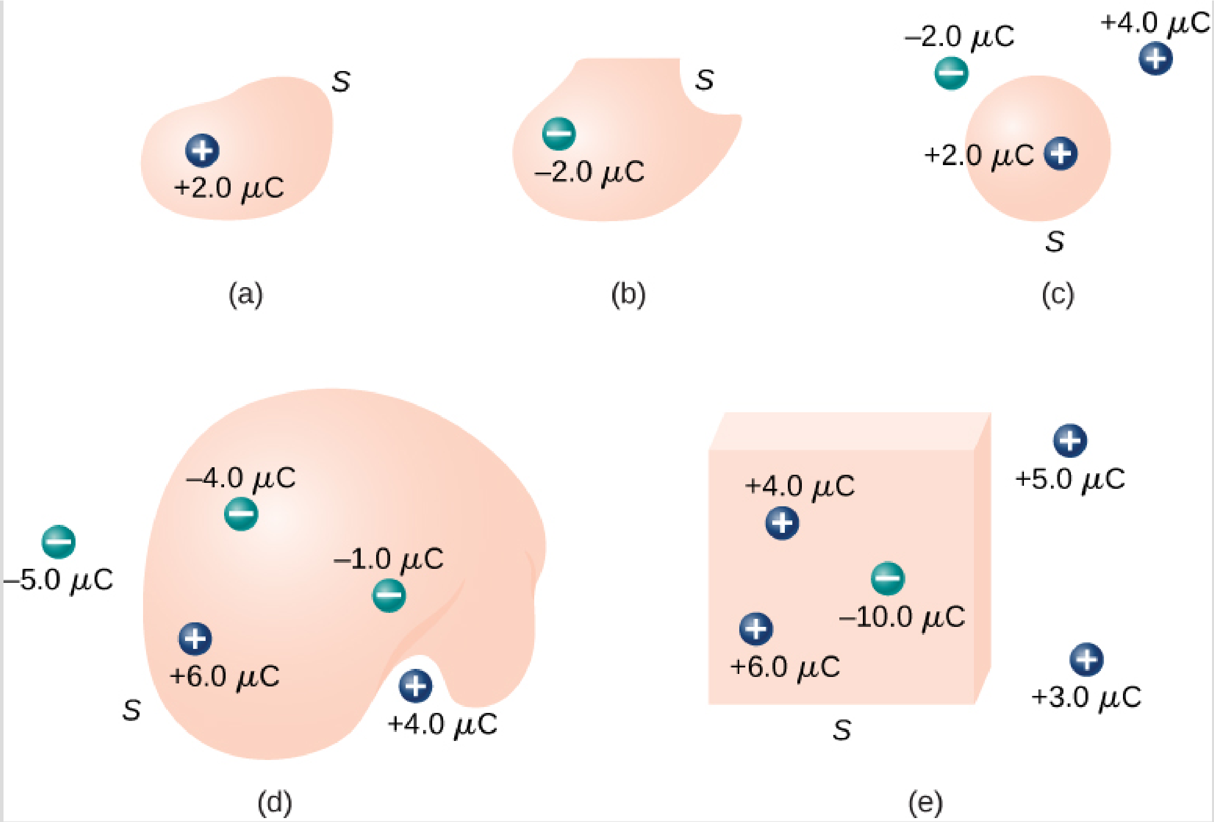
To use Gauss's law effectively, you must have a clear understanding of what each term in the equation represents. The field $\vec{\mathbf{E}}$ is the *total electric field* at every point on the Gaussian surface. This total field includes contributions from charges both inside and outside the Gaussian surface. However, q_{enc} is just the charge *inside* the Gaussian surface. Finally, the Gaussian surface is any closed surface in space. That surface can coincide with the actual surface of a conductor, or it can be an imaginary geometric surface. The only requirement imposed on a Gaussian surface is that it be closed ([link]).

A Klein bottle partially filled with a liquid. Could the Klein bottle be used as a Gaussian surface?

Example:

Electric Flux through Gaussian Surfaces

Calculate the electric flux through each Gaussian surface shown in [link].



Various Gaussian surfaces and charges.

Strategy

From Gauss's law, the flux through each surface is given by $q_{\text{enc}}/\varepsilon_0$, where q_{enc} is the charge enclosed by that surface.

Solution

For the surfaces and charges shown, we find

a. $\Phi = \frac{2.0\,\mu\text{C}}{\varepsilon_0} = 2.3 \times 10^5\,\text{N}\cdot\text{m}^2/\text{C}.$

b. $\Phi = \frac{-2.0\,\mu\text{C}}{\varepsilon_0} = -2.3 \times 10^5\,\text{N}\cdot\text{m}^2/\text{C}.$

c. $\Phi = \frac{2.0\,\mu\text{C}}{\varepsilon_0} = 2.3 \times 10^5\,\text{N}\cdot\text{m}^2/\text{C}.$

d. $\Phi = \frac{-4.0\,\mu\text{C}+6.0\,\mu\text{C}-1.0\,\mu\text{C}}{\varepsilon_0} = 1.1 \times 10^5\,\text{N}\cdot\text{m}^2/\text{C}.$

e. $\Phi = \frac{4.0\,\mu\text{C}+6.0\,\mu\text{C}-10.0\,\mu\text{C}}{\varepsilon_0} = 0.$

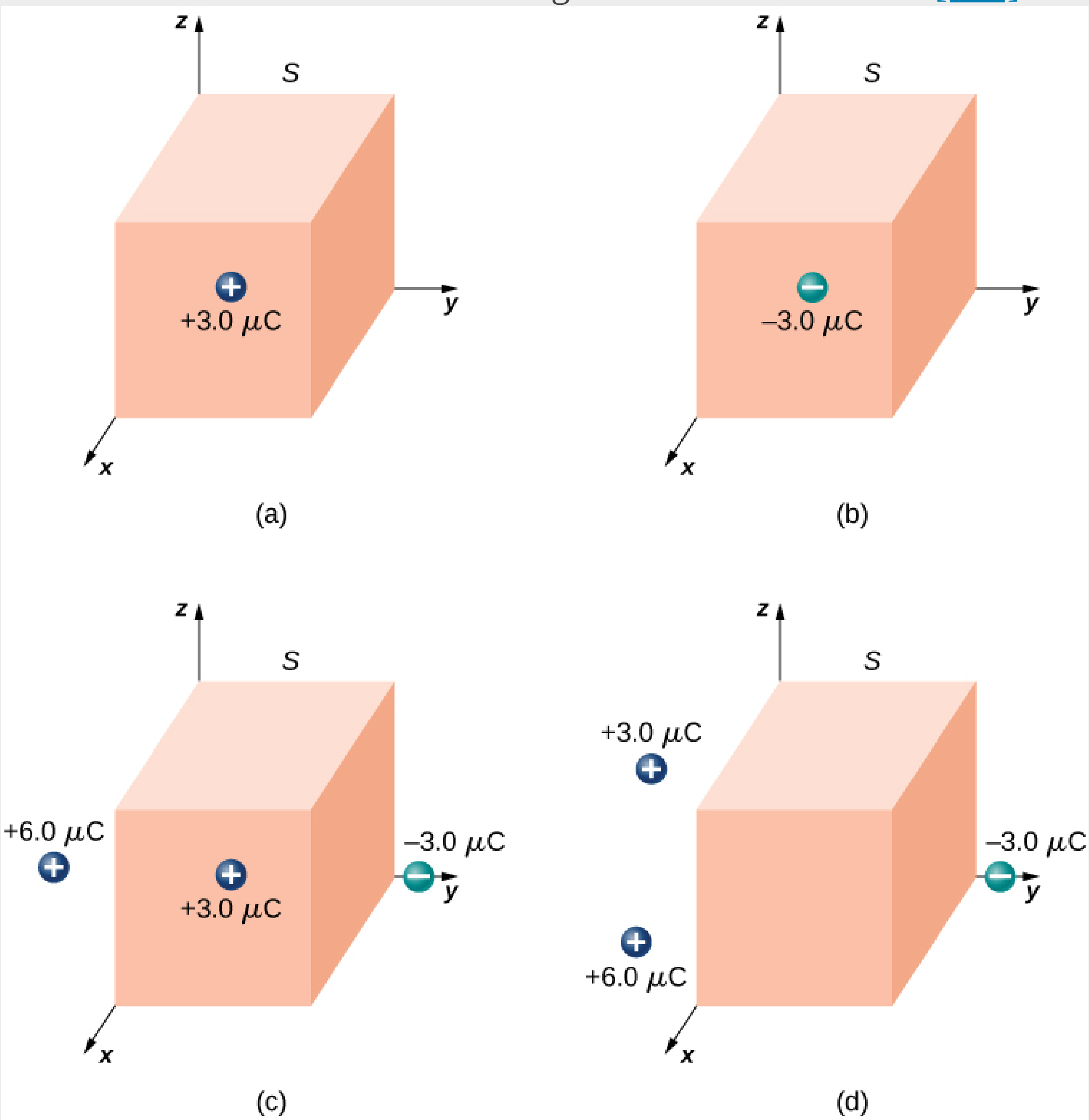
Significance

In the special case of a closed surface, the flux calculations become a sum of charges. In the next section, this will allow us to work with more complex systems.

Note:
Exercise:

Problem:

Check Your Understanding Calculate the electric flux through the closed cubical surface for each charge distribution shown in [link].



A cubical Gaussian surface with various charge distributions.

Solution:

a. $3.4 \times 10^5 \, \text{N} \cdot \text{m}^2/\text{C}$; b. $-3.4 \times 10^5 \, \text{N} \cdot \text{m}^2/\text{C}$; c. $3.4 \times 10^5 \, \text{N} \cdot \text{m}^2/\text{C}$; d. 0

Note:
Use this simulation to adjust the magnitude of the charge and the radius of the Gaussian surface around it. See how this affects the total flux and the magnitude of the electric field at the Gaussian surface.

Summary

- Gauss's law relates the electric flux through a closed surface to the net charge within that surface,
  Equation:

  $$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} \, dA = \frac{q_{\text{enc}}}{\varepsilon_0},$$

  where q_{enc} is the total charge inside the Gaussian surface *S*.
- All surfaces that include the same amount of charge have the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surfaces enclose the same amount of charge.

Conceptual Questions

Exercise:

Problem:

Two concentric spherical surfaces enclose a point charge q. The radius of the outer sphere is twice that of the inner one. Compare the electric fluxes crossing the two surfaces.

Solution:

Since the electric field vector has a $\frac{1}{r^2}$ dependence, the fluxes are the same since $A = 4\pi r^2$.

Exercise:

Problem:

Compare the electric flux through the surface of a cube of side length a that has a charge q at its center to the flux through a spherical surface of radius a with a charge q at its center.

Exercise:

Problem:

(a) If the electric flux through a closed surface is zero, is the electric field necessarily zero at all points on the surface? (b) What is the net charge inside the surface?

Solution:

a. no; b. zero

Exercise:

Problem:

Discuss how Gauss's law would be affected if the electric field of a point charge did not vary as $1/r^2$.

Exercise:

Problem:

Discuss the similarities and differences between the gravitational field of a point mass *m* and the electric field of a point charge *q*.

Solution:

Both fields vary as $\frac{1}{r^2}$. Because the gravitational constant is so much smaller than $\frac{1}{4\pi\varepsilon_0}$, the gravitational field is orders of magnitude weaker than the electric field. Also, the gravitational flux through a closed surface is zero or positive; however, the electric flux is positive, negative, or zero, depending on the definition of flux for the given situation.

Exercise:

Problem:

Discuss whether Gauss's law can be applied to other forces, and if so, which ones.

Exercise:

Problem:

Is the term $\overrightarrow{\mathbf{E}}$ in Gauss's law the electric field produced by just the charge inside the Gaussian surface?

Solution:

No, it is produced by all charges both inside and outside the Gaussian surface.

Exercise:

Problem:

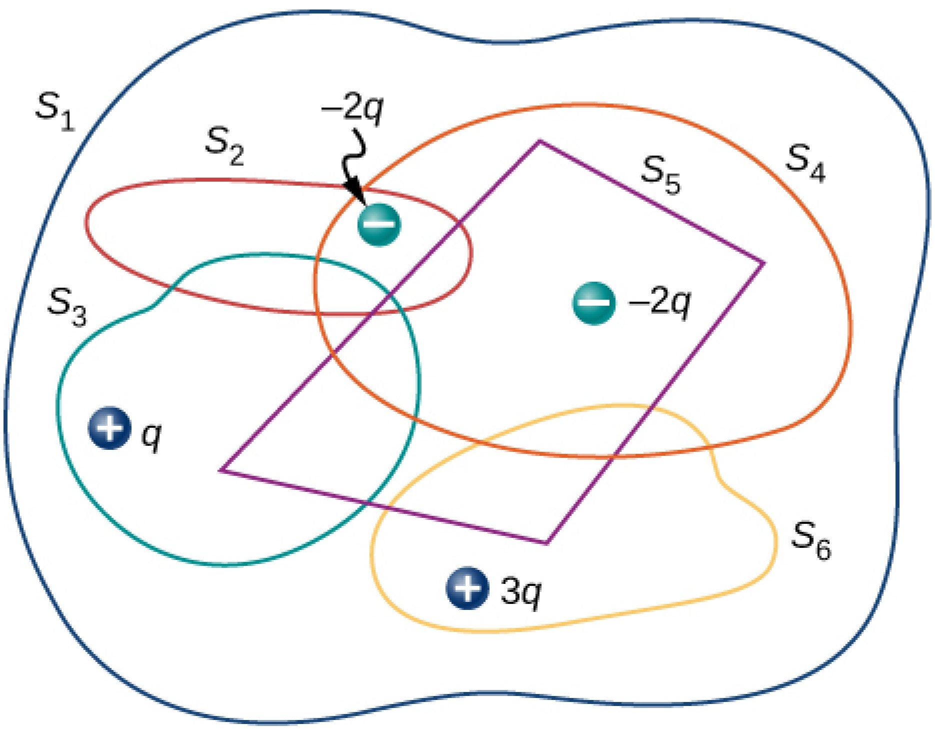
Reformulate Gauss's law by choosing the unit normal of the Gaussian surface to be the one directed inward.

Problems

Exercise:

Problem:

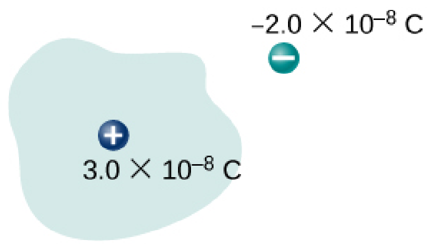
Determine the electric flux through each closed surface where the cross-section inside the surface is shown below.



Exercise:

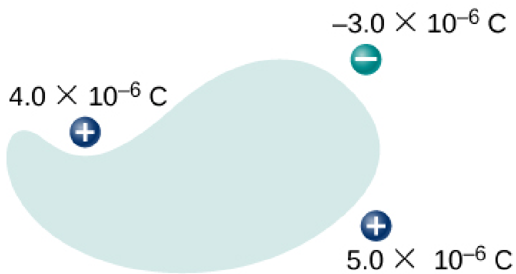
Problem:

Find the electric flux through the closed surface whose cross-sections are shown below.



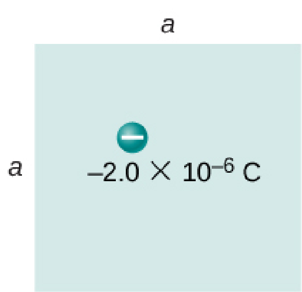
–2.0 × 10–8 C
3.0 × 10–8 C

(a)



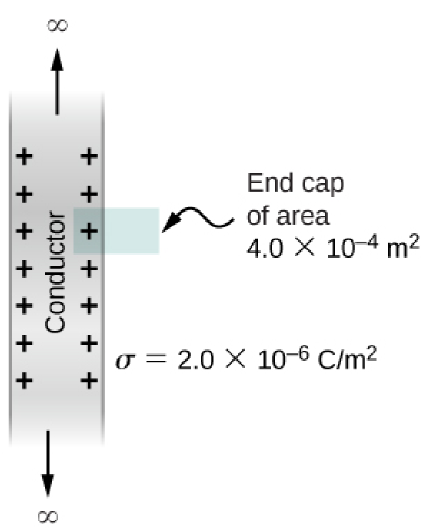
–3.0 × 10–6 C
4.0 × 10–6 C
5.0 × 10–6 C

(b)



a
a
–2.0 × 10–6 C

(c)



∞
Conductor
End cap
of area
4.0 × 10–4 m2
σ = 2.0 × 10–6 C/m2
∞

(d)

Solution:

a. $\Phi = 3.39 \times 10^3\ \mathrm{N \cdot m^2/C}$; b. $\Phi = 0$;
c. $\Phi = -2.25 \times 10^5\ \mathrm{N \cdot m^2/C}$;
d. $\Phi = 90.4\ \mathrm{N \cdot m^2/C}$

Exercise:

Problem:

A point charge q is located at the center of a cube whose sides are of length a. If there are no other charges in this system, what is the electric flux through one face of the cube?

Exercise:

Problem:

A point charge of 10 μC is at an unspecified location inside a cube of side 2 cm. Find the net electric flux though the surfaces of the cube.

Solution:

$\Phi = 1.13 \times 10^6\ \mathrm{N \cdot m^2/C}$

Exercise:

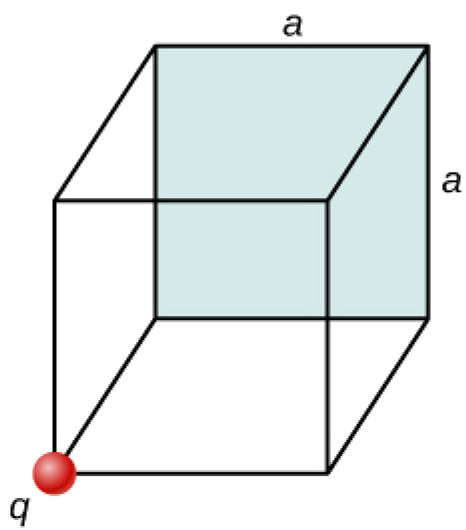
Problem:

A net flux of $1.0 \times 10^4\ \mathrm{N \cdot m^2/C}$ passes inward through the surface of a sphere of radius 5 cm. (a) How much charge is inside the sphere? (b) How precisely can we determine the location of the charge from this information?

Exercise:

Problem:

A charge q is placed at one of the corners of a cube of side a, as shown below. Find the magnitude of the electric flux through the shaded face due to q. Assume $q > 0$.



Solution:

Make a cube with q at the center, using the cube of side a. This would take four cubes of side a to make one side of the large cube. The shaded side of the small cube would be 1/24th of the total area of the large cube; therefore, the flux through the shaded area would be $\Phi = \frac{1}{24} \frac{q}{\varepsilon_0}$.

Exercise:

Problem:

The electric flux through a cubical box 8.0 cm on a side is $1.2 \times 10^3 \text{ N} \cdot \text{m}^2/\text{C}$. What is the total charge enclosed by the box?

Exercise:

Problem:

The electric flux through a spherical surface is $4.0 \times 10^4 \text{ N} \cdot \text{m}^2/\text{C}$. What is the net charge enclosed by the surface?

Solution:

$q = 3.54 \times 10^{-7} \text{ C}$

Exercise:

Problem:

A cube whose sides are of length *d* is placed in a uniform electric field of magnitude $E = 4.0 \times 10^3 \ \mathrm{N/C}$ so that the field is perpendicular to two opposite faces of the cube. What is the net flux through the cube?

Exercise:

Problem:

Repeat the previous problem, assuming that the electric field is directed along a body diagonal of the cube.

Solution:

zero, also because flux in equals flux out

Exercise:

Problem:

A total charge $5.0 \times 10^{-6} \ \mathrm{C}$ is distributed uniformly throughout a cubical volume whose edges are 8.0 cm long. (a) What is the charge density in the cube? (b) What is the electric flux through a cube with 12.0-cm edges that is concentric with the charge distribution? (c) Do the same calculation for cubes whose edges are 10.0 cm long and 5.0 cm long. (d) What is the electric flux through a spherical surface of radius 3.0 cm that is also concentric with the charge distribution?

Glossary

Gaussian surface
: any enclosed (usually imaginary) surface

Applying Gauss’s Law
By the end of this section, you will be able to:

- Explain what spherical, cylindrical, and planar symmetry are
- Recognize whether or not a given system possesses one of these symmetries
- Apply Gauss’s law to determine the electric field of a system with one of these symmetries

Gauss’s law is very helpful in determining expressions for the electric field, even though the law is not directly about the electric field; it is about the electric flux. It turns out that in situations that have certain symmetries (spherical, cylindrical, or planar) in the charge distribution, we can deduce the electric field based on knowledge of the electric flux. In these systems, we can find a Gaussian surface *S* over which the electric field has constant magnitude. Furthermore, if $\overrightarrow{\mathbf{E}}$ is parallel to $\widehat{\mathbf{n}}$ everywhere on the surface, then $\overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}} = E$. (If $\overrightarrow{\mathbf{E}}$ and $\widehat{\mathbf{n}}$ are antiparallel everywhere on the surface, then $\overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}} = -E$.) Gauss’s law then simplifies to

Note:
Equation:

$$\Phi = \oint_S \overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = E \oint_S \, dA = EA = \frac{q_{\text{enc}}}{\varepsilon_0},$$

where *A* is the area of the surface. Note that these symmetries lead to the transformation of the flux integral into a product of the magnitude of the electric field and an appropriate area. When you use this flux in the expression for Gauss’s law, you obtain an algebraic equation that you can solve for the magnitude of the electric field, which looks like
Equation:

$$E \sim \frac{q_{\text{enc}}}{\varepsilon_0 \text{ area}}.$$

The direction of the electric field at point *P* is obtained from the symmetry of the charge distribution and the type of charge in the distribution. Therefore, Gauss’s law can be used to determine $\overrightarrow{\mathbf{E}}$. Here is a summary of the steps we will follow:

Note:
Problem-Solving Strategy: Gauss’s Law

1. *Identify the spatial symmetry of the charge distribution*. This is an important first step that allows us to choose the appropriate Gaussian surface. As examples, an isolated point charge

has spherical symmetry, and an infinite line of charge has cylindrical symmetry.
2. *Choose a Gaussian surface with the same symmetry as the charge distribution and identify its consequences*. With this choice, $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is easily determined over the Gaussian surface.
3. *Evaluate the integral* $\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA$ *over the Gaussian surface, that is, calculate the flux through the surface*. The symmetry of the Gaussian surface allows us to factor $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ outside the integral.
4. *Determine the amount of charge enclosed by the Gaussian surface*. This is an evaluation of the right-hand side of the equation representing Gauss's law. It is often necessary to perform an integration to obtain the net enclosed charge.
5. *Evaluate the electric field of the charge distribution*. The field may now be found using the results of steps 3 and 4.

Basically, there are only three types of symmetry that allow Gauss's law to be used to deduce the electric field. They are

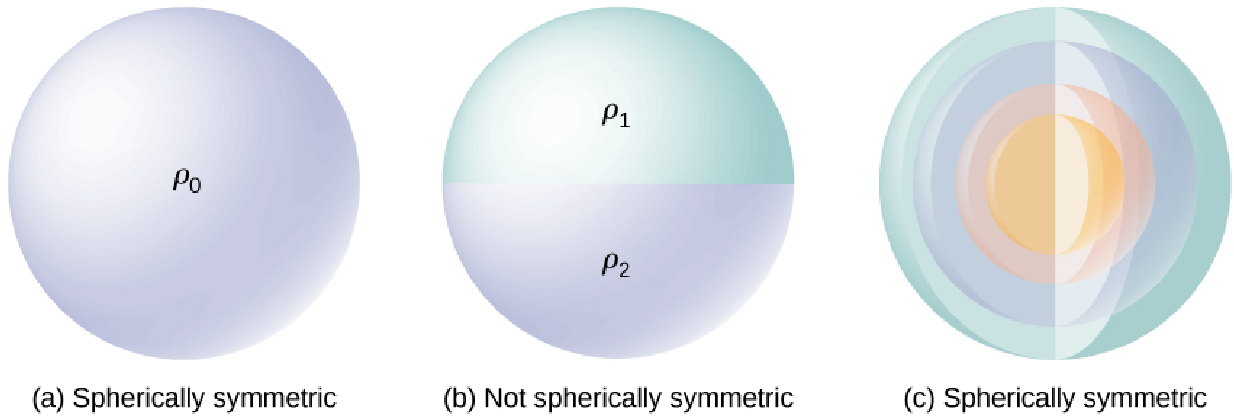
- A charge distribution with spherical symmetry
- A charge distribution with cylindrical symmetry
- A charge distribution with planar symmetry

To exploit the symmetry, we perform the calculations in appropriate coordinate systems and use the right kind of Gaussian surface for that symmetry, applying the remaining four steps.

Charge Distribution with Spherical Symmetry

A charge distribution has **spherical symmetry** if the density of charge depends only on the distance from a point in space and not on the direction. In other words, if you rotate the system, it doesn't look different. For instance, if a sphere of radius *R* is uniformly charged with charge density ρ_0 then the distribution has spherical symmetry ([link](a)). On the other hand, if a sphere of radius *R* is charged so that the top half of the sphere has uniform charge density ρ_1 and the bottom half has a uniform charge density $\rho_2 \neq \rho_1$, then the sphere does not have spherical symmetry because the charge density depends on the direction ([link](b)). Thus, it is not the shape of the object but rather the shape of the charge distribution that determines whether or not a system has spherical symmetry.

[link](c) shows a sphere with four different shells, each with its own uniform charge density. Although this is a situation where charge density in the full sphere is not uniform, the charge density function depends only on the distance from the center and not on the direction. Therefore, this charge distribution does have spherical symmetry.



Illustrations of spherically symmetrical and nonsymmetrical systems. Different shadings indicate different charge densities. Charges on spherically shaped objects do not necessarily mean the charges are distributed with spherical symmetry. The spherical symmetry occurs only when the charge density does not depend on the direction. In (a), charges are distributed uniformly in a sphere. In (b), the upper half of the sphere has a different charge density from the lower half; therefore, (b) does not have spherical symmetry. In (c), the charges are in spherical shells of different charge densities, which means that charge density is only a function of the radial distance from the center; therefore, the system has spherical symmetry.

One good way to determine whether or not your problem has spherical symmetry is to look at the charge density function in spherical coordinates, $\rho\left(r, \theta, \phi\right)$. If the charge density is only a function of *r*, that is $\rho = \rho\left(r\right)$, then you have spherical symmetry. If the density depends on θ or ϕ, you could change it by rotation; hence, you would not have spherical symmetry.

Consequences of symmetry

In all spherically symmetrical cases, the electric field at any point must be radially directed, because the charge and, hence, the field must be invariant under rotation. Therefore, using spherical coordinates with their origins at the center of the spherical charge distribution, we can write down the expected form of the electric field at a point *P* located at a distance *r* from the center:

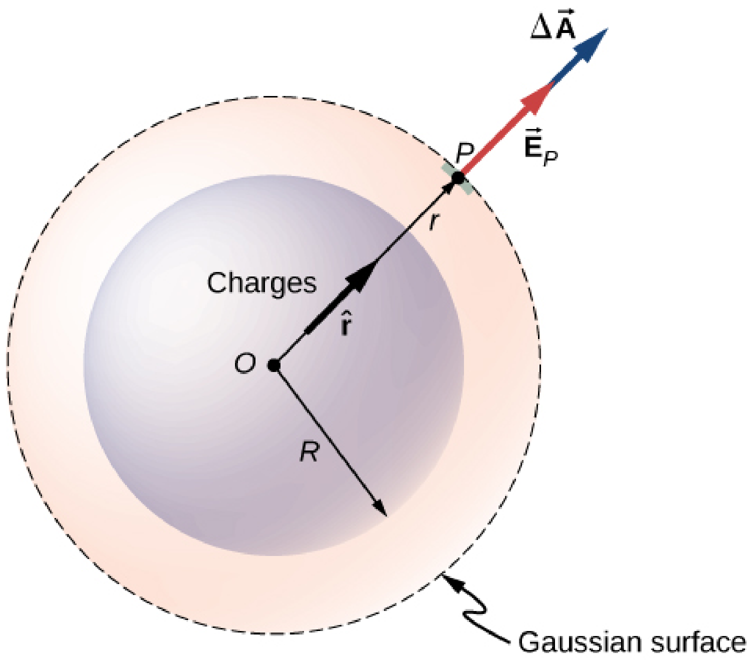
Equation:

$$\text{Spherical symmetry: } \vec{\mathbf{E}}_P = E_P(r)\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector pointed in the direction from the origin to the field point *P*. The radial component E_P of the electric field can be positive or negative. When $E_P > 0$, the electric field at *P* points away from the origin, and when $E_P < 0$, the electric field at *P* points toward the origin.

Gaussian surface and flux calculations

We can now use this form of the electric field to obtain the flux of the electric field through the Gaussian surface. For spherical symmetry, the Gaussian surface is a closed spherical surface that has the same center as the center of the charge distribution. Thus, the direction of the area vector of an area element on the Gaussian surface at any point is parallel to the direction of the electric field at that point, since they are both radially directed outward ([link]).



The electric field at any point of the spherical Gaussian surface for a spherically symmetrical charge distribution is parallel to the area element vector at that point, giving flux as the product of the magnitude of electric field and the value of the area. Note that the radius R of the charge distribution and the radius r of the Gaussian surface are different quantities.

The magnitude of the electric field $\overrightarrow{\mathbf{E}}$ must be the same everywhere on a spherical Gaussian surface concentric with the distribution. For a spherical surface of radius r,

Equation:

$$\Phi = \oint_S \vec{\mathbf{E}}_P \cdot \widehat{\mathbf{n}}\, dA = E_P \oint_S dA = E_P\, 4\pi r^2.$$

Using Gauss's law

According to Gauss's law, the flux through a closed surface is equal to the total charge enclosed within the closed surface divided by the permittivity of vacuum ε_0. Let q_{enc} be the total charge enclosed inside the distance *r* from the origin, which is the space inside the Gaussian spherical surface of radius *r*. This gives the following relation for Gauss's law:

Equation:

$$4\pi r^2 E = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

Hence, the electric field at point *P* that is a distance *r* from the center of a spherically symmetrical charge distribution has the following magnitude and direction:

Equation:

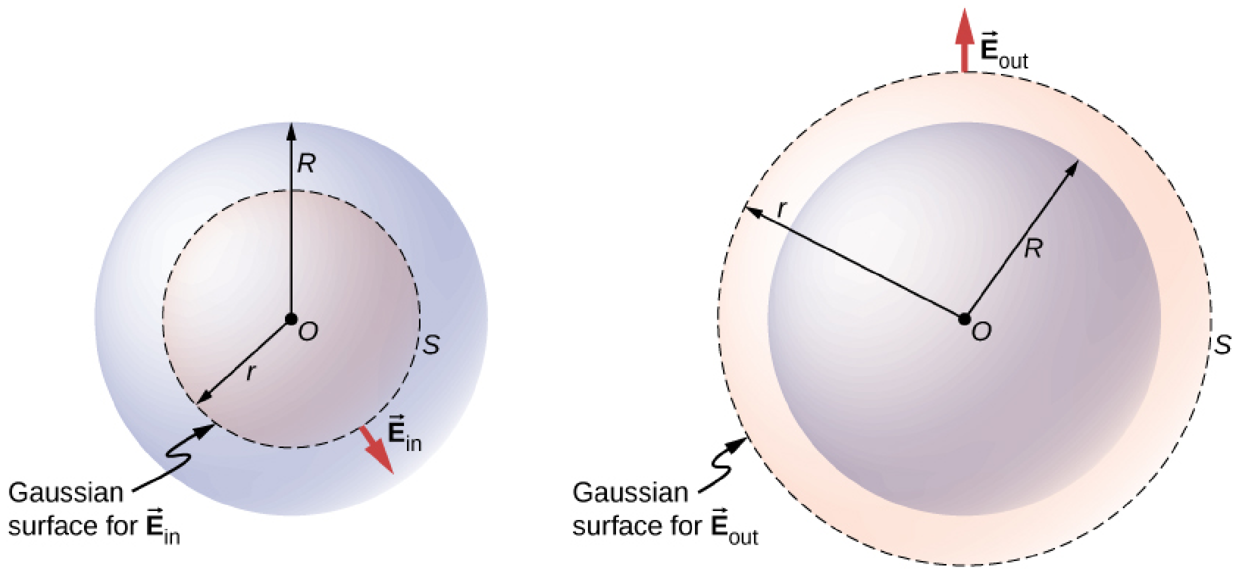
$$\text{Magnitude: } E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{enc}}}{r^2}$$

Direction: radial from *O* to *P* or from *P* to *O*.

The direction of the field at point *P* depends on whether the charge in the sphere is positive or negative. For a net positive charge enclosed within the Gaussian surface, the direction is from *O* to *P*, and for a net negative charge, the direction is from *P* to *O*. This is all we need for a point charge, and you will notice that the result above is identical to that for a point charge. However, Gauss's law becomes truly useful in cases where the charge occupies a finite volume.

Computing enclosed charge

The more interesting case is when a spherical charge distribution occupies a volume, and asking what the electric field inside the charge distribution is thus becomes relevant. In this case, the charge enclosed depends on the distance *r* of the field point relative to the radius of the charge distribution *R*, such as that shown in [link].



A spherically symmetrical charge distribution and the Gaussian surface used for finding the field (a) inside and (b) outside the distribution.

If point P is located outside the charge distribution—that is, if $r \geq R$—then the Gaussian surface containing P encloses all charges in the sphere. In this case, q_{enc} equals the total charge in the sphere. On the other hand, if point P is within the spherical charge distribution, that is, if $r < R$, then the Gaussian surface encloses a smaller sphere than the sphere of charge distribution. In this case, q_{enc} is less than the total charge present in the sphere. Referring to [link], we can write q_{enc} as

Equation:

$$q_{\text{enc}} = \begin{cases} q_{\text{tot}}(\text{total charge}) \text{ if } r \geq R \\ q_{\text{within } r < R}(\text{only charge within } r < R) \text{ if } r < R \end{cases}.$$

The field at a point outside the charge distribution is also called $\vec{\mathbf{E}}_{\text{out}}$, and the field at a point inside the charge distribution is called $\vec{\mathbf{E}}_{\text{in}}$. Focusing on the two types of field points, either inside or outside the charge distribution, we can now write the magnitude of the electric field as

Equation:

$$P \text{ outside sphere } E_{\text{out}} = \frac{1}{4\pi\varepsilon_0} \frac{q_{\text{tot}}}{r^2}$$

Equation:

$$P \text{ inside sphere } E_{\text{in}} = \frac{1}{4\pi\varepsilon_0} \frac{q_{\text{within } r < R}}{r^2}.$$

Note that the electric field outside a spherically symmetrical charge distribution is identical to that of a point charge at the center that has a charge equal to the total charge of the spherical charge distribution. This is remarkable since the charges are not located at the center only. We now work out specific examples of spherical charge distributions, starting with the case of a uniformly charged sphere.

Example:

Uniformly Charged Sphere

A sphere of radius R, such as that shown in [link], has a uniform volume charge density ρ_0. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss's law problem-solving strategy, where we have already worked out the flux calculation.

Solution

The charge enclosed by the Gaussian surface is given by

Equation:

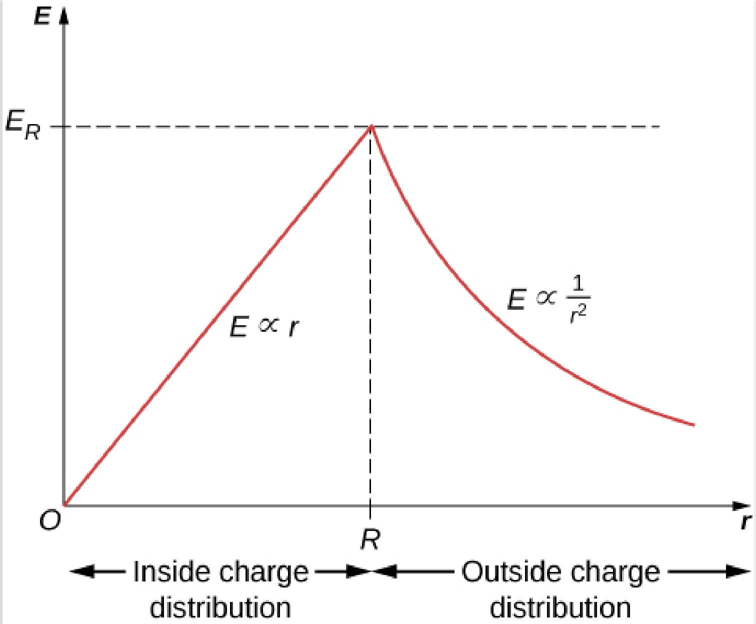
$$q_{\text{enc}} = \int_0 \rho \, dV = \int_0^r \rho_0 4\pi r'^2 dr' = \rho_0 \left(\frac{4}{3}\pi r^3\right).$$

The answer for electric field amplitude can then be written down immediately for a point outside the sphere, labeled E_{out}, and a point inside the sphere, labeled E_{in}.

Equation:

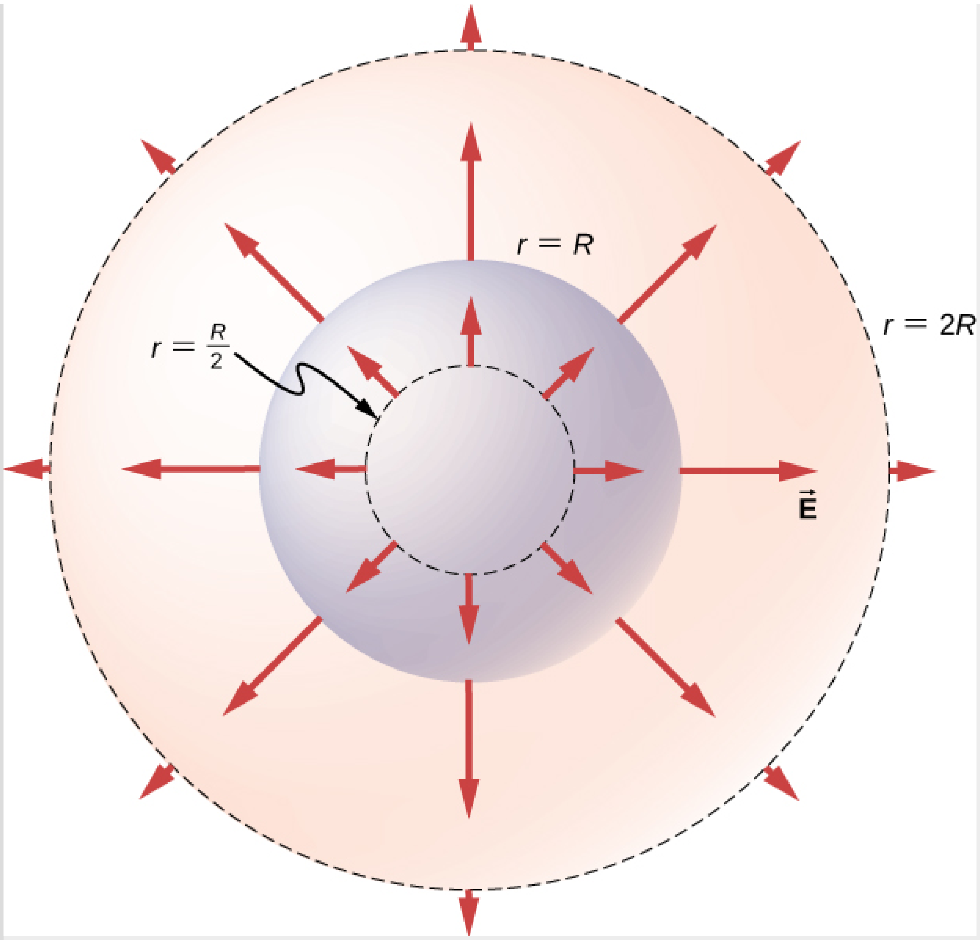
$$\begin{aligned} E_{\text{out}} &= \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{tot}}}{r^2}, \; q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho_0, \\ E_{\text{in}} &= \frac{q_{\text{enc}}}{4\pi\varepsilon_0 r^2} = \frac{\rho_0 r}{3\varepsilon_0}, \; \text{since } q_{\text{enc}} = \frac{4}{3}\pi r^3 \rho_0. \end{aligned}$$

It is interesting to note that the magnitude of the electric field increases inside the material as you go out, since the amount of charge enclosed by the Gaussian surface increases with the volume. Specifically, the charge enclosed grows $\propto r^3$, whereas the field from each infinitesimal element of charge drops off $\propto 1/r^2$ with the net result that the electric field within the distribution increases in strength linearly with the radius. The magnitude of the electric field outside the sphere decreases as you go away from the charges, because the included charge remains the same but the distance increases. [link] displays the variation of the magnitude of the electric field with distance from the center of a uniformly charged sphere.



Electric field of a uniformly charged, non-conducting sphere increases inside the sphere to a maximum at the surface and then decreases as $1/r^2$. Here, $E_R = \frac{\rho_0 R}{3\varepsilon_0}$. The electric field is due to a spherical charge distribution of uniform charge density and total charge Q as a function of distance from the center of the distribution.

The direction of the electric field at any point P is radially outward from the origin if ρ_0 is positive, and inward (i.e., toward the center) if ρ_0 is negative. The electric field at some representative space points are displayed in [link] whose radial coordinates r are $r = R/2$, $r = R$, and $r = 2R$.



Electric field vectors inside and outside a uniformly charged sphere.

Significance
Notice that E_{out} has the same form as the equation of the electric field of an isolated point charge. In determining the electric field of a uniform spherical charge distribution, we can therefore assume that all of the charge inside the appropriate spherical Gaussian surface is located at the center of the distribution.

Example:
Non-Uniformly Charged Sphere
A non-conducting sphere of radius R has a non-uniform charge density that varies with the distance from its center as given by
Equation:

$$\rho(r) = ar^n \; (r \leq R; n \geq 0),$$

where a is a constant. We require $n \geq 0$ so that the charge density is not undefined at $r = 0$. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss’s law strategy given above, where we work out the enclosed charge integrals separately for cases inside and outside the sphere.

Solution

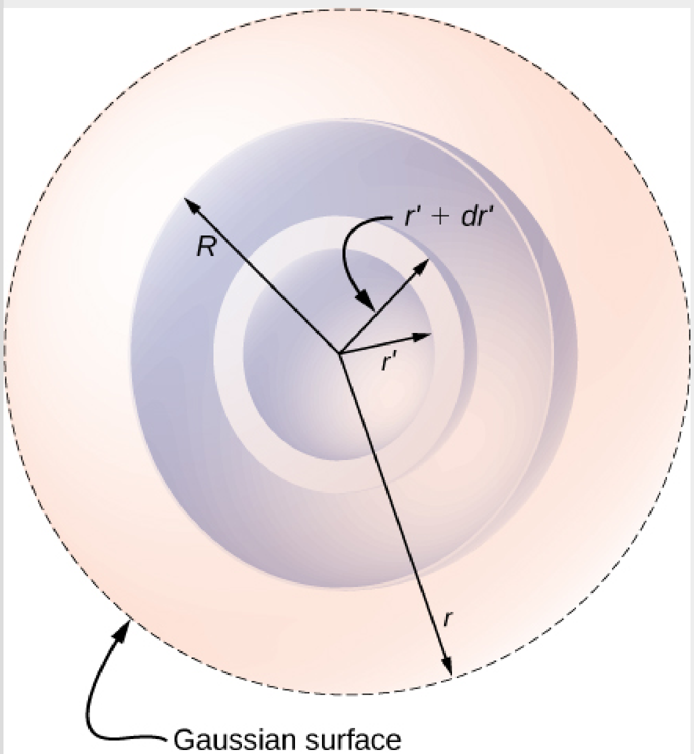
Since the given charge density function has only a radial dependence and no dependence on direction, we have a spherically symmetrical situation. Therefore, the magnitude of the electric field at any point is given above and the direction is radial. We just need to find the enclosed charge q_{enc}, which depends on the location of the field point.

A note about symbols: We use r' for locating charges in the charge distribution and *r* for locating the field point(s) at the Gaussian surface(s). The letter *R* is used for the radius of the charge distribution.

As charge density is not constant here, we need to integrate the charge density function over the volume enclosed by the Gaussian surface. Therefore, we set up the problem for charges in one spherical shell, say between r' and $r' + dr'$, as shown in [link]. The volume of charges in the shell of infinitesimal width is equal to the product of the area of surface $4\pi r'^2$ and the thickness dr'. Multiplying the volume with the density at this location, which is ar'^n, gives the charge in the shell:

Equation:

$$dq = ar'^n \, 4\pi r'^2 dr'.$$



Spherical symmetry with non-uniform charge distribution. In this type of problem, we need four radii: *R* is the radius of the charge distribution, *r* is the radius of the Gaussian surface, r' is the inner radius of the spherical

shell, and $r' + dr'$ is the outer radius of the spherical shell. The spherical shell is used to calculate the charge enclosed within the Gaussian surface. The range for r' is from 0 to r for the field at a point inside the charge distribution and from 0 to R for the field at a point outside the charge distribution. If $r > R$, then the Gaussian surface encloses more volume than the charge distribution, but the additional volume does not contribute to q_{enc}.

(a) **Field at a point outside the charge distribution.** In this case, the Gaussian surface, which contains the field point P, has a radius r that is greater than the radius R of the charge distribution, $r > R$. Therefore, all charges of the charge distribution are enclosed within the Gaussian surface. Note that the space between $r' = R$ and $r' = r$ is empty of charges and therefore does not contribute to the integral over the volume enclosed by the Gaussian surface:
Equation:

$$q_{\text{enc}} = \int dq = \int_0^R ar'^n \, 4\pi r'^2 dr' = \frac{4\pi a}{n+3} R^{n+3}.$$

This is used in the general result for $\vec{\mathbf{E}}_{\text{out}}$ above to obtain the electric field at a point outside the charge distribution as
Equation:

$$\vec{\mathbf{E}}_{\text{out}} = \left[\frac{aR^{n+3}}{\varepsilon_0(n+3)}\right]\frac{1}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction from the origin to the field point at the Gaussian surface.
(b) **Field at a point inside the charge distribution.** The Gaussian surface is now buried inside the charge distribution, with $r < R$. Therefore, only those charges in the distribution that are within a distance r of the center of the spherical charge distribution count in r_{enc}:
Equation:

$$q_{\text{enc}} = \int_0^r ar'^n 4\pi r'^2 \, dr' = \frac{4\pi a}{n+3} r^{n+3}.$$

Now, using the general result above for $\vec{\mathbf{E}}_{\text{in}}$, we find the electric field at a point that is a distance r from the center and lies within the charge distribution as
Equation:

$$\vec{\mathbf{E}}_{\text{in}} = \left[\frac{a}{\varepsilon_0(n+3)}\right] r^{n+1}\hat{\mathbf{r}},$$

where the direction information is included by using the unit radial vector.

Note:
Exercise:

Problem:

Check Your Understanding Check that the electric fields for the sphere reduce to the correct values for a point charge.

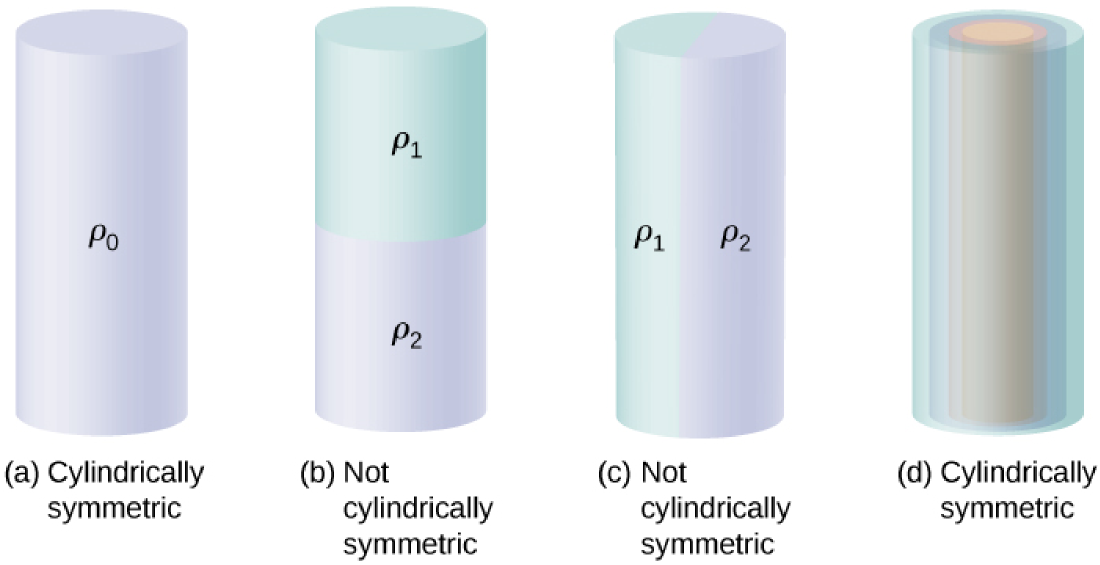
Solution:

In this case, there is only $\vec{\mathbf{E}}_{\text{out}}.$ So, yes.

Charge Distribution with Cylindrical Symmetry

A charge distribution has **cylindrical symmetry** if the charge density depends only upon the distance *r* from the axis of a cylinder and must not vary along the axis or with direction about the axis. In other words, if your system varies if you rotate it around the axis, or shift it along the axis, you do not have cylindrical symmetry.

[link] shows four situations in which charges are distributed in a cylinder. A uniform charge density ρ_0. in an infinite straight wire has a cylindrical symmetry, and so does an infinitely long cylinder with constant charge density ρ_0. An infinitely long cylinder that has different charge densities along its length, such as a charge density ρ_1 for $z > 0$ and $\rho_2 \neq \rho_1$ for $z < 0$, does not have a usable cylindrical symmetry for this course. Neither does a cylinder in which charge density varies with the direction, such as a charge density ρ_1 for $0 \leq \theta < \pi$ and $\rho_2 \neq \rho_1$ for $\pi \leq \theta < 2\pi$. A system with concentric cylindrical shells, each with uniform charge densities, albeit different in different shells, as in [link](d), does have cylindrical symmetry if they are infinitely long. The infinite length requirement is due to the charge density changing along the axis of a finite cylinder. In real systems, we don’t have infinite cylinders; however, if the cylindrical object is considerably longer than the radius from it that we are interested in, then the approximation of an infinite cylinder becomes useful.



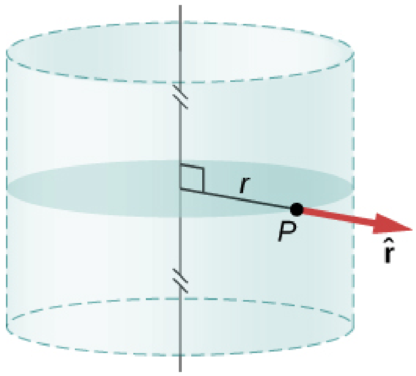
To determine whether a given charge distribution has cylindrical symmetry, look at the cross-section of an “infinitely long” cylinder. If the charge density does not depend on the polar angle of the cross-section or along the axis, then you have cylindrical symmetry. (a) Charge density is constant in the cylinder; (b) upper half of the cylinder has a different charge density from the lower half; (c) left half of the cylinder has a different charge density from the right half; (d) charges are constant in different cylindrical rings, but the density does not depend on the polar angle. Cases (a) and (d) have cylindrical symmetry, whereas (b) and (c) do not.

Consequences of symmetry

In all cylindrically symmetrical cases, the electric field $\overrightarrow{\mathbf{E}}_P$ at any point *P* must also display cylindrical symmetry.

Cylindrical symmetry: $\vec{\mathbf{E}}_P = E_P(r)\hat{\mathbf{r}}$,

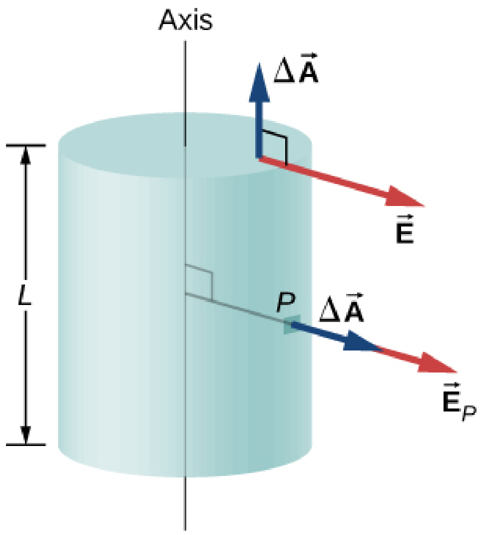
where *r* is the distance from the axis and $\hat{\mathbf{r}}$ is a unit vector directed perpendicularly away from the axis ([link]).



The electric field in a cylindrically symmetrical situation depends only on the distance from the axis. The direction of the electric field is pointed away from the axis for positive charges and toward the axis for negative charges.

Gaussian surface and flux calculation

To make use of the direction and functional dependence of the electric field, we choose a closed Gaussian surface in the shape of a cylinder with the same axis as the axis of the charge distribution. The flux through this surface of radius *s* and height *L* is easy to compute if we divide our task into two parts: (a) a flux through the flat ends and (b) a flux through the curved surface ([link]).



The Gaussian surface in the case of cylindrical symmetry. The electric field at a patch is either parallel or perpendicular to the normal to the patch of the Gaussian surface.

The electric field is perpendicular to the cylindrical side and parallel to the planar end caps of the surface. The flux through the cylindrical part is

Equation:

$$\int_S \overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = E \int_S dA = E(2\pi r L),$$

whereas the flux through the end caps is zero because $\overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}} = 0$ there. Thus, the flux is

Equation:

$$\int_S \overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = E(2\pi r L) + 0 + 0 = 2\pi r L E.$$

Using Gauss’s law

According to Gauss’s law, the flux must equal the amount of charge within the volume enclosed by this surface, divided by the permittivity of free space. When you do the calculation for a cylinder of length L, you find that q_{enc} of Gauss’s law is directly proportional to L. Let us write it as charge per unit length (λ_{enc}) times length L:

Equation:

$$q_{\text{enc}} = \lambda_{\text{enc}} L.$$

Hence, Gauss's law for any cylindrically symmetrical charge distribution yields the following magnitude of the electric field a distance *s* away from the axis:

Equation:

$$\text{Magnitude: } E(r) = \frac{\lambda_{\text{enc}}}{2\pi\varepsilon_0}\frac{1}{r}.$$

The charge per unit length λ_{enc} depends on whether the field point is inside or outside the cylinder of charge distribution, just as we have seen for the spherical distribution.

Computing enclosed charge

Let *R* be the radius of the cylinder within which charges are distributed in a cylindrically symmetrical way. Let the field point *P* be at a distance *s* from the axis. (The side of the Gaussian surface includes the field point *P*.) When $r > R$ (that is, when *P* is outside the charge distribution), the Gaussian surface includes all the charge in the cylinder of radius *R* and length *L*. When $r < R$ (*P* is located inside the charge distribution), then only the charge within a cylinder of radius *s* and length *L* is enclosed by the Gaussian surface:

Equation:

$$\lambda_{\text{enc}} L = \begin{cases} (\text{total charge}) \text{ if } r \geq R \\ (\text{only charge within } r < R) \text{ if } r < R \end{cases}.$$

Example:

Uniformly Charged Cylindrical Shell

A very long non-conducting cylindrical shell of radius *R* has a uniform surface charge density σ_0. Find the electric field (a) at a point outside the shell and (b) at a point inside the shell.

Strategy

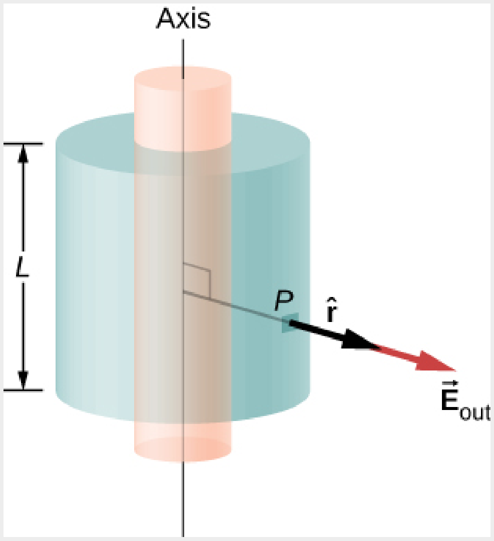
Apply the Gauss's law strategy given earlier, where we treat the cases inside and outside the shell separately.

Solution

a. **Electric field at a point outside the shell.** For a point outside the cylindrical shell, the Gaussian surface is the surface of a cylinder of radius $r > R$ and length *L*, as shown in [link]. The charge enclosed by the Gaussian cylinder is equal to the charge on the cylindrical shell of length *L*. Therefore, λ_{enc} is given by

   Equation:

$$\lambda_{\text{enc}} = \frac{\sigma_0 2\pi\, RL}{L} = 2\pi\, R\sigma_0.$$



A Gaussian surface surrounding a cylindrical shell.

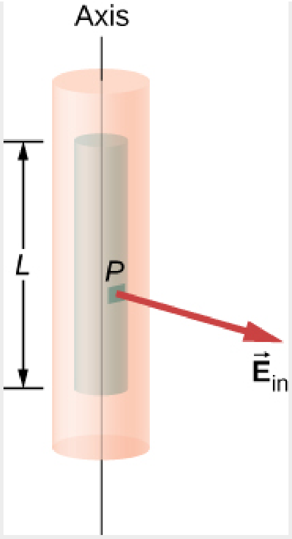
Hence, the electric field at a point *P* outside the shell at a distance *r* away from the axis is
Equation:

$$\vec{\mathbf{E}} = \frac{2\pi R\sigma_0}{2\pi\varepsilon_o}\frac{1}{r}\hat{\mathbf{r}} = \frac{R\sigma_0}{\varepsilon_o}\frac{1}{r}\hat{\mathbf{r}}\;(r > R)$$

where $\hat{\mathbf{r}}$ is a unit vector, perpendicular to the axis and pointing away from it, as shown in the figure. The electric field at *P* points in the direction of $\hat{\mathbf{r}}$ given in [link] if $\sigma_0 > 0$ and in the opposite direction to $\hat{\mathbf{r}}$ if $\sigma_0 < 0$.

b. **Electric field at a point inside the shell.** For a point inside the cylindrical shell, the Gaussian surface is a cylinder whose radius *r* is less than *R* ([link]). This means no charges are included inside the Gaussian surface:
Equation:

$$\lambda_{\text{enc}} = 0.$$



A Gaussian surface within a cylindrical shell.

This gives the following equation for the magnitude of the electric field E_{in} at a point whose *r* is less than *R* of the shell of charges.
Equation:

$$E_{\text{in}}\, 2\pi r L = 0\ (r < R),$$

This gives us
Equation:

$$E_{\text{in}} = 0\ (r < R).$$

Significance
Notice that the result inside the shell is exactly what we should expect: No enclosed charge means zero electric field. Outside the shell, the result becomes identical to a wire with uniform charge $R\sigma_0$.

Note:
Exercise:

Problem:

Check Your Understanding A thin straight wire has a uniform linear charge density λ_0. Find the electric field at a distance *d* from the wire, where *d* is much less than the length of the wire.

Solution:

$\overrightarrow{\mathbf{E}} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{1}{d}\hat{\mathbf{r}}$; This agrees with the calculation of [link] where we found the electric field by integrating over the charged wire. Notice how much simpler the calculation of this electric field is with Gauss's law.

Charge Distribution with Planar Symmetry

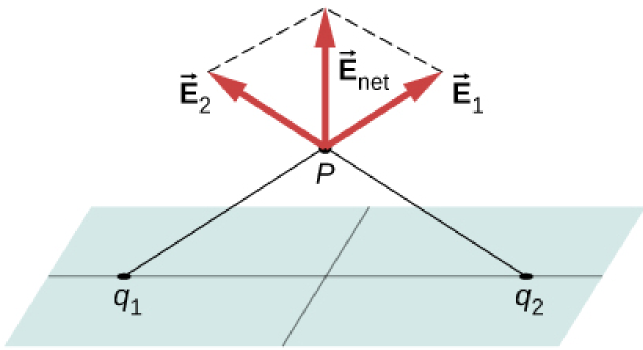
A **planar symmetry** of charge density is obtained when charges are uniformly spread over a large flat surface. In planar symmetry, all points in a plane parallel to the plane of charge are identical with respect to the charges.

Consequences of symmetry

We take the plane of the charge distribution to be the *xy*-plane and we find the electric field at a space point *P* with coordinates (*x*, *y*, *z*). Since the charge density is the same at all (*x*, *y*)-coordinates in the $z = 0$ plane, by symmetry, the electric field at *P* cannot depend on the *x*- or *y*-coordinates of point *P*, as shown in [link]. Therefore, the electric field at *P* can only depend on the distance from the plane and has a direction either toward the plane or away from the plane. That is, the electric field at *P* has only a nonzero *z*-component.

Uniform charges in *xy* plane: $\overrightarrow{\mathbf{E}} = E(z)\widehat{\mathbf{z}}$

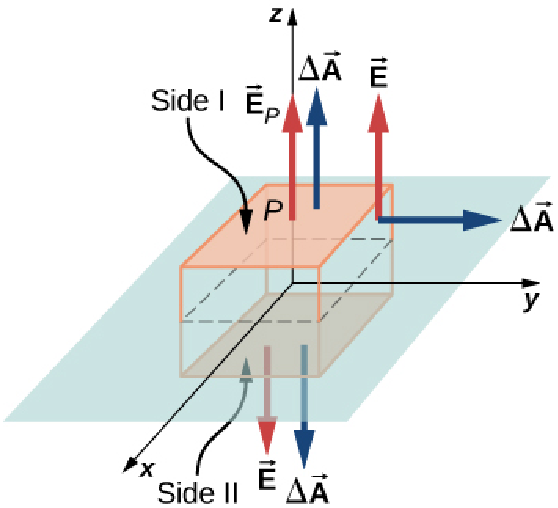
where *z* is the distance from the plane and $\widehat{\mathbf{z}}$ is the unit vector normal to the plane. Note that in this system, $E\left(z\right) = E\left(-z\right)$, although of course they point in opposite directions.



The components of the electric field parallel to a plane of charges cancel out the two charges located symmetrically from the field point P. Therefore, the field at any point is pointed vertically from the plane of charges. For any point P and charge q_1, we can always find a q_2 with this effect.

Gaussian surface and flux calculation

In the present case, a convenient Gaussian surface is a box, since the expected electric field points in one direction only. To keep the Gaussian box symmetrical about the plane of charges, we take it to straddle the plane of the charges, such that one face containing the field point P is taken parallel to the plane of the charges. In [link], sides I and II of the Gaussian surface (the box) that are parallel to the infinite plane have been shaded. They are the only surfaces that give rise to nonzero flux because the electric field and the area vectors of the other faces are perpendicular to each other.



A thin charged sheet and the Gaussian box for finding the electric field at the field point *P*. The normal to each face of the box is from inside the box to outside. On two faces of the box, the electric fields are parallel to the area vectors, and on the other four faces, the electric fields are perpendicular to the area vectors.

Let *A* be the area of the shaded surface on each side of the plane and E_P be the magnitude of the electric field at point *P*. Since sides I and II are at the same distance from the plane, the electric field has the same magnitude at points in these planes, although the directions of the electric field at these points in the two planes are opposite to each other.

Magnitude at I or II: $E(z) = E_P$.

If the charge on the plane is positive, then the direction of the electric field and the area vectors are as shown in [link]. Therefore, we find for the flux of electric field through the box
Equation:

$$\Phi = \oint_S \vec{\mathbf{E}}_P \cdot \hat{\mathbf{n}}\, dA = E_P A + E_P A + 0 + 0 + 0 + 0 = 2E_P A$$

where the zeros are for the flux through the other sides of the box. Note that if the charge on the plane is negative, the directions of electric field and area vectors for planes I and II are opposite to each other, and we get a negative sign for the flux. According to Gauss's law, the flux must equal $q_{\text{enc}}/\varepsilon_0$. From [link], we see that the charges inside the volume enclosed by the Gaussian box reside on an area *A* of the *xy*-plane. Hence,

Equation:

$$q_{\text{enc}} = \sigma_0 A.$$

Using the equations for the flux and enclosed charge in Gauss's law, we can immediately determine the electric field at a point at height *z* from a uniformly charged plane in the *xy*-plane:
Equation:

$$\vec{\mathbf{E}}_P = \frac{\sigma_0}{2\varepsilon_0}\widehat{\mathbf{n}}.$$

The direction of the field depends on the sign of the charge on the plane and the side of the plane where the field point *P* is located. Note that above the plane, $\widehat{\mathbf{n}} = +\widehat{\mathbf{z}}$, while below the plane, $\widehat{\mathbf{n}} = -\widehat{\mathbf{z}}$.

You may be surprised to note that the electric field does not actually depend on the distance from the plane; this is an effect of the assumption that the plane is infinite. In practical terms, the result given above is still a useful approximation for finite planes near the center.

Summary

- For a charge distribution with certain spatial symmetries (spherical, cylindrical, and planar), we can find a Gaussian surface over which $\overrightarrow{\mathbf{E}} \cdot \widehat{\mathbf{n}} = E$, where E is constant over the surface. The electric field is then determined with Gauss's law.
- For spherical symmetry, the Gaussian surface is also a sphere, and Gauss's law simplifies to $4\pi r^2 E = \frac{q_{\text{enc}}}{\varepsilon_0}$.
- For cylindrical symmetry, we use a cylindrical Gaussian surface, and find that Gauss's law simplifies to $2\pi r L E = \frac{q_{\text{enc}}}{\varepsilon_0}$.
- For planar symmetry, a convenient Gaussian surface is a box penetrating the plane, with two faces parallel to the plane and the remainder perpendicular, resulting in Gauss's law being $2AE = \frac{q_{\text{enc}}}{\varepsilon_0}$.

Conceptual Questions

Exercise:

Problem:

Would Gauss's law be helpful for determining the electric field of two equal but opposite charges a fixed distance apart?

Solution:

No, since the situation does not have symmetry, making Gauss's law challenging to simplify.

Exercise:

Problem:

Discuss the role that symmetry plays in the application of Gauss's law. Give examples of continuous charge distributions in which Gauss's law is useful and not useful in determining the electric field.

Exercise:

Problem:

Discuss the restrictions on the Gaussian surface used to discuss planar symmetry. For example, is its length important? Does the cross-section have to be square? Must the end faces be on opposite sides of the sheet?

Solution:

Any shape of the Gaussian surface can be used. The only restriction is that the Gaussian integral must be calculable; therefore, a box or a cylinder are the most convenient geometrical shapes for the Gaussian surface.

Problems

Exercise:

Problem:

Recall that in the example of a uniform charged sphere, $\rho_0 = Q/(\frac{4}{3}\pi R^3)$. Rewrite the answers in terms of the total charge Q on the sphere.

Solution:

$r > R,\ E = \frac{Q}{4\pi\varepsilon_0 r^2};\ r < R,\ E = \frac{qr}{4\pi\varepsilon_0 R^3}$

Exercise:

Problem:

Suppose that the charge density of the spherical charge distribution shown in [link] is $\rho(r) = \rho_0 r/R$ for $r \leq R$ and zero for $r > R$. Obtain expressions for the electric field both inside and outside the distribution.

Exercise:

Problem:

A very long, thin wire has a uniform linear charge density of 50 $\mu\mathrm{C/m}$. What is the electric field at a distance 2.0 cm from the wire?

Solution:

$EA = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = 4.50 \times 10^7 \text{ N/C}$

Exercise:

Problem:

A charge of $-30\ \mu\text{C}$ is distributed uniformly throughout a spherical volume of radius 10.0 cm. Determine the electric field due to this charge at a distance of (a) 2.0 cm, (b) 5.0 cm, and (c) 20.0 cm from the center of the sphere.

Exercise:

Problem:

Repeat your calculations for the preceding problem, given that the charge is distributed uniformly over the surface of a spherical conductor of radius 10.0 cm.

Solution:

a. 0; b. 0; c. $\vec{\mathbf{E}} = 6.74 \times 10^6 \text{ N/C}(-\hat{\mathbf{r}})$

Exercise:

Problem:

A total charge Q is distributed uniformly throughout a spherical shell of inner and outer radii r_1 and r_2, respectively. Show that the electric field due to the charge is

$$\begin{array}{ll} \vec{\mathbf{E}} = \vec{\mathbf{0}} & (r \le r_1); \\ \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^3 - {r_1}^3}{{r_2}^3 - {r_1}^3}\right)\hat{\mathbf{r}} & (r_1 \le r \le r_2); \\ \vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & (r \ge r_2). \end{array}$$

Exercise:

Problem:

When a charge is placed on a metal sphere, it ends up in equilibrium at the outer surface. Use this information to determine the electric field of $+3.0\ \mu\text{C}$ charge put on a 5.0-cm aluminum spherical ball at the following two points in space: (a) a point 1.0 cm from the center of the ball (an inside point) and (b) a point 10 cm from the center of the ball (an outside point).

Solution:

a. 0; b. $E = 2.70 \times 10^6 \text{ N/C}$

Exercise:

Problem:

A large sheet of charge has a uniform charge density of 10 $\mu\mathrm{C/m^2}$. What is the electric field due to this charge at a point just above the surface of the sheet?

Exercise:

Problem:

Determine if approximate cylindrical symmetry holds for the following situations. State why or why not. (a) A 300-cm long copper rod of radius 1 cm is charged with +500 nC of charge and we seek electric field at a point 5 cm from the center of the rod. (b) A 10-cm long copper rod of radius 1 cm is charged with +500 nC of charge and we seek electric field at a point 5 cm from the center of the rod. (c) A 150-cm wooden rod is glued to a 150-cm plastic rod to make a 300-cm long rod, which is then painted with a charged paint so that one obtains a uniform charge density. The radius of each rod is 1 cm, and we seek an electric field at a point that is 4 cm from the center of the rod. (d) Same rod as (c), but we seek electric field at a point that is 500 cm from the center of the rod.

Solution:

a. Yes, the length of the rod is much greater than the distance to the point in question. b. No, The length of the rod is of the same order of magnitude as the distance to the point in question. c. Yes, the length of the rod is much greater than the distance to the point in question. d. No. The length of the rod is of the same order of magnitude as the distance to the point in question.

Exercise:

Problem:

A long silver rod of radius 3 cm has a charge of $-5\ \mu\mathrm{C/cm}$ on its surface. (a) Find the electric field at a point 5 cm from the center of the rod (an outside point). (b) Find the electric field at a point 2 cm from the center of the rod (an inside point).

Exercise:

Problem:

The electric field at 2 cm from the center of long copper rod of radius 1 cm has a magnitude 3 N/C and directed outward from the axis of the rod. (a) How much charge per unit length exists on the copper rod? (b) What would be the electric flux through a cube of side 5 cm situated such that the rod passes through opposite sides of the cube perpendicularly?

Solution:

a. $\overrightarrow{\mathbf{E}} = \frac{R\sigma_0}{\varepsilon_0}\ \frac{1}{r}\hat{\mathbf{r}} \Rightarrow \sigma_0 = 5.31\ \times\ 10^{-11}\ \mathrm{C/m^2}$,
$\lambda = 3.33\ \times\ 10^{-12}\ \mathrm{C/m}$;
b. $\Phi = \frac{q_{\mathrm{enc}}}{\varepsilon_0} = \frac{3.33\times10^{-12}\ \mathrm{C/m}(0.05\ \mathrm{m})}{\varepsilon_0} = 0.019\ \mathrm{N\cdot m^2/C}$

Exercise:

Problem:

A long copper cylindrical shell of inner radius 2 cm and outer radius 3 cm surrounds concentrically a charged long aluminum rod of radius 1 cm with a charge density of 4 pC/m. All charges on the aluminum rod reside at its surface. The inner surface of the copper shell has exactly opposite charge to that of the aluminum rod while the outer surface of the copper shell has the same charge as the aluminum rod. Find the magnitude and direction of the electric field at points that are at the following distances from the center of the aluminum rod: (a) 0.5 cm, (b) 1.5 cm, (c) 2.5 cm, (d) 3.5 cm, and (e) 7 cm.

Exercise:

Problem:

Charge is distributed uniformly with a density ρ throughout an infinitely long cylindrical volume of radius *R*. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\rho r}{2\varepsilon_0} \qquad (r \le R);$$
$$E = \frac{\rho R^2}{2\varepsilon_0 r} \qquad (r \ge R).$$

Solution:

$$E 2\pi r l = \frac{\rho \pi r^2 l}{\varepsilon_0} \Rightarrow E = \frac{\rho r}{2\varepsilon_0}\ (r \le R);$$
$$E 2\pi r l = \frac{\rho \pi R^2 l}{\varepsilon_0} \Rightarrow E = \frac{\rho R^2}{2\varepsilon_0 r}\ (r \ge R)$$

Exercise:

Problem:

Charge is distributed throughout a very long cylindrical volume of radius *R* such that the charge density increases with the distance *r* from the central axis of the cylinder according to $\rho = \alpha r$, where α is a constant. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\alpha r^2}{3\varepsilon_0} \qquad (r \le R);$$
$$E = \frac{\alpha R^3}{3\varepsilon_0 r} \qquad (r \ge R).$$

Exercise:

Problem:

The electric field 10.0 cm from the surface of a copper ball of radius 5.0 cm is directed toward the ball’s center and has magnitude $4.0 \times 10^2\ \text{N/C}$. How much charge is on the surface of the ball?

Solution:

$\Phi = \frac{q_{\text{enc}}}{\varepsilon_0} \Rightarrow q_{\text{enc}} = -1.0 \times 10^{-9}\text{ C}$

Exercise:

Problem:

Charge is distributed throughout a spherical shell of inner radius r_1 and outer radius r_2 with a volume density given by $\rho = \rho_0 r_1/r$, where ρ_0 is a constant. Determine the electric field due to this charge as a function of *r*, the distance from the center of the shell.

Exercise:

Problem:

Charge is distributed throughout a spherical volume of radius *R* with a density $\rho = \alpha r^2$, where α is a constant. Determine the electric field due to the charge at points both inside and outside the sphere.

Solution:

$$q_{\text{enc}} = \frac{4}{5}\pi\alpha r^5,$$
$$E4\pi r^2 = \frac{4\pi\alpha r^5}{5\varepsilon_0} \Rightarrow E = \frac{\alpha r^3}{5\varepsilon_0}\ (r \le R),$$
$$q_{\text{enc}} = \frac{4}{5}\pi\alpha R^5,\ E4\pi r^2 = \frac{4\pi\alpha R^5}{5\varepsilon_0} \Rightarrow E = \frac{\alpha R^5}{5\varepsilon_0 r^2}\ (r \ge R)$$

Exercise:

Problem:

Consider a uranium nucleus to be sphere of radius $R = 7.4 \times 10^{-15}$ m with a charge of 92*e* distributed uniformly throughout its volume. (a) What is the electric force exerted on an electron when it is 3.0×10^{-15}m from the center of the nucleus? (b) What is the acceleration of the electron at this point?

Exercise:

Problem:

The volume charge density of a spherical charge distribution is given by $\rho(r) = \rho_0 e^{-\alpha r}$, where ρ_0 and α are constants. What is the electric field produced by this charge distribution?

Solution:

integrate by parts:

$$q_{\text{enc}} = 4\pi\rho_0 \left[-e^{-\alpha r}(\frac{(r)^2}{\alpha} + \frac{2r}{\alpha^2} + \frac{2}{\alpha^3}) + \frac{2}{\alpha^3}\right] \Rightarrow E = \frac{\rho_0}{r^2\varepsilon_0}\left[-e^{-\alpha r}(\frac{(r)^2}{\alpha} + \frac{2r}{\alpha^2} + \frac{2}{\alpha^3}) + \frac{2}{\alpha^3}\right]$$

Glossary

cylindrical symmetry

system only varies with distance from the axis, not direction

planar symmetry

system only varies with distance from a plane

spherical symmetry

system only varies with the distance from the origin, not in direction

Conductors in Electrostatic Equilibrium
By the end of this section, you will be able to:

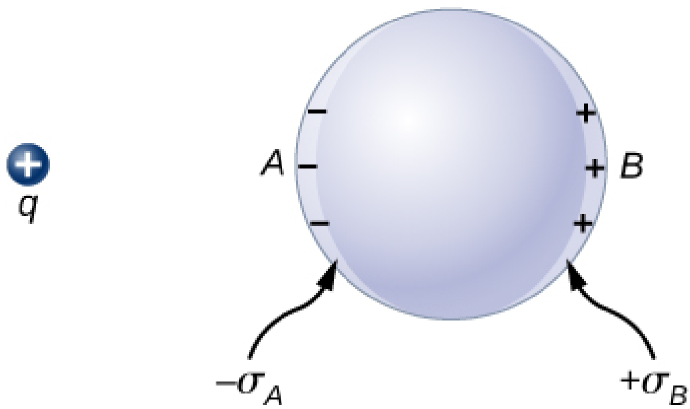
- Describe the electric field within a conductor at equilibrium
- Describe the electric field immediately outside the surface of a charged conductor at equilibrium
- Explain why if the field is not as described in the first two objectives, the conductor is not at equilibrium

So far, we have generally been working with charges occupying a volume within an insulator. We now study what happens when free charges are placed on a conductor. Generally, in the presence of a (generally external) electric field, the free charge in a conductor redistributes and very quickly reaches electrostatic equilibrium. The resulting charge distribution and its electric field have many interesting properties, which we can investigate with the help of Gauss's law and the concept of electric potential.

The Electric Field inside a Conductor Vanishes

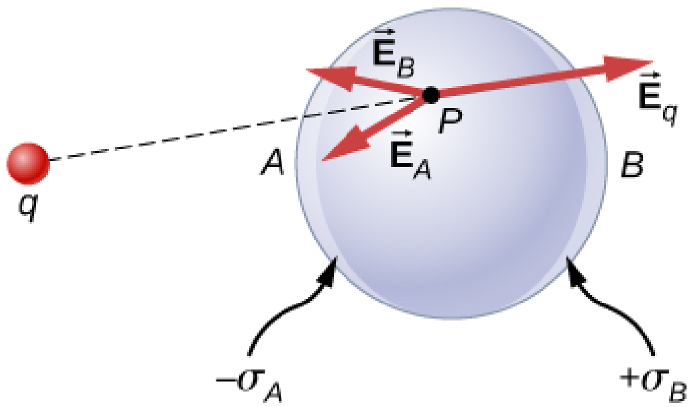
If an electric field is present inside a conductor, it exerts forces on the **free electrons** (also called conduction electrons), which are electrons in the material that are not bound to an atom. These free electrons then accelerate. However, moving charges by definition means nonstatic conditions, contrary to our assumption. Therefore, when electrostatic equilibrium is reached, the charge is distributed in such a way that the electric field inside the conductor vanishes.

If you place a piece of a metal near a positive charge, the free electrons in the metal are attracted to the external positive charge and migrate freely toward that region. The region the electrons move to then has an excess of electrons over the protons in the atoms and the region from where the electrons have migrated has more protons than electrons. Consequently, the metal develops a negative region near the charge and a positive region at the far end ([link]). As we saw in the preceding chapter, this separation of equal magnitude and opposite type of electric charge is called polarization. If you remove the external charge, the electrons migrate back and neutralize the positive region.



Polarization of a metallic sphere by an external point charge $+q$. The near side of the metal has an opposite surface charge compared to the far side of the metal. The sphere is said to be polarized. When you remove the external charge, the polarization of the metal also disappears.

The polarization of the metal happens only in the presence of external charges. You can think of this in terms of electric fields. The external charge creates an external electric field. When the metal is placed in the region of this electric field, the electrons and protons of the metal experience electric forces due to this external electric field, but only the conduction electrons are free to move in the metal over macroscopic distances. The movement of the conduction electrons leads to the polarization, which creates an induced electric field in addition to the external electric field ([link]). The net electric field is a vector sum of the fields of $+q$ and the surface charge densities $-\sigma_A$ and $+\sigma_B$. This means that the net field inside the conductor is different from the field outside the conductor.



In the presence of an external charge q, the charges in a metal redistribute. The electric field at any point has three contributions, from $+q$ and the induced charges $-\sigma_A$ and $+\sigma_B$. Note that the surface charge distribution will not be uniform in this case.

The redistribution of charges is such that the sum of the three contributions at any point P inside the conductor is

Equation:

$$\vec{\mathbf{E}}_P = \vec{\mathbf{E}}_q + \vec{\mathbf{E}}_B + \vec{\mathbf{E}}_A = \vec{\mathbf{0}}.$$

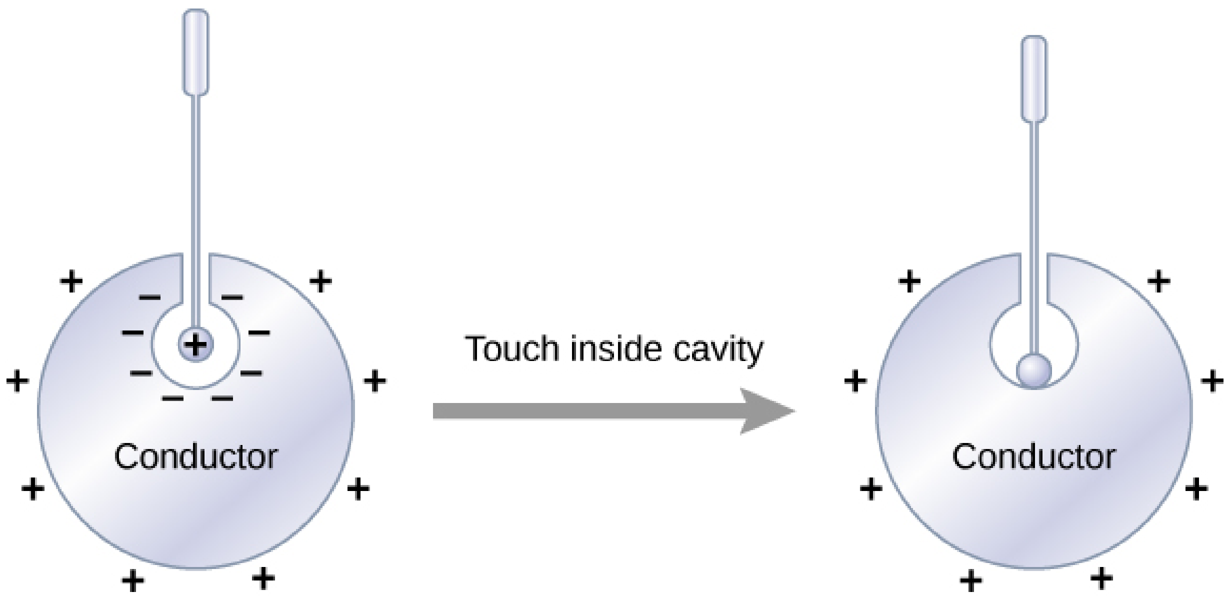
Now, thanks to Gauss’s law, we know that there is no net charge enclosed by a Gaussian surface that is solely within the volume of the conductor at equilibrium. That is, $q_{\text{enc}} = 0$ and hence

Equation:

$$\vec{\mathbf{E}}_{\text{net}} = \vec{\mathbf{0}} \text{ (at points inside a conductor).}$$

Charge on a Conductor

An interesting property of a conductor in static equilibrium is that extra charges on the conductor end up on the outer surface of the conductor, regardless of where they originate. [link] illustrates a system in which we bring an external positive charge inside the cavity of a metal and then touch it to the inside surface. Initially, the inside surface of the cavity is negatively charged and the outside surface of the conductor is positively charged. When we touch the inside surface of the cavity, the induced charge is neutralized, leaving the outside surface and the whole metal charged with a net positive charge.



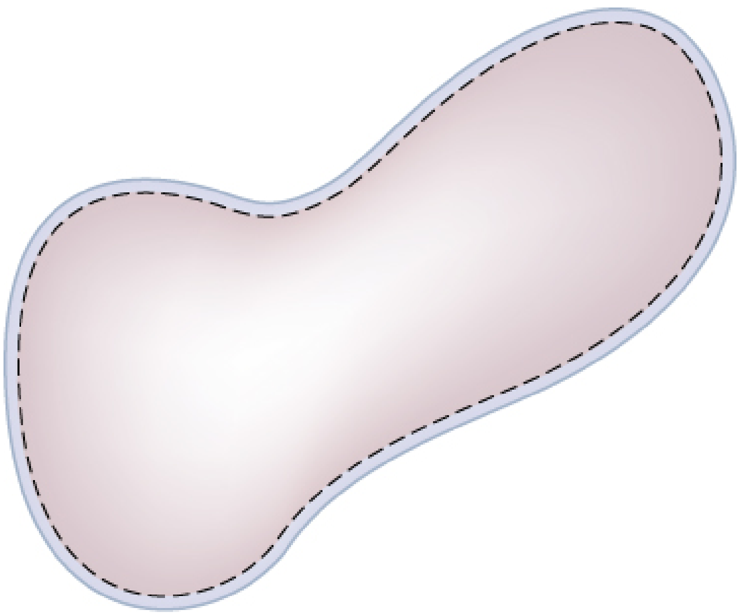
Electric charges on a conductor migrate to the outside surface no matter where you put them initially.

To see why this happens, note that the Gaussian surface in [link] (the dashed line) follows the contour of the actual surface of the conductor and is located an infinitesimal distance *within* it. Since $E = 0$ everywhere inside a conductor,

Equation:

$$\oint_s \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}} dA = 0.$$

Thus, from Gauss's law, there is no net charge inside the Gaussian surface. But the Gaussian surface lies just below the actual surface of the conductor; consequently, there is no net charge inside the conductor. Any excess charge must lie on its surface.



The dashed line represents a Gaussian surface that is just beneath the actual surface of the conductor.

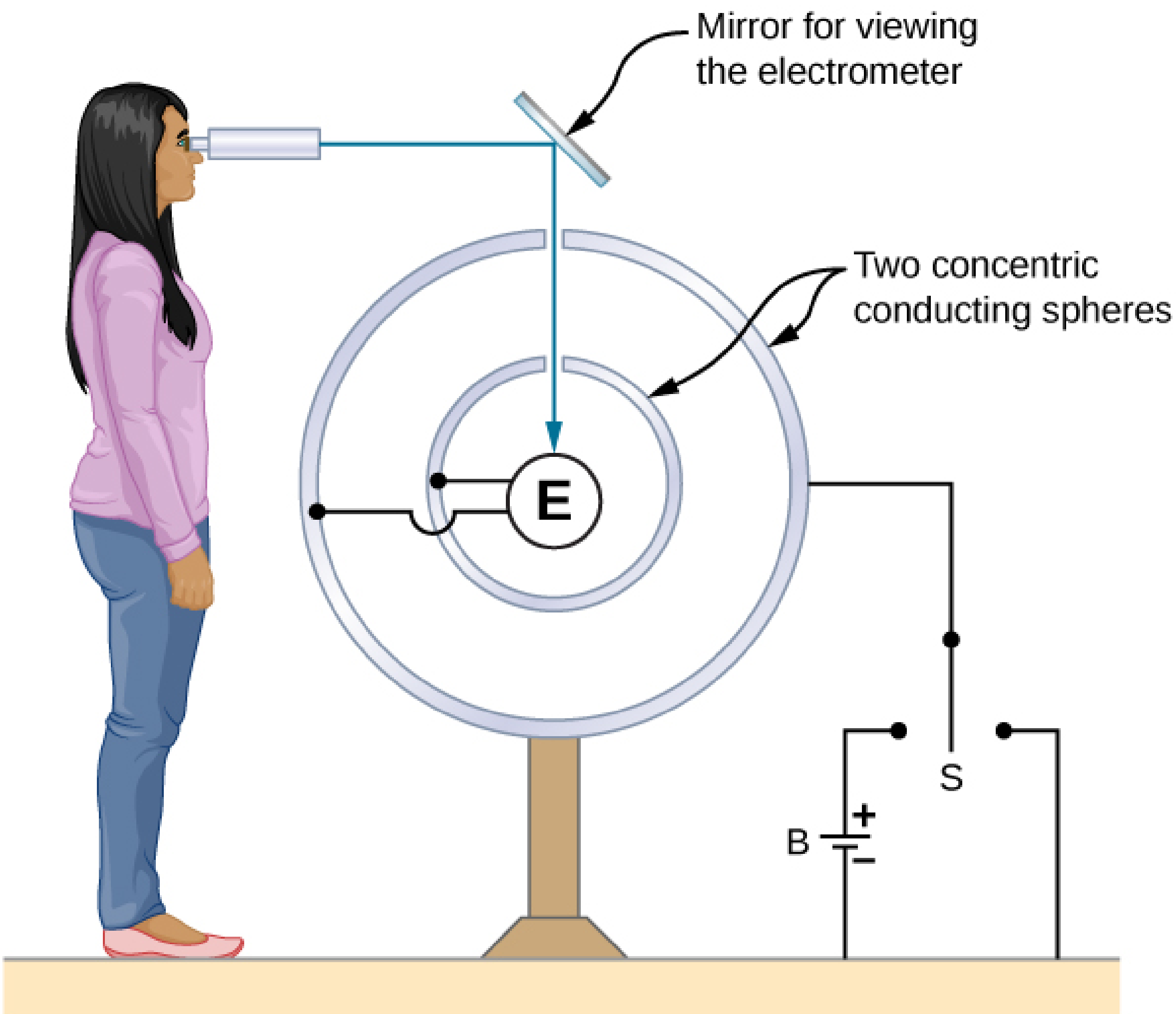
This particular property of conductors is the basis for an extremely accurate method developed by Plimpton and Lawton in 1936 to verify Gauss's law and, correspondingly, Coulomb's law. A sketch of their apparatus is shown in [link]. Two spherical shells are connected to one another through an electrometer E, a device that can detect a very slight amount of charge flowing from one shell to the other. When switch S is thrown to the left,

charge is placed on the outer shell by the battery B. Will charge flow through the electrometer to the inner shell?

No. Doing so would mean a violation of Gauss's law. Plimpton and Lawton did not detect any flow and, knowing the sensitivity of their electrometer, concluded that if the radial dependence in Coulomb's law were $1/r^{(2+\delta)}$, δ would be less than 2×10^{-9}[footnote]. More recent measurements place δ at less than 3×10^{-16}[footnote], a number so small that the validity of Coulomb's law seems indisputable.

S. Plimpton and W. Lawton. 1936. "A Very Accurate Test of Coulomb's Law of Force between Charges." *Physical Review* 50, No. 11: 1066, doi:10.1103/PhysRev.50.1066

E. Williams, J. Faller, and H. Hill. 1971. "New Experimental Test of Coulomb's Law: A Laboratory Upper Limit on the Photon Rest Mass." *Physical Review Letters* 26 , No. 12: 721, doi:10.1103/PhysRevLett.26.721



A representation of the apparatus used by Plimpton and Lawton. Any transfer of charge between the spheres is detected by the electrometer E.

The Electric Field at the Surface of a Conductor

If the electric field had a component parallel to the surface of a conductor, free charges on the surface would move, a situation contrary to the assumption of electrostatic equilibrium. Therefore, the electric field is always perpendicular to the surface of a conductor.

At any point just above the surface of a conductor, the surface charge density σ and the magnitude of the electric field E are related by

Note:
Equation:

$$E = \frac{\sigma}{\varepsilon_0}.$$

To see this, consider an infinitesimally small Gaussian cylinder that surrounds a point on the surface of the conductor, as in [link]. The cylinder has one end face inside and one end face outside the surface. The height and cross-sectional area of the cylinder are δ and ΔA, respectively. The cylinder's sides are perpendicular to the surface of the conductor, and its end faces are parallel to the surface. Because the cylinder is infinitesimally small, the charge density σ is essentially constant over the surface enclosed, so the total charge inside the Gaussian cylinder is $\sigma \Delta A$. Now E is perpendicular to the surface of the conductor outside the conductor and vanishes within it, because otherwise, the charges would accelerate, and we would not be in equilibrium. Electric flux therefore crosses only the outer end face of the Gaussian surface and may be written as $E \Delta A$, since the cylinder is assumed to be small enough that E is approximately constant over that area. From Gauss' law,

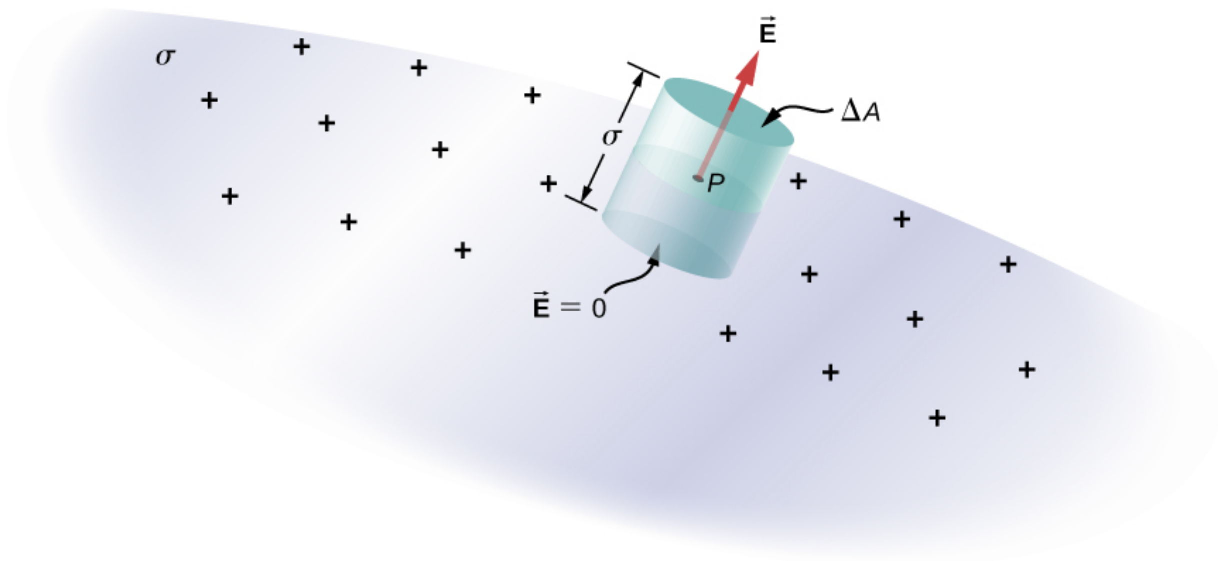
Equation:

$$E \Delta A = \frac{\sigma \Delta A}{\varepsilon_0}.$$

Thus,

Equation:

$$E = \frac{\sigma}{\varepsilon_0}.$$

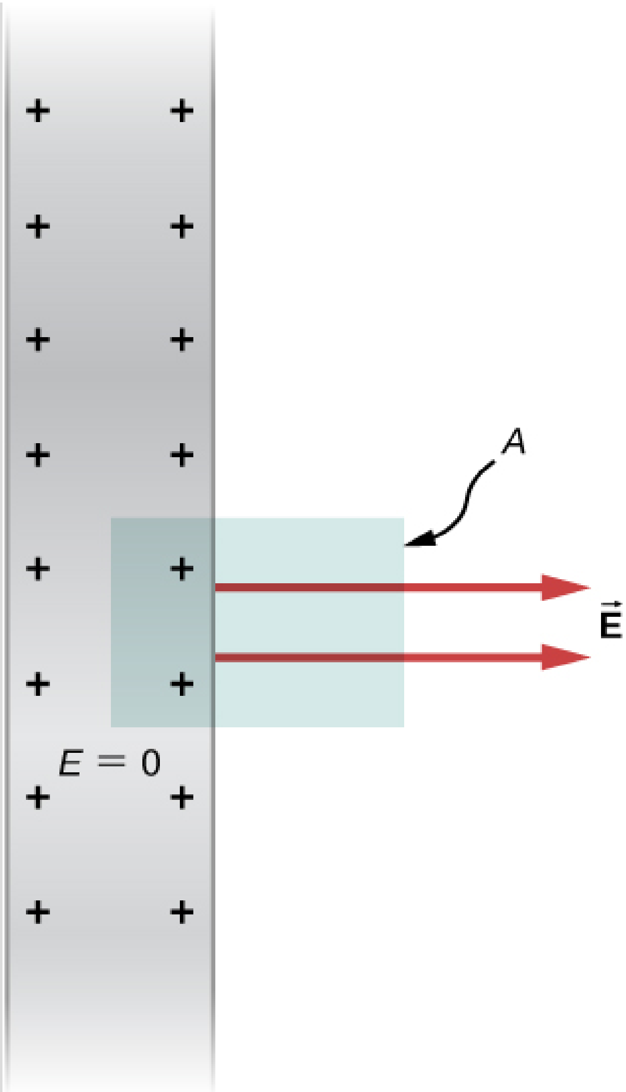


An infinitesimally small cylindrical Gaussian surface surrounds point *P*, which is on the surface of the conductor. The field $\vec{\mathbf{E}}$ is perpendicular to the surface of the conductor outside the conductor and vanishes within it.

Example:

Electric Field of a Conducting Plate

The infinite conducting plate in [link] has a uniform surface charge density σ. Use Gauss' law to find the electric field outside the plate. Compare this result with that previously calculated directly.



A side view of an infinite conducting plate and Gaussian cylinder with cross-sectional area *A*.

Strategy
For this case, we use a cylindrical Gaussian surface, a side view of which is shown.
Solution
The flux calculation is similar to that for an infinite sheet of charge from the previous chapter with one major exception: The left face of the Gaussian

surface is inside the conductor where $\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{0}}$, so the total flux through the Gaussian surface is *EA* rather than 2*EA*. Then from Gauss' law,

Equation:

$$EA = \frac{\sigma A}{\varepsilon_0}$$

and the electric field outside the plate is

Equation:

$$E = \frac{\sigma}{\varepsilon_0}.$$

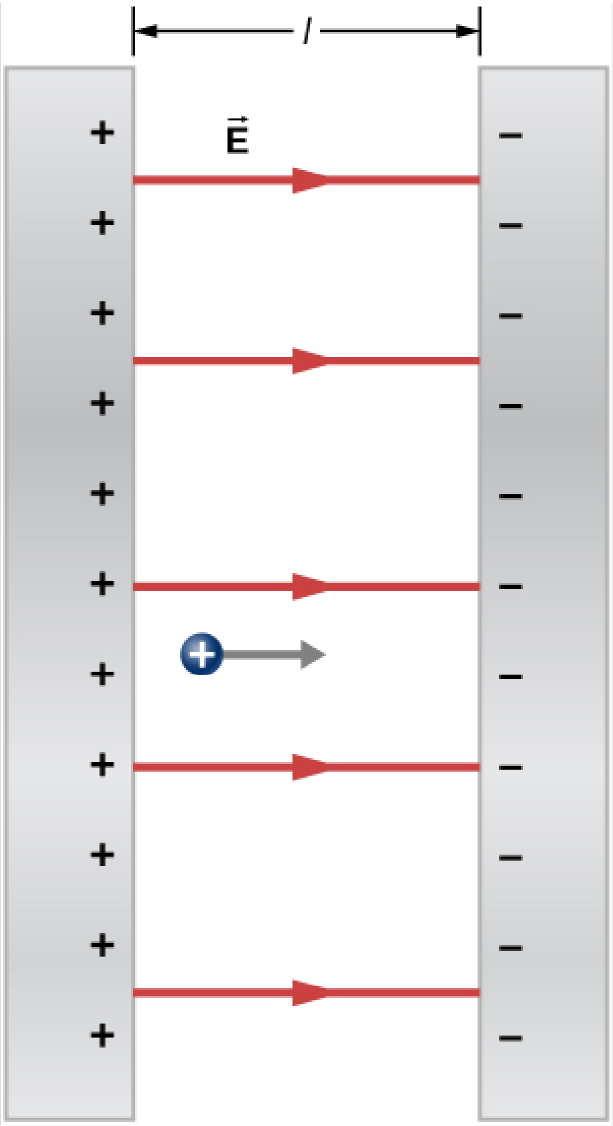
Significance

This result is in agreement with the result from the previous section, and consistent with the rule stated above.

Example:

Electric Field between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7}\ \text{C/m}^2$, as shown in [link]. The separation between the plates is $l = 6.50\ \text{mm}$. What is the electric field between the plates?



The electric field between oppositely charged parallel plates. A test charge is released at the positive plate.

Strategy
Note that the electric field at the surface of one plate only depends on the charge on that plate. Thus, apply $E = \sigma/\varepsilon_0$ with the given values.
Solution

The electric field is directed from the positive to the negative plate, as shown in the figure, and its magnitude is given by

Equation:

$$E=\frac{\sigma}{\varepsilon_0}=\frac{6.81\ \times\ 10^{-7}\,\mathrm{C/m^2}}{8.85\ \times\ 10^{-12}\,\mathrm{C^2/N\,m^2}}=7.69\ \times\ 10^4\,\mathrm{N/C}.$$

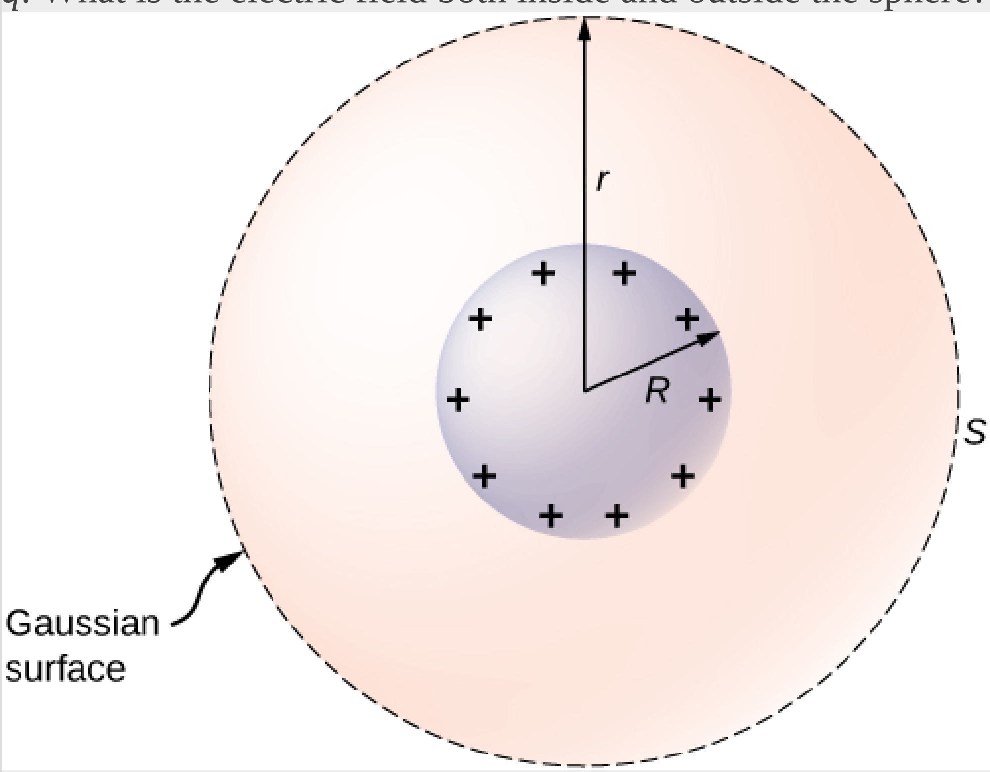
Significance

This formula is applicable to more than just a plate. Furthermore, two-plate systems will be important later.

Example:

A Conducting Sphere

The isolated conducting sphere ([link]) has a radius *R* and an excess charge *q*. What is the electric field both inside and outside the sphere?



An isolated conducting sphere.

Strategy

The sphere is isolated, so its surface change distribution and the electric field of that distribution are spherically symmetrical. We can therefore represent the field as $\vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$. To calculate $E(r)$, we apply Gauss's law over a closed spherical surface S of radius r that is concentric with the conducting sphere.

Solution

Since r is constant and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the sphere,

Equation:

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = E(r) \oint_S dA = E(r)\, 4\pi r^2.$$

For $r < R$, S is within the conductor, so $q_{\text{enc}} = 0$, and Gauss's law gives

Equation:

$$E(r) = 0,$$

as expected inside a conductor. If $r > R$, S encloses the conductor so $q_{\text{enc}} = q$. From Gauss's law,

Equation:

$$E(r)\, 4\pi r^2 = \frac{q}{\varepsilon_0}.$$

The electric field of the sphere may therefore be written as

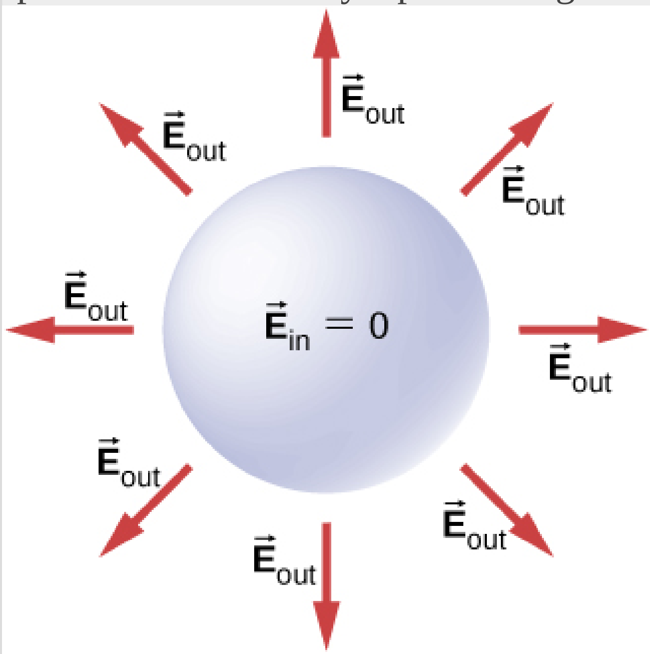
Equation:

$$\begin{aligned} \vec{\mathbf{E}} &= \vec{\mathbf{0}} && (r < R), \\ \vec{\mathbf{E}} &= \tfrac{1}{4\pi\varepsilon_0} \tfrac{q}{r^2}\hat{\mathbf{r}} && (r \geq R). \end{aligned}$$

Significance

Notice that in the region $r \geq R$, the electric field due to a charge q placed on an isolated conducting sphere of radius R is identical to the electric field of a point charge q located at the center of the sphere. The difference between the charged metal and a point charge occurs only at the space

points inside the conductor. For a point charge placed at the center of the sphere, the electric field is not zero at points of space occupied by the sphere, but a conductor with the same amount of charge has a zero electric field at those points ([link]). However, there is no distinction at the outside points in space where $r > R$, and we can replace the isolated charged spherical conductor by a point charge at its center with impunity.



Electric field of a positively charged metal sphere. The electric field inside is zero, and the electric field outside is same as the electric field of a point charge at the center, although the charge on the metal sphere is at the surface.

Note:
Exercise:

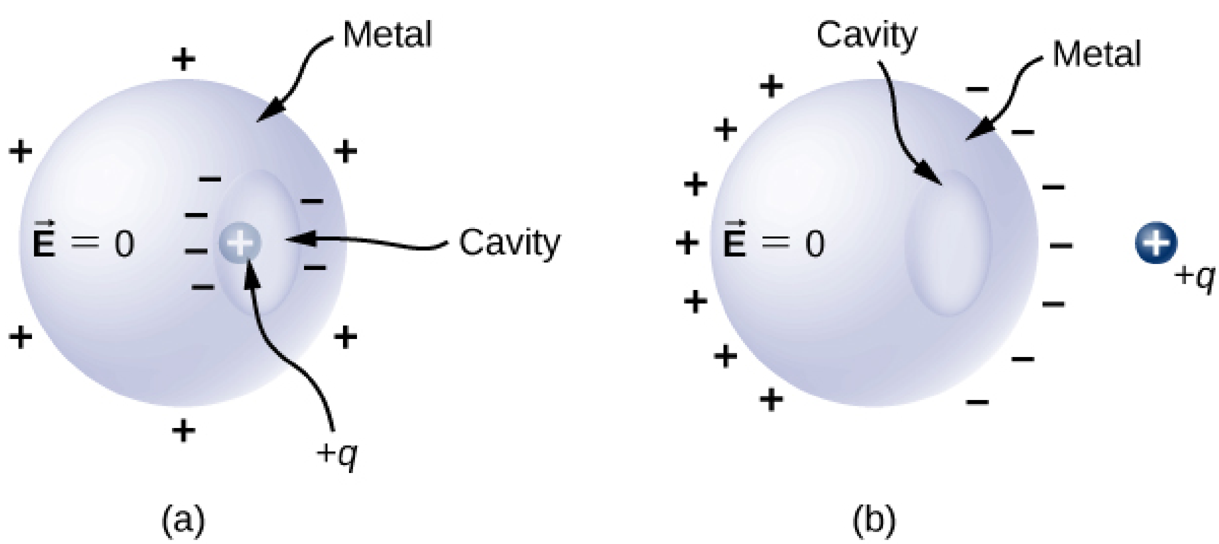
Problem:

Check Your Understanding How will the system above change if there are charged objects external to the sphere?

Solution:

If there are other charged objects around, then the charges on the surface of the sphere will not necessarily be spherically symmetrical; there will be more in certain direction than in other directions.

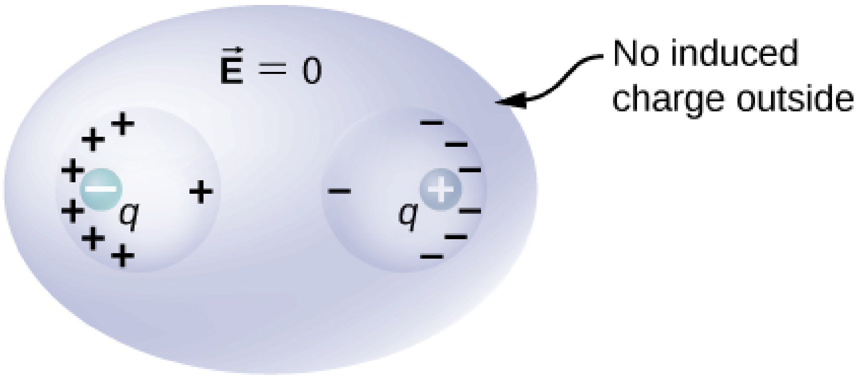
For a conductor with a cavity, if we put a charge $+q$ inside the cavity, then the charge separation takes place in the conductor, with $-q$ amount of charge on the inside surface and a $+q$ amount of charge at the outside surface ([link](a)). For the same conductor with a charge $+q$ outside it, there is no excess charge on the inside surface; both the positive and negative induced charges reside on the outside surface ([link](b)).



(a) A charge inside a cavity in a metal. The distribution of charges at the outer surface does not depend on how the charges are distributed at the inner surface, since the *E*-field inside the body of the metal is zero.

That magnitude of the charge on the outer surface does depend on the magnitude of the charge inside, however. (b) A charge outside a conductor containing an inner cavity. The cavity remains free of charge. The polarization of charges on the conductor happens at the surface.

If a conductor has two cavities, one of them having a charge $+q_a$ inside it and the other a charge $-q_b$, the polarization of the conductor results in $-q_a$ on the inside surface of the cavity a, $+q_b$ on the inside surface of the cavity b, and $q_a - q_b$ on the outside surface ([link]). The charges on the surfaces may not be uniformly spread out; their spread depends upon the geometry. The only rule obeyed is that when the equilibrium has been reached, the charge distribution in a conductor is such that the electric field by the charge distribution in the conductor cancels the electric field of the external charges at all space points inside the body of the conductor.



The charges induced by two equal and opposite charges in two separate cavities of a conductor. If the net charge on the cavity is nonzero, the external surface becomes charged to the amount of the net charge.

Summary

- The electric field inside a conductor vanishes.
- Any excess charge placed on a conductor resides entirely on the surface of the conductor.
- The electric field is perpendicular to the surface of a conductor everywhere on that surface.
- The magnitude of the electric field just above the surface of a conductor is given by $E = \frac{\sigma}{\varepsilon_0}$.

Key Equations

Definition of electric flux, for uniform electric field	$\Phi = \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{A}} \to EA\cos\theta$
Electric flux through an open surface	$\Phi = \int_S \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \int_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}}$
Electric flux through a closed surface	$\Phi = \oint_S \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \oint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}}$
Gauss's law	$\Phi = \oint_S \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \frac{q_{\text{enc}}}{\varepsilon_0}$
Gauss's Law for systems with symmetry	$\Phi = \oint_S \overrightarrow{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = E \oint_S dA = EA = \frac{q_{\text{enc}}}{\varepsilon_0}$
The magnitude of the electric field	$E = \frac{\sigma}{\varepsilon_0}$

just outside the surface of a conductor	

Conceptual Questions

Exercise:

Problem: Is the electric field inside a metal always zero?

Exercise:

Problem:

Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all the conduction electrons in a conductor are on the surface?

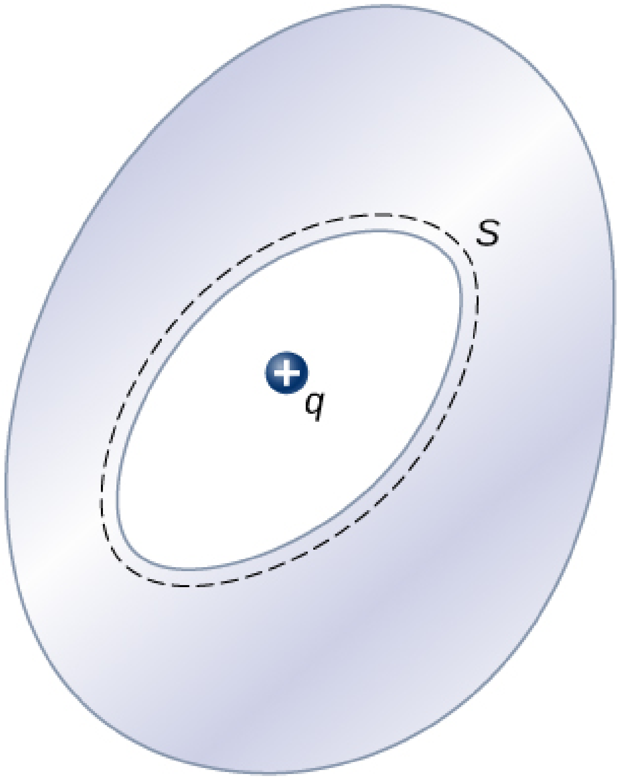
Solution:

No. If a metal was in a region of zero electric field, all the conduction electrons would be distributed uniformly throughout the metal.

Exercise:

Problem:

A charge q is placed in the cavity of a conductor as shown below. Will a charge outside the conductor experience an electric field due to the presence of q?



Exercise:

Problem:

The conductor in the preceding figure has an excess charge of $-5.0\ \mu\text{C}$. If a 2.0-μC point charge is placed in the cavity, what is the net charge on the surface of the cavity and on the outer surface of the conductor?

Solution:

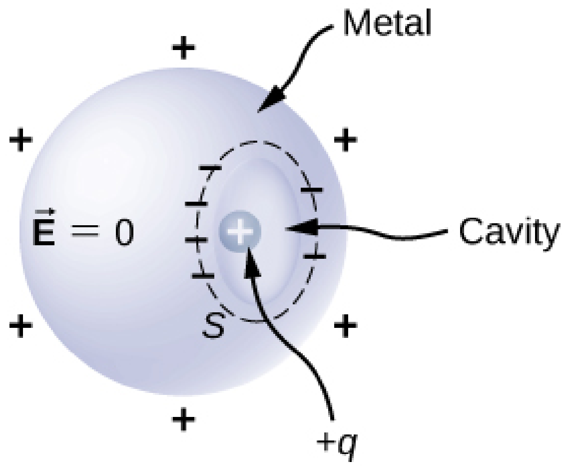
Since the electric field is zero inside a conductor, a charge of $-2.0\ \mu\text{C}$ is induced on the inside surface of the cavity. This will put a charge of $+2.0\ \mu C$ on the outside surface leaving a net charge of $-3.0\ \mu\text{C}$ on the surface.

Problems

Exercise:

Problem:

An uncharged conductor with an internal cavity is shown in the following figure. Use the closed surface *S* along with Gauss' law to show that when a charge *q* is placed in the cavity a total charge –*q* is induced on the inner surface of the conductor. What is the charge on the outer surface of the conductor?

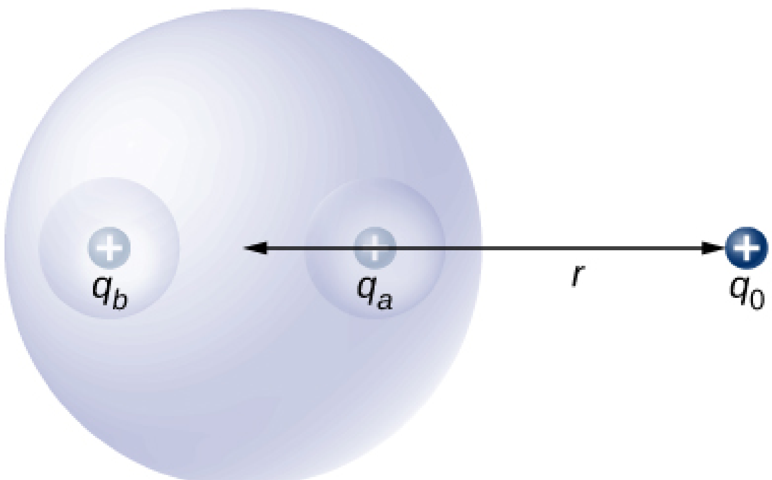


A charge inside a cavity of a metal. Charges at the outer surface do not depend on how the charges are distributed at the inner surface since E field inside the body of the metal is zero.

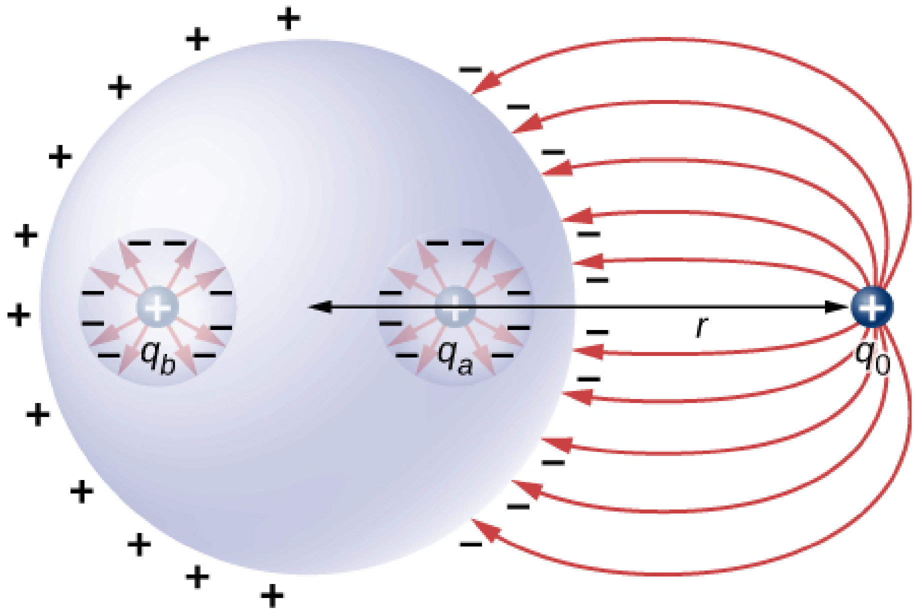
Exercise:

Problem:

An uncharged spherical conductor S of radius R has two spherical cavities A and B of radii a and b, respectively as shown below. Two point charges $+q_a$ and $+q_b$ are placed at the center of the two cavities by using non-conducting supports. In addition, a point charge $+q_0$ is placed outside at a distance r from the center of the sphere. (a) Draw approximate charge distributions in the metal although metal sphere has no net charge. (b) Draw electric field lines. Draw enough lines to represent all distinctly different places.



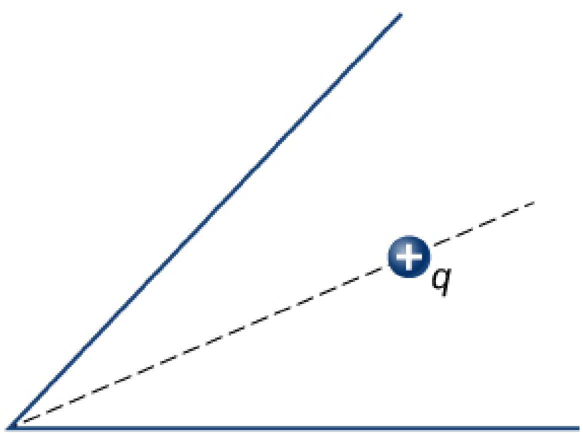
Solution:



Exercise:

Problem:

A positive point charge is placed at the angle bisector of two uncharged plane conductors that make an angle of 45°. See below. Draw the electric field lines.



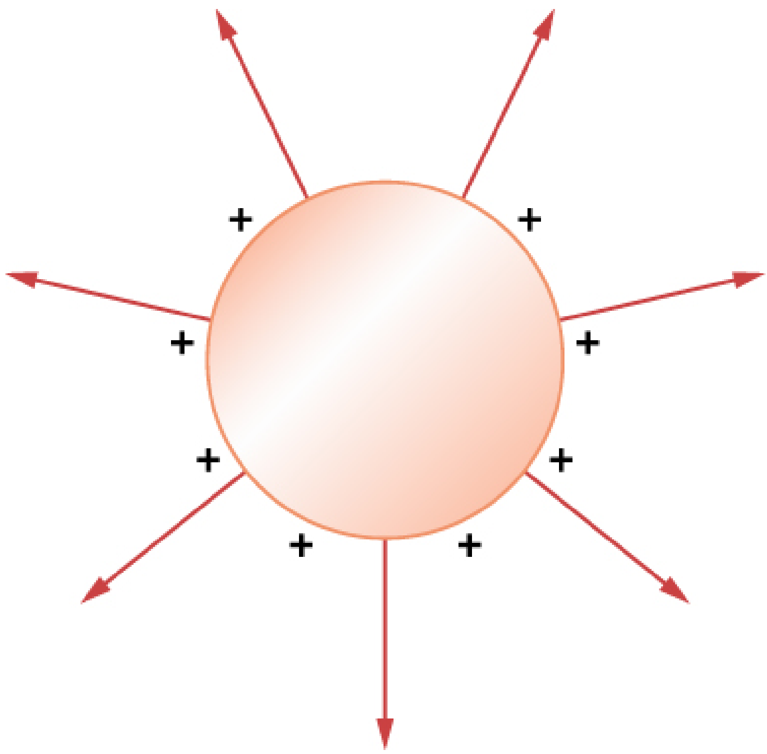
Exercise:

Problem:

A long cylinder of copper of radius 3 cm is charged so that it has a uniform charge per unit length on its surface of 3 C/m. (a) Find the electric field inside and outside the cylinder. (b) Draw electric field lines in a plane perpendicular to the rod.

Solution:

a. Outside: $E2\pi rl = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{3.0\,\mathrm{C/m}}{2\pi\varepsilon_0 r}$; Inside $E_{\text{in}} = 0$; b.



Exercise:

Problem:

An aluminum spherical ball of radius 4 cm is charged with $5\ \mu\text{C}$ of charge. A copper spherical shell of inner radius 6 cm and outer radius 8 cm surrounds it. A total charge of $-8\ \mu\text{C}$ is put on the copper shell. (a) Find the electric field at all points in space, including points inside the aluminum and copper shell when copper shell and aluminum sphere are concentric. (b) Find the electric field at all points in space, including points inside the aluminum and copper shell when the centers of copper shell and aluminum sphere are 1 cm apart.

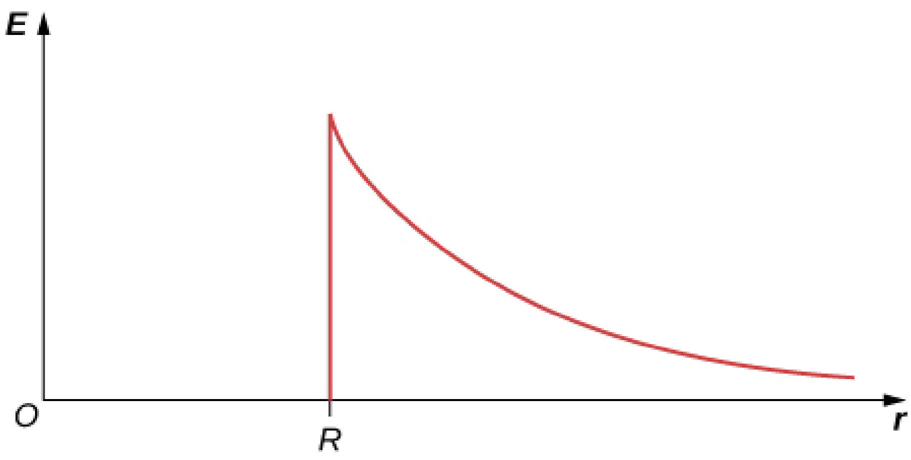
Exercise:

Problem:

A long cylinder of aluminum of radius *R* meters is charged so that it has a uniform charge per unit length on its surface of λ. (a) Find the electric field inside and outside the cylinder. (b) Plot electric field as a function of distance from the center of the rod.

Solution:

a. $E2\pi rl = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$ $r \geq R$ E inside equals 0; b.



Exercise:

Problem:

At the surface of any conductor in electrostatic equilibrium, $E = \sigma/\varepsilon_0$. Show that this equation is consistent with the fact that $E = kq/r^2$ at the surface of a spherical conductor.

Exercise:

Problem:

Two parallel plates 10 cm on a side are given equal and opposite charges of magnitude 5.0×10^{-9} C. The plates are 1.5 mm apart. What is the electric field at the center of the region between the plates?

Solution:

$E = 5.65 \times 10^4 \text{ N/C}$

Exercise:

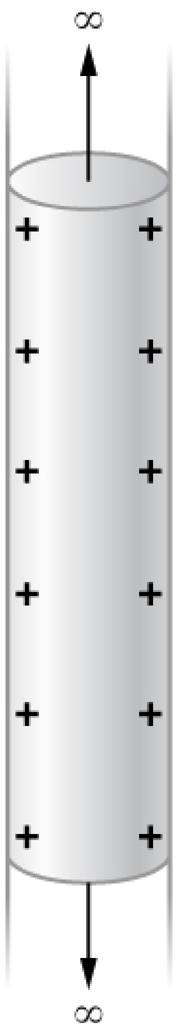
Problem:

Two parallel conducting plates, each of cross-sectional area 400 cm^2, are 2.0 cm apart and uncharged. If 1.0×10^{12} electrons are transferred from one plate to the other, what are (a) the charge density on each plate? (b) The electric field between the plates?

Exercise:

Problem:

The surface charge density on a long straight metallic pipe is σ. What is the electric field outside and inside the pipe? Assume the pipe has a diameter of $2a$.



Solution:

$\lambda = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{a\sigma}{\varepsilon_0 r}$ $r \geq a$, $E = 0$ inside since q enclosed $= 0$

Exercise:

Problem:

A point charge $q = -5.0 \times 10^{-12}$ C is placed at the center of a spherical conducting shell of inner radius 3.5 cm and outer radius 4.0 cm. The electric field just above the surface of the conductor is directed radially outward and has magnitude 8.0 N/C. (a) What is the charge density on the inner surface of the shell? (b) What is the charge density on the outer surface of the shell? (c) What is the net charge on the conductor?

Exercise:

Problem:

A solid cylindrical conductor of radius *a* is surrounded by a concentric cylindrical shell of inner radius *b*. The solid cylinder and the shell carry charges +*Q* and –*Q*, respectively. Assuming that the length *L* of both conductors is much greater than *a* or *b*, determine the electric field as a function of *r*, the distance from the common central axis of the cylinders, for (a) $r < a$; (b) $a < r < b$; and (c)$r > b$.

Solution:

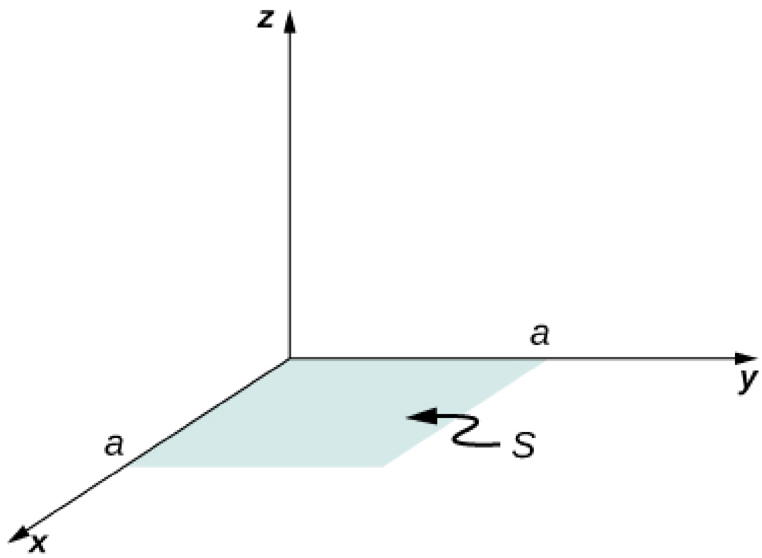
a. $E = 0$; b. $E2\pi rL = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{2\pi\varepsilon_0 rL}$; c. $E = 0$ since *r* would be either inside the second shell or if outside then *q* enclosed equals 0.

Additional Problems

Exercise:

Problem:

A vector field $\vec{\mathbf{E}}$ (not necessarily an electric field; note units) is given by $\vec{\mathbf{E}} = 3x^2\widehat{\mathbf{k}}$. Calculate $\int_S \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, da$, where *S* is the area shown below. Assume that $\widehat{\mathbf{n}} = \widehat{\mathbf{k}}$.



Exercise:

Problem: Repeat the preceding problem, with $\vec{\mathbf{E}} = 2x\hat{\mathbf{i}} + 3x^2\widehat{\mathbf{k}}$.

Solution:

$$\int \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = a^4$$

Exercise:

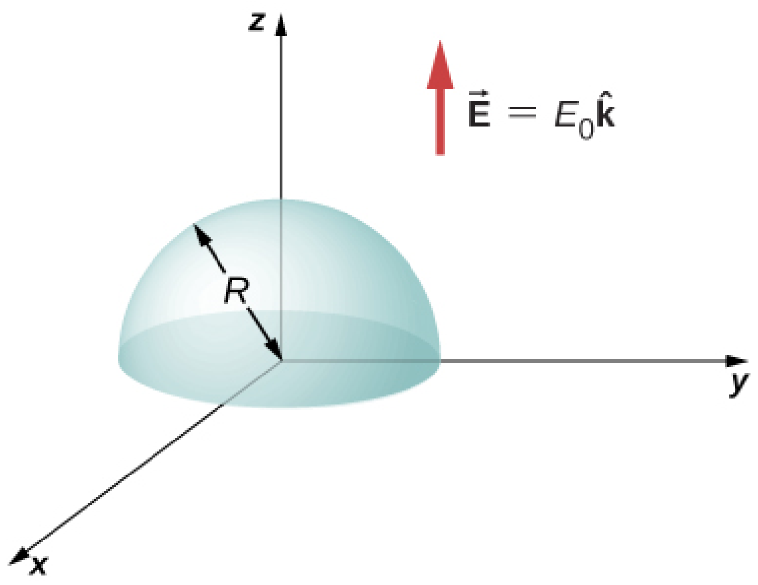
Problem:

A circular area *S* is concentric with the origin, has radius *a*, and lies in the *yz*-plane. Calculate $\int_S \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA$ for $\vec{\mathbf{E}} = 3z^2\hat{\mathbf{i}}$.

Exercise:

Problem:

(a) Calculate the electric flux through the open hemispherical surface due to the electric field $\vec{\mathbf{E}} = E_0\widehat{\mathbf{k}}$ (see below). (b) If the hemisphere is rotated by 90° around the *x*-axis, what is the flux through it?



Solution:

a. $\int \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, dA = E_0 r^2 \pi$; b. zero, since the flux through the upper half cancels the flux through the lower half of the sphere

Exercise:

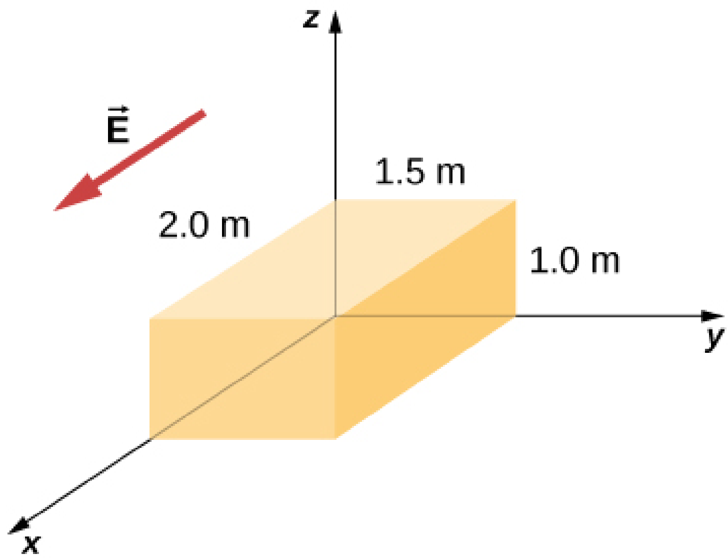
Problem:

Suppose that the electric field of an isolated point charge were proportional to $1/r^{2+\sigma}$ rather than $1/r^2$. Determine the flux that passes through the surface of a sphere of radius *R* centered at the charge. Would Gauss's law remain valid?

Exercise:

Problem:

The electric field in a region is given by $\vec{\mathbf{E}} = a/(b + cx)\hat{\mathbf{i}}$, where $a = 200\,\mathrm{N} \cdot \mathrm{m/C}, b = 2.0\,\mathrm{m}$, and $c = 2.0$. What is the net charge enclosed by the shaded volume shown below?



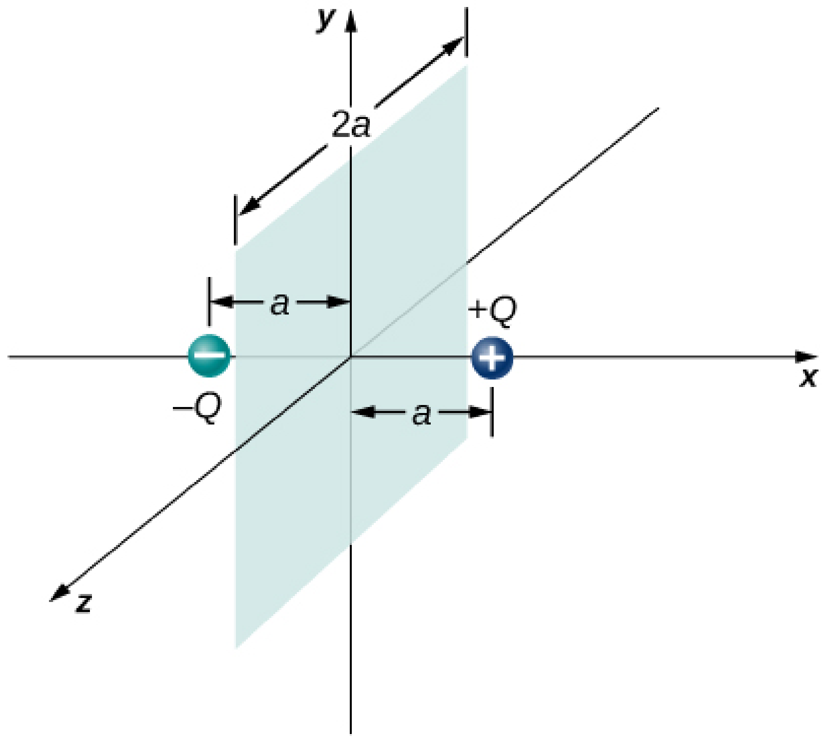
Solution:

$\Phi = \frac{q_{\text{enc}}}{\varepsilon_0}$; There are two contributions to the surface integral: one at the side of the rectangle at $x = 0$ and the other at the side at $x = 2.0\,\text{m}$;
$-E(0)[1.5\,\text{m}^2] + E(2.0\,\text{m})[1.5\,\text{m}^2] = \frac{q_{\text{enc}}}{\varepsilon_0} = -100\,\text{Nm}^2/\text{C}$
where the minus sign indicates that at $x = 0$, the electric field is along positive *x* and the unit normal is along negative *x*. At $x = 2$, the unit normal and the electric field vector are in the same direction:
$q_{\text{enc}} = \varepsilon_0\,\Phi = -8.85\ \times\ 10^{-10}\,\text{C}.$

Exercise:

Problem:

Two equal and opposite charges of magnitude *Q* are located on the *x*-axis at the points +*a* and –*a*, as shown below. What is the net flux due to these charges through a square surface of side 2*a* that lies in the *yz*-plane and is centered at the origin? (*Hint:* Determine the flux due to each charge separately, then use the principle of superposition. You may be able to make a symmetry argument.)



Exercise:

Problem:

A fellow student calculated the flux through the square for the system in the preceding problem and got 0. What went wrong?

Solution:

didn’t keep consistent directions for the area vectors, or the electric fields

Exercise:

Problem:

A $10\,\text{cm} \times 10\,\text{cm}$ piece of aluminum foil of 0.1 mm thickness has a charge of $20\,\mu\text{C}$ that spreads on both wide side surfaces evenly. You may ignore the charges on the thin sides of the edges. (a) Find the charge density. (b) Find the electric field 1 cm from the center, assuming approximate planar symmetry.

Exercise:

Problem:

Two $10 \text{ cm } \times 10 \text{ cm}$ pieces of aluminum foil of thickness 0.1 mm face each other with a separation of 5 mm. One of the foils has a charge of $+30 \ \mu\text{C}$ and the other has $-30 \ \mu\text{C}$. (a) Find the charge density at all surfaces, i.e., on those facing each other and those facing away. (b) Find the electric field between the plates near the center assuming planar symmetry.

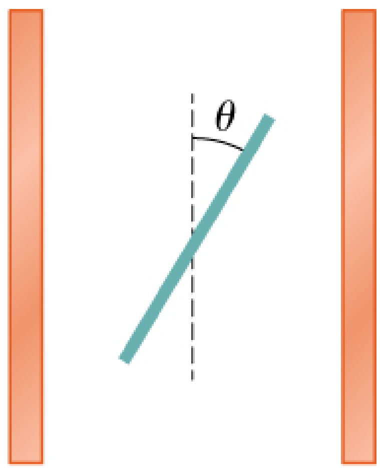
Solution:

a. $\sigma = 3.0 \times 10^{-3} \text{ C/m}^2, +3 \times 10^{-3} \text{ C/m}^2$ on one and $-3 \times 10^{-3} \text{ C/m}^2$ on the other; b. $E = 3.39 \times 10^8 \text{ N/C}$

Exercise:

Problem:

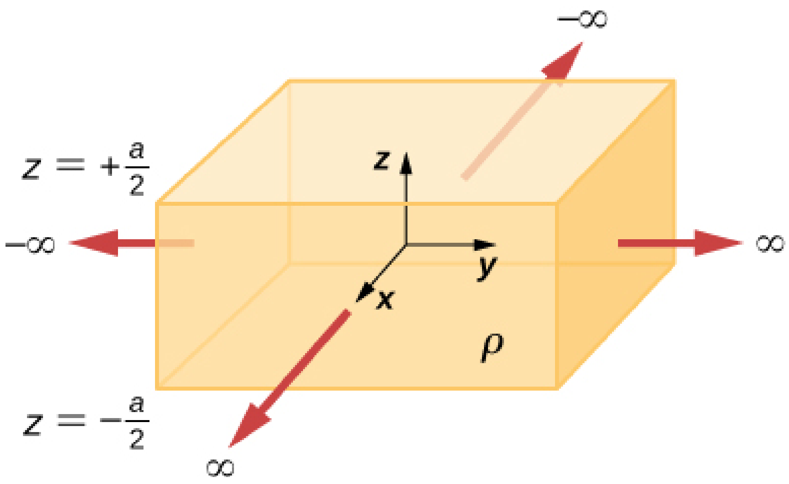
Two large copper plates facing each other have charge densities $\pm 4.0 \text{ C/m}^2$ on the surface facing the other plate, and zero in between the plates. Find the electric flux through a $3 \text{ cm } \times 4 \text{ cm}$ rectangular area between the plates, as shown below, for the following orientations of the area. (a) If the area is parallel to the plates, and (b) if the area is tilted $\theta = 30^\circ$ from the parallel direction. Note, this angle can also be $\theta = 180^\circ + 30^\circ$.



Exercise:

Problem:

The infinite slab between the planes defined by $z = -a/2$ and $z = a/2$ contains a uniform volume charge density ρ (see below). What is the electric field produced by this charge distribution, both inside and outside the distribution?



Solution:

Construct a Gaussian cylinder along the z-axis with cross-sectional area A.
$|z| \geq \frac{a}{2}\ q_{\text{enc}} = \rho A a,\ \Phi = \frac{\rho A a}{\varepsilon_0} \Rightarrow E = \frac{\rho a}{2\varepsilon_0},$
$|z| \leq \frac{a}{2}\ q_{\text{enc}} = \rho A 2z,\ E(2A) = \frac{\rho A 2z}{\varepsilon_0} \Rightarrow E = \frac{\rho z}{\varepsilon_0}$

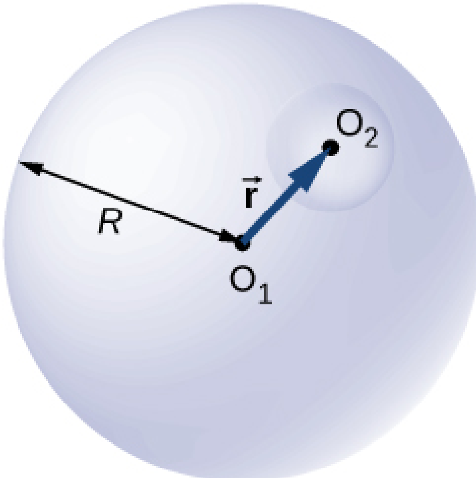
Exercise:

Problem:

A total charge Q is distributed uniformly throughout a spherical volume that is centered at O_1 and has a radius R. Without disturbing the charge remaining, charge is removed from the spherical volume that is centered at O_2 (see below). Show that the electric field everywhere in the empty region is given by

$$\vec{\mathbf{E}} = \frac{Q\vec{\mathbf{r}}}{4\pi\varepsilon_0 R^3},$$

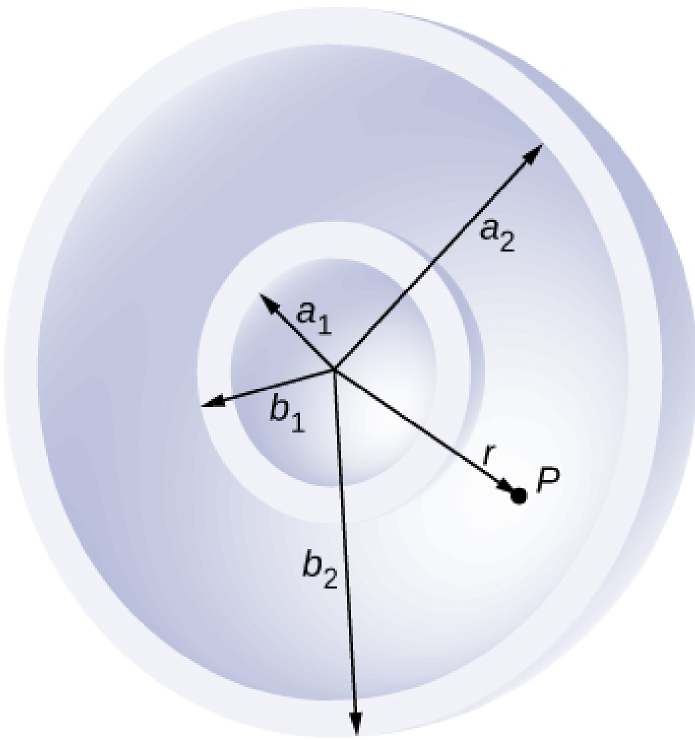
where $\vec{\mathbf{r}}$ is the displacement vector directed from O_1 to O_2.



Exercise:

Problem:

A non-conducting spherical shell of inner radius a_1 and outer radius b_1 is uniformly charged with charged density ρ_1 inside another non-conducting spherical shell of inner radius a_2 and outer radius b_2 that is also uniformly charged with charge density ρ_2. See below. Find the electric field at space point *P* at a distance *r* from the common center such that (a) $r > b_2$, (b) $a_2 < r < b_2$, (c) $b_1 < r < a_2$, (d) $a_1 < r < b_1$, and (e) $r < a_1$.



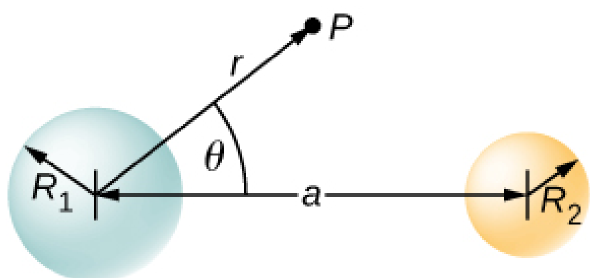
Solution:

a. $r > b_2$ $E4\pi r^2 = \frac{\frac{4}{3}\pi[\rho_1(b_1^3 - a_1^3) + \rho_2(b_2^3 - a_2^3)}{\varepsilon_0} \Rightarrow E = \frac{\rho_1(b_1^3 - a_1^3) + \rho_2(b_2^3 - a_2^3)}{3\varepsilon_0 r^2}$;
b.
$a_2 < r < b_2 \quad E4\pi r^2 = \frac{\frac{4}{3}\pi[\rho_1(b_1^3 - a_1^3) + \rho_2(r^3 - a_2^3)]}{\varepsilon_0} \Rightarrow E = \frac{\rho_1(b_1^3 - a_1^3) + \rho_2(r^3 - a_2^3)}{3\varepsilon_0 r^2}$
;
c. $b_1 < r < a_2 \quad E4\pi r^2 = \frac{\frac{4}{3}\pi\rho_1(b_1^3 - a_1^3)}{\varepsilon_0} \Rightarrow E = \frac{\rho_1(b_1^3 - a_1^3)}{3\varepsilon_0 r^2}$;
d. $a_1 < r < b_1 \quad E4\pi r^2 = \frac{\frac{4}{3}\pi\rho_1(r^3 - a_1^3)}{\varepsilon_0} \Rightarrow E = \frac{\rho_1(r^3 - a_1^3)}{3\varepsilon_0 r^2}$; e. 0

Exercise:

Problem:

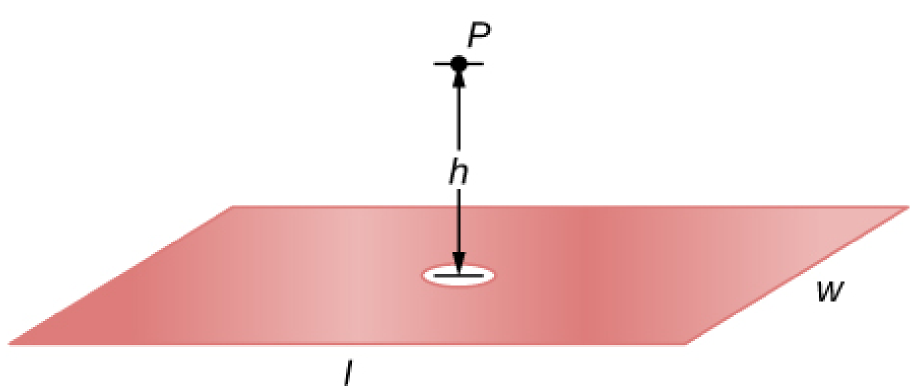
Two non-conducting spheres of radii R_1 and R_2 are uniformly charged with charge densities ρ_1 and ρ_2, respectively. They are separated at center-to-center distance a (see below). Find the electric field at point P located at a distance r from the center of sphere 1 and is in the direction θ from the line joining the two spheres assuming their charge densities are not affected by the presence of the other sphere. (*Hint:* Work one sphere at a time and use the superposition principle.)



Exercise:

Problem:

A disk of radius R is cut in a non-conducting large plate that is uniformly charged with charge density σ (coulomb per square meter). See below. Find the electric field at a height h above the center of the disk. $(h \;>>\; R, h \;<<\; l$ or $w)$. (*Hint:* Fill the hole with $\pm\sigma$.)



Solution:

Electric field due to plate without hole: $E = \frac{\sigma}{2\varepsilon_0}$.

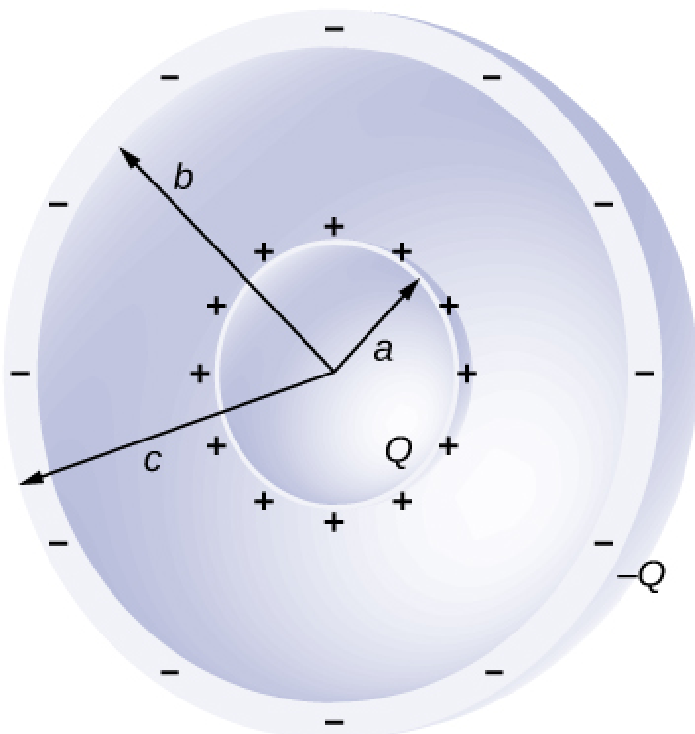
Electric field of just hole filled with $-\sigma$ $E = \frac{-\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$.

Thus, $E_{\mathbf{net}} = \frac{\sigma}{2\varepsilon_0}\,\frac{h}{\sqrt{R^2+h^2}}$.

Exercise:

Problem:

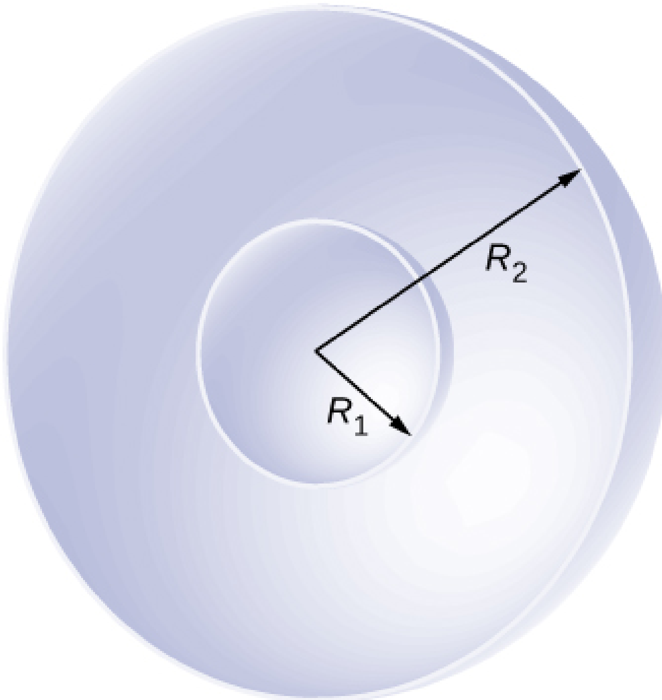
Concentric conducting spherical shells carry charges Q and –Q, respectively (see below). The inner shell has negligible thickness. Determine the electric field for (a) $r < a$; (b) $a < r < b$; (c) $b < r < c$; and (d) $r > c$.



Exercise:

Problem:

Shown below are two concentric conducting spherical shells of radii R_1 and R_2, each of finite thickness much less than either radius. The inner and outer shell carry net charges q_1 and q_2, respectively, where both q_1 and q_2 are positive. What is the electric field for (a) $r < R_1$; (b) $R_1 < r < R_2$; and (c) $r > R_2$? (d) What is the net charge on the inner surface of the inner shell, the outer surface of the inner shell, the inner surface of the outer shell, and the outer surface of the outer shell?



Solution:

a. $E = 0$; b. $E = \frac{q_1}{4\pi\varepsilon_0 r^2}$; c. $E = \frac{q_1+q_2}{4\pi\varepsilon_0 r^2}$; d. $0,\ q_1, -q_1,\ q_1 + q_2$

Exercise:

Problem:

A point charge of $q = 5.0 \times 10^{-8}\,\text{C}$ is placed at the center of an uncharged spherical conducting shell of inner radius 6.0 cm and outer radius 9.0 cm. Find the electric field at (a) $r = 4.0\,\text{cm}$, (b) $r = 8.0\,\text{cm}$, and (c) $r = 12.0\,\text{cm}$. (d) What are the charges induced on the inner and outer surfaces of the shell?

Challenge Problems

Exercise:

Problem:

The Hubble Space Telescope can measure the energy flux from distant objects such as supernovae and stars. Scientists then use this data to calculate the energy emitted by that object. Choose an interstellar object which scientists have observed the flux at the Hubble with (for example, Vega[footnote]), find the distance to that object and the size of Hubble's primary mirror, and calculate the total energy flux. (*Hint:* The Hubble intercepts only a small part of the total flux.)
http://adsabs.harvard.edu/abs/2004AJ....127.3508B

Solution:

Given the referenced link, using a distance to Vega of 237×10^{15} m[footnote] and a diameter of 2.4 m for the primary mirror,[footnote] we find that at a wavelength of 555.6 nm, Vega is emitting $2.44 \times 10^{24}\,\text{J/s}$ at that wavelength. Note that the flux through the mirror is essentially constant.
http://webviz.u-strasbg.fr/viz-bin/VizieR-5?-source=I/311&HIP=91262
http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19910003124.pdf

Exercise:

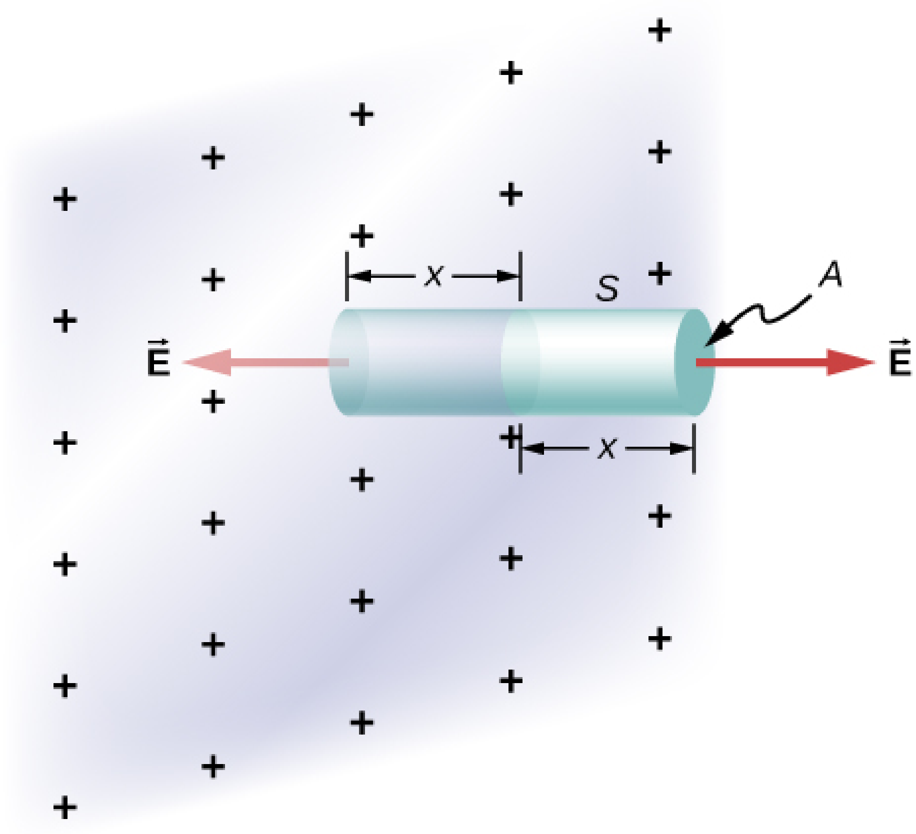
Problem:

Re-derive Gauss's law for the gravitational field, with $\vec{\mathbf{g}}$ directed positively outward.

Exercise:

Problem:

An infinite plate sheet of charge of surface charge density σ is shown below. What is the electric field at a distance x from the sheet? Compare the result of this calculation with that of worked out in the text.



Solution:

The symmetry of the system forces $\overrightarrow{\mathbf{E}}$ to be perpendicular to the sheet and constant over any plane parallel to the sheet. To calculate the

electric field, we choose the cylindrical Gaussian surface shown. The cross-section area and the height of the cylinder are A and $2x$, respectively, and the cylinder is positioned so that it is bisected by the plane sheet. Since E is perpendicular to each end and parallel to the side of the cylinder, we have EA as the flux through each end and there is no flux through the side. The charge enclosed by the cylinder is σA, so from Gauss's law, $2EA = \frac{\sigma A}{\varepsilon_0}$, and the electric field of an infinite sheet of charge is
$E = \frac{\sigma}{2\varepsilon_0}$, in agreement with the calculation of in the text.

Exercise:

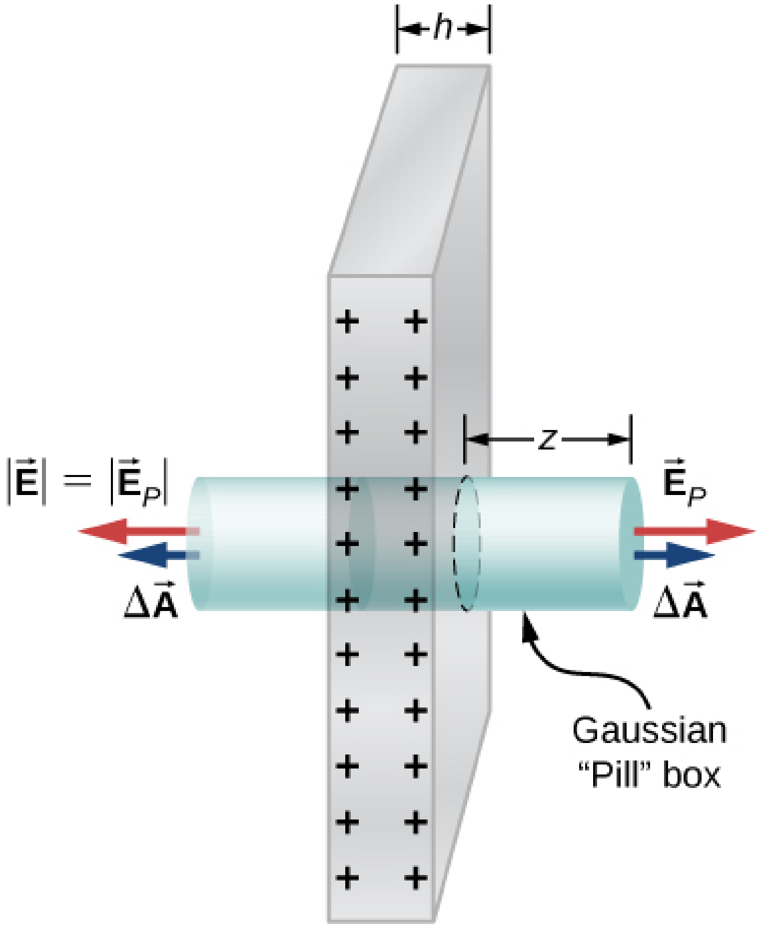
Problem:

A spherical rubber balloon carries a total charge Q distributed uniformly over its surface. At $t = 0$, the radius of the balloon is R. The balloon is then slowly inflated until its radius reaches $2R$ at the time t_0. Determine the electric field due to this charge as a function of time (a) at the surface of the balloon, (b) at the surface of radius R, and (c) at the surface of radius $2R$. Ignore any effect on the electric field due to the material of the balloon and assume that the radius increases uniformly with time.

Exercise:

Problem:

Find the electric field of a large conducting plate containing a net charge q. Let A be area of one side of the plate and h the thickness of the plate (see below). The charge on the metal plate will distribute mostly on the two planar sides and very little on the edges if the plate is thin.



Solution:

There is $Q/2$ on each side of the plate since the net charge is Q: $\sigma = \frac{Q}{2A}$

,

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \frac{2\sigma\Delta A}{\varepsilon_0} \Rightarrow E_P = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 2A}$$

Glossary

free electrons
: also called conduction electrons, these are the electrons in a conductor that are not bound to any particular atom, and hence are free to move around

Introduction class="introduction"

The energy released in a lightning strike is an excellent illustration of the vast quantities of energy that may be stored and released by an electric potential difference. In this chapter, we calculate just how much energy can be released in a lightning strike and how this varies with the height of the clouds from the ground. (credit: modification of work

by Anthony Quintano)

In Electric Charges and Fields, we just scratched the surface (or at least rubbed it) of electrical phenomena. Two terms commonly used to describe electricity are its energy and *voltage*, which we show in this chapter is directly related to the potential energy in a system.

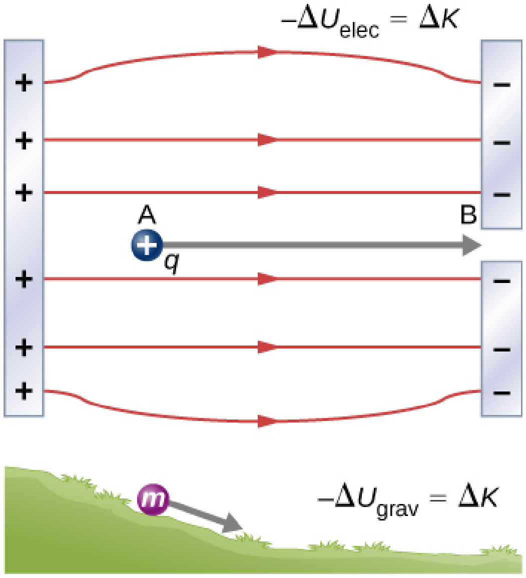
We know, for example, that great amounts of electrical energy can be stored in batteries, are transmitted cross-country via currents through power lines, and may jump from clouds to explode the sap of trees. In a similar manner, at the molecular level, ions cross cell membranes and transfer information.

We also know about voltages associated with electricity. Batteries are typically a few volts, the outlets in your home frequently produce 120 volts, and power lines can be as high as hundreds of thousands of volts. But energy and voltage are not the same thing. A motorcycle battery, for example, is small and would not be very successful in replacing a much larger car battery, yet each has the same voltage. In this chapter, we examine the relationship between voltage and electrical energy, and begin to explore some of the many applications of electricity.

Electric Potential Energy
By the end of this section, you will be able to:

- Define the work done by an electric force
- Define electric potential energy
- Apply work and potential energy in systems with electric charges

When a free positive charge q is accelerated by an electric field, it is given kinetic energy ([link]). The process is analogous to an object being accelerated by a gravitational field, as if the charge were going down an electrical hill where its electric potential energy is converted into kinetic energy, although of course the sources of the forces are very different. Let us explore the work done on a charge q by the electric field in this process, so that we may develop a definition of electric potential energy.

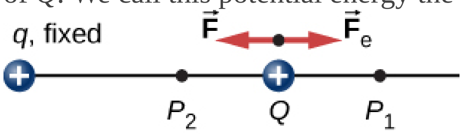


A charge accelerated by an electric field is analogous to a mass going down a hill. In both cases, potential energy decreases as kinetic energy increases, $-\Delta U = \Delta K$. Work is done by a force, but since this force is conservative, we can write $W = -\Delta U$.

The electrostatic or Coulomb force is conservative, which means that the work done on q is independent of the path taken, as we will demonstrate later. This is exactly analogous to the gravitational force. When a force is conservative, it is possible to define a potential energy associated with the force. It is usually easier to work with the potential energy (because it depends only on position) than to calculate the work directly.

To show this explicitly, consider an electric charge $+q$ fixed at the origin and move another charge $+Q$ toward q in such a manner that, at each instant, the applied force $\vec{\mathbf{F}}$ exactly balances the electric force

$\vec{\mathbf{F}}_e$ on Q ([link]). The work done by the applied force $\overrightarrow{\mathbf{F}}$ on the charge Q changes the potential energy of Q. We call this potential energy the **electrical potential energy** of Q.



Displacement of "test" charge Q in the presence of fixed "source" charge q.

The work W_{12} done by the applied force $\overrightarrow{\mathbf{F}}$ when the particle moves from P_1 to P_2 may be calculated by

Equation:

$$W_{12} = \int_{P_1}^{P_2} \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{l}}.$$

Since the applied force $\overrightarrow{\mathbf{F}}$ balances the electric force $\vec{\mathbf{F}}_e$ on Q, the two forces have equal magnitude and opposite directions. Therefore, the applied force is

Equation:

$$\overrightarrow{\mathbf{F}} = -\mathbf{F}_e = -\frac{kqQ}{r^2}\hat{\mathbf{r}},$$

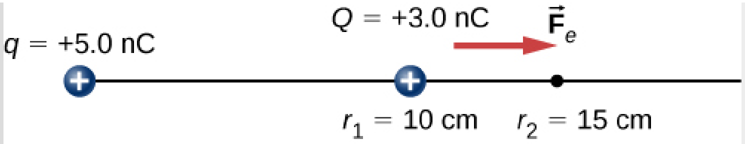
where we have defined positive to be pointing away from the origin and r is the distance from the origin. The directions of both the displacement and the applied force in the system in [link] are parallel, and thus the work done on the system is positive.

We use the letter U to denote electric potential energy, which has units of joules (J). When a conservative force does negative work, the system gains potential energy. When a conservative force does positive work, the system loses potential energy, $\Delta U = -W$. In the system in [link], the Coulomb force acts in the opposite direction to the displacement; therefore, the work is negative. However, we have increased the potential energy in the two-charge system.

Example:

Kinetic Energy of a Charged Particle

A $+3.0\text{-nC}$ charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0\text{-nC}$ charge q fixed at the origin ([link]). Naturally, the Coulomb force accelerates Q away from q, eventually reaching 15 cm (r_2).



The charge Q is repelled by q, thus having work done on it and gaining kinetic energy.

a. What is the work done by the electric field between r_1 and r_2?
b. How much kinetic energy does Q have at r_2?

Strategy
Calculate the work with the usual definition. Since Q started from rest, this is the same as the kinetic energy.
Solution
Integrating force over distance, we obtain
Equation:

$$\begin{aligned} W_{12} &= \int_{r_1}^{r_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{r_1}^{r_2} \frac{kqQ}{r^2}\, dr = \left[-\frac{kqQ}{r}\right]_{r_1}^{r_2} = kqQ\left[\frac{-1}{r_2} + \frac{1}{r_1}\right] \\ &= \left(8.99 \times 10^9\, \mathrm{Nm^2/C^2}\right)\left(5.0 \times 10^{-9}\, \mathrm{C}\right)\left(3.0 \times 10^{-9}\, \mathrm{C}\right)\left[\frac{-1}{0.15\, \mathrm{m}} + \frac{1}{0.10\, \mathrm{m}}\right] \\ &= 4.5 \times 10^{-7}\, \mathrm{J}. \end{aligned}$$

This is also the value of the kinetic energy at r_2.
Significance
Charge Q was initially at rest; the electric field of q did work on Q, so now Q has kinetic energy equal to the work done by the electric field.

Note:
Exercise:

Problem: Check Your Understanding If Q has a mass of 4.00 μg, what is the speed of Q at r_2?

Solution:

$$K = \frac{1}{2}mv^2,\ v = \sqrt{2\frac{K}{m}} = \sqrt{2\frac{4.5 \times 10^{-7}\, \mathrm{J}}{4.00 \times 10^{-9}\, \mathrm{kg}}} = 15\, \mathrm{m/s}$$

In this example, the work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. A value for U can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Note:
Electric Potential Energy
Work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. Mathematically,
Equation:

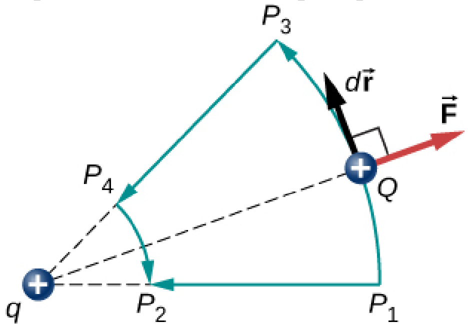
$$W = -\Delta U.$$

Gravitational potential energy and electric potential energy are quite analogous. Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of electric potential energy than to deal with the Coulomb force directly in real-world applications.

In polar coordinates with q at the origin and Q located at r, the displacement element vector is $d\vec{\mathbf{l}} = \hat{\mathbf{r}}\, dr$ and thus the work becomes
Equation:

$$W_{12} = kqQ \int_{r_1}^{r_2} \frac{1}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr = kqQ \frac{1}{r_2} - kqQ \frac{1}{r_1}.$$

Notice that this result only depends on the endpoints and is otherwise independent of the path taken. To explore this further, compare path P_1 to P_2 with path $P_1P_3P_4P_2$ in [link].



Two paths for displacement P_1 to P_2. The work on segments P_1P_3 and P_4P_2 are zero due to the electrical force being perpendicular to the displacement along these paths. Therefore, work on paths P_1P_2 and $P_1P_3P_4P_2$ are equal.

The segments P_1P_3 and P_4P_2 are arcs of circles centered at q. Since the force on Q points either toward or away from q, no work is done by a force balancing the electric force, because it is

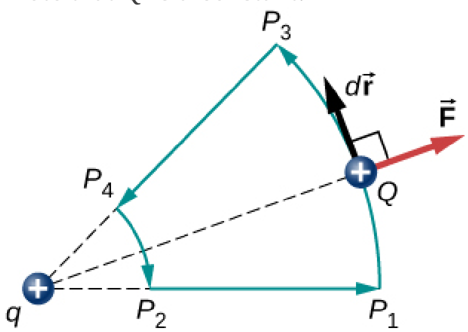
perpendicular to the displacement along these arcs. Therefore, the only work done is along segment P_3P_4, which is identical to P_1P_2.

One implication of this work calculation is that if we were to go around the path $P_1P_3P_4P_2P_1$, the net work would be zero ([link]). Recall that this is how we determine whether a force is conservative or not. Hence, because the electric force is related to the electric field by $\overrightarrow{\mathbf{F}} = q\overrightarrow{\mathbf{E}}$, the electric field is itself conservative. That is,

Equation:

$$\oint \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0.$$

Note that Q is a constant.



A closed path in an electric field. The net work around this path is zero.

Another implication is that we may define an electric potential energy. Recall that the work done by a conservative force is also expressed as the difference in the potential energy corresponding to that force. Therefore, the work W_{ref} to bring a charge from a reference point to a point of interest may be written as

Equation:

$$W_{\text{ref}} = \int_{r_{\text{ref}}}^{r} \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

and, by [link], the difference in potential energy $(U_2 - U_1)$ of the test charge Q between the two points is

Equation:

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{l}}.$$

Therefore, we can write a general expression for the potential energy of two point charges (in spherical coordinates):

Equation:

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \frac{kqQ}{r^2}dr = -\left[-\frac{kqQ}{r}\right]_{r_{\text{ref}}}^{r} = kqQ\left[\frac{1}{r} - \frac{1}{r_{\text{ref}}}\right].$$

We may take the second term to be an arbitrary constant reference level, which serves as the zero reference:

Equation:

$$U(r) = k\frac{qQ}{r} - U_{\text{ref}}.$$

A convenient choice of reference that relies on our common sense is that when the two charges are infinitely far apart, there is no interaction between them. (Recall the discussion of reference potential energy in Potential Energy and Conservation of Energy.) Taking the potential energy of this state to be zero removes the term U_{ref} from the equation (just like when we say the ground is zero potential energy in a gravitational potential energy problem), and the potential energy of Q when it is separated from q by a distance r assumes the form

Note:

Equation:

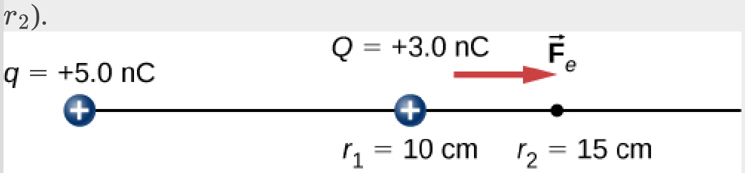
$$U(r) = k\frac{qQ}{r} \text{ (zero reference at } r = \infty).$$

This formula is symmetrical with respect to q and Q, so it is best described as the potential energy of the two-charge system.

Example:

Potential Energy of a Charged Particle

A $+3.0\text{-nC}$ charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0\text{-nC}$ charge q fixed at the origin ([link]). Naturally, the Coulomb force accelerates Q away from q, eventually reaching 15 cm (r_2).



The charge Q is repelled by q, thus having work done on it and losing potential energy.

What is the change in the potential energy of the two-charge system from r_1 to r_2?
Strategy
Calculate the potential energy with the definition given above: $\Delta U_{12} = -\int_{r_1}^{r_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Since Q started from rest, this is the same as the kinetic energy.
Solution
We have
Equation:

$$\begin{aligned}\Delta U_{12} &= -\int_{r_1}^{r_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{r_1}^{r_2} \frac{kqQ}{r^2} dr = -\left[-\frac{kqQ}{r}\right]_{r_1}^{r_2} = kqQ\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \\ &= \left(8.99 \times 10^9 \, \mathrm{Nm}^2/\mathrm{C}^2\right)\left(5.0 \times 10^{-9} \, \mathrm{C}\right)\left(3.0 \times 10^{-9} \, \mathrm{C}\right)\left[\frac{1}{0.15 \, \mathrm{m}} - \frac{1}{0.10 \, \mathrm{m}}\right] \\ &= -4.5 \times 10^{-7} \, \mathrm{J}.\end{aligned}$$

Significance
The change in the potential energy is negative, as expected, and equal in magnitude to the change in kinetic energy in this system. Recall from [link] that the change in kinetic energy was positive.

Note:
Exercise:

Problem:

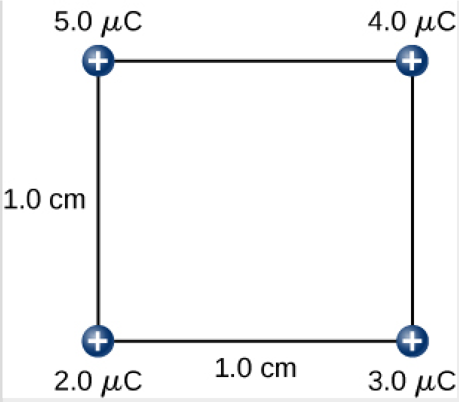
Check Your Understanding What is the potential energy of Q relative to the zero reference at infinity at r_2 in the above example?

Solution:

It has kinetic energy of 4.5×10^{-7} J at point r_2 and potential energy of 9.0×10^{-7} J, which means that as Q approaches infinity, its kinetic energy totals three times the kinetic energy at r_2, since all of the potential energy gets converted to kinetic.

Due to Coulomb's law, the forces due to multiple charges on a test charge Q superimpose; they may be calculated individually and then added. This implies that the work integrals and hence the resulting potential energies exhibit the same behavior. To demonstrate this, we consider an example of assembling a system of four charges.

Example:
Assembling Four Positive Charges
Find the amount of work an external agent must do in assembling four charges $+2.0 \, \mu\mathrm{C}$, $+3.0 \, \mu\mathrm{C}$, $+4.0 \, \mu\mathrm{C}$, and $+5.0 \, \mu\mathrm{C}$ at the vertices of a square of side 1.0 cm, starting each charge from infinity ([link]).



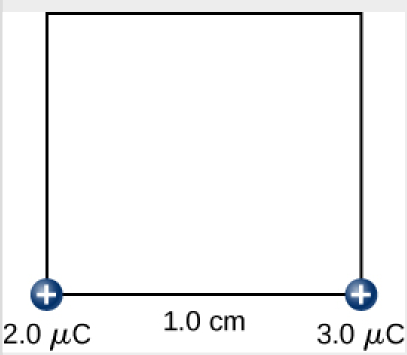
How much work is needed to assemble this charge configuration?

Strategy
We bring in the charges one at a time, giving them starting locations at infinity and calculating the work to bring them in from infinity to their final location. We do this in order of increasing charge.
Solution
Step 1. First bring the $+2.0\text{-}\mu\text{C}$ charge to the origin. Since there are no other charges at a finite distance from this charge yet, no work is done in bringing it from infinity,
Equation:

$$W_1 = 0.$$

Step 2. While keeping the $+2.0\text{-}\mu\text{C}$ charge fixed at the origin, bring the $+3.0\text{-}\mu\text{C}$ charge to $(x, y, z) = (1.0\ \text{cm}, 0, 0)$ ([link]). Now, the applied force must do work against the force exerted by the $+2.0\text{-}\mu\text{C}$ charge fixed at the origin. The work done equals the change in the potential energy of the $+3.0\text{-}\mu\text{C}$ charge:
Equation:

$$W_2 = k\frac{q_1 q_2}{r_{12}} = \left(9.0 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}\right) \frac{\left(2.0 \times 10^{-6}\ \text{C}\right)\left(3.0 \times 10^{-6}\ \text{C}\right)}{1.0 \times 10^{-2}\ \text{m}} = 5.4\ \text{J}.$$



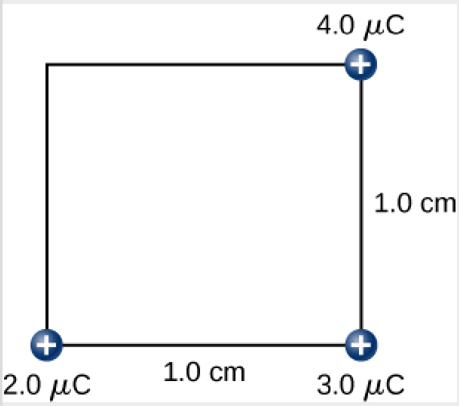
Step 2. Work W_2 to bring the $+3.0\text{-}\mu\text{C}$ charge from

infinity.

Step 3. While keeping the charges of $+2.0\ \mu\text{C}$ and $+3.0\ \mu\text{C}$ fixed in their places, bring in the $+4.0\text{-}\mu\text{C}$ charge to $(x, y, z) = (1.0\text{ cm}, 1.0\text{ cm}, 0)$ ([link]). The work done in this step is

Equation:

$$\begin{aligned} W_3 &= k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}} \\ &= \left(9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(2.0 \times 10^{-6}\text{ C})(4.0 \times 10^{-6}\text{ C})}{\sqrt{2} \times 10^{-2}\text{ m}} + \frac{(3.0 \times 10^{-6}\text{ C})(4.0 \times 10^{-6}\text{ C})}{1.0 \times 10^{-2}\text{ m}}\right] = 15.9\text{ J.} \end{aligned}$$

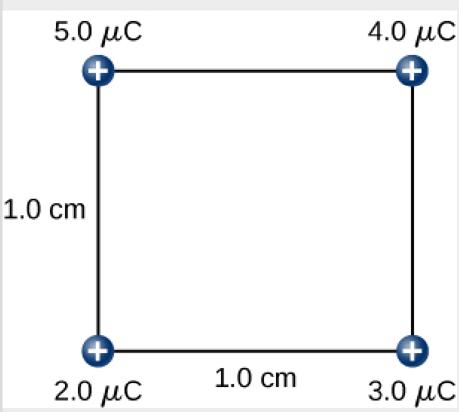


Step 3. The work W_3 to bring the $+4.0\text{-}\mu\text{C}$ charge from infinity.

Step 4. Finally, while keeping the first three charges in their places, bring the $+5.0\text{-}\mu\text{C}$ charge to $(x, y, z) = (0,\ 1.0\text{ cm},\ 0)$ ([link]). The work done here is

Equation:

$$\begin{aligned} W_4 &= kq_4\left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right], \\ &= \left(9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(5.0 \times 10^{-6}\text{ C})\left[\frac{(2.0 \times 10^{-6}\text{ C})}{1.0 \times 10^{-2}\text{ m}} + \frac{(3.0 \times 10^{-6}\text{ C})}{\sqrt{2} \times 10^{-2}\text{ m}} + \frac{(4.0 \times 10^{-6}\text{ C})}{1.0 \times 10^{-2}\text{ m}}\right] = 36.5\text{ J.} \end{aligned}$$



Step 4. The work W_4 to bring the $+5.0\text{-}\mu\text{C}$ charge from infinity.

Hence, the total work done by the applied force in assembling the four charges is equal to the sum of the work in bringing each charge from infinity to its final position:

Equation:

$$W_{\text{T}} = W_1 + W_2 + W_3 + W_4 = 0 + 5.4\,\text{J} + 15.9\,\text{J} + 36.5\,\text{J} = 57.8\,\text{J}.$$

Significance

The work on each charge depends only on its pairwise interactions with the other charges. No more complicated interactions need to be considered; the work on the third charge only depends on its interaction with the first and second charges, the interaction between the first and second charge does not affect the third.

Note:

Exercise:

Problem:

Check Your Understanding Is the electrical potential energy of two point charges positive or negative if the charges are of the same sign? Opposite signs? How does this relate to the work necessary to bring the charges into proximity from infinity?

Solution:

positive, negative, and these quantities are the same as the work you would need to do to bring the charges in from infinity

Note that the electrical potential energy is positive if the two charges are of the same type, either positive or negative, and negative if the two charges are of opposite types. This makes sense if you think of the change in the potential energy ΔU as you bring the two charges closer or move them farther apart. Depending on the relative types of charges, you may have to work on the system or the system would do work on you, that is, your work is either positive or negative. If you have to do positive work on the system (actually push the charges closer), then the energy of the system should increase. If you bring two positive charges or two negative charges closer, you have to do positive work on the system, which raises their potential energy. Since potential energy is proportional to $1/r$, the potential energy goes up when r goes down between two positive or two negative charges.

On the other hand, if you bring a positive and a negative charge nearer, you have to do negative work on the system (the charges are pulling you), which means that you take energy away from the system. This reduces the potential energy. Since potential energy is negative in the case of a positive and a negative charge pair, the increase in $1/r$ makes the potential energy more negative, which is the same as a reduction in potential energy.

The result from [link] may be extended to systems with any arbitrary number of charges. In this case, it is most convenient to write the formula as

Note:
Equation:

$$W_{12\cdots N}=\frac{k}{2}\sum_{i}^{N}\sum_{j}^{N}\frac{q_i q_j}{r_{ij}}\text{ for }i\neq j.$$

The factor of 1/2 accounts for adding each pair of charges twice.

Summary

- The work done to move a charge from point A to B in an electric field is path independent, and the work around a closed path is zero. Therefore, the electric field and electric force are conservative.
- We can define an electric potential energy, which between point charges is $U(r)=k\frac{qQ}{r}$, with the zero reference taken to be at infinity.
- The superposition principle holds for electric potential energy; the potential energy of a system of multiple charges is the sum of the potential energies of the individual pairs.

Conceptual Questions

Exercise:

Problem:

Would electric potential energy be meaningful if the electric field were not conservative?

Solution:

No. We can only define potential energies for conservative fields.

Exercise:

Problem:

Why do we need to be careful about work done *on* the system versus work done *by* the system in calculations?

Exercise:

Problem:

Does the order in which we assemble a system of point charges affect the total work done?

Solution:

No, though certain orderings may be simpler to compute.

Problems

Exercise:

Problem:

Consider a charge $Q_1 (+5.0\ \mu\text{C})$ fixed at a site with another charge Q_2 (charge $+3.0\ \mu$C, mass $6.0\ \mu\text{g}$) moving in the neighboring space. (a) Evaluate the potential energy of Q_2 when it is 4.0 cm from Q_1. (b) If Q_2 starts from rest from a point 4.0 cm from Q_1, what will be its speed when it is 8.0 cm from Q_1? (*Note:* Q_1 is held fixed in its place.)

Solution:

a. $U = 3.4$ J;
b. $\frac{1}{2}mv^2 = Q_1 Q_2 \left(\frac{1}{r_i} - \frac{1}{r_f}\right) \to v = 2.4 \text{ x } 10^4 \text{ m/s}$

Exercise:

Problem:

Two charges $Q_1 (+2.00\ \mu\text{C})$ and $Q_2 (+2.00\ \mu\text{C})$ are placed symmetrically along the *x*-axis at $x = \pm 3.00$ cm. Consider a charge Q_3 of charge $+4.00\ \mu$C and mass 10.0 mg moving along the *y*-axis. If Q_3 starts from rest at $y = 2.00$ cm, what is its speed when it reaches $y = 4.00$ cm?

Exercise:

Problem:

To form a hydrogen atom, a proton is fixed at a point and an electron is brought from far away to a distance of 0.529×10^{-10} m, the average distance between proton and electron in a hydrogen atom. How much work is done?

Solution:

$U = 4.36 \times 10^{-18}$ J

Exercise:

Problem:

(a) What is the average power output of a heart defibrillator that dissipates 400 J of energy in 10.0 ms? (b) Considering the high-power output, why doesn’t the defibrillator produce serious burns?

Glossary

electric potential energy
: potential energy stored in a system of charged objects due to the charges

Electric Potential and Potential Difference

By the end of this section, you will be able to:

- Define electric potential, voltage, and potential difference
- Define the electron-volt
- Calculate electric potential and potential difference from potential energy and electric field
- Describe systems in which the electron-volt is a useful unit
- Apply conservation of energy to electric systems

Recall that earlier we defined electric field to be a quantity independent of the test charge in a given system, which would nonetheless allow us to calculate the force that would result on an arbitrary test charge. (The default assumption in the absence of other information is that the test charge is positive.) We briefly defined a field for gravity, but gravity is always attractive, whereas the electric force can be either attractive or repulsive. Therefore, although potential energy is perfectly adequate in a gravitational system, it is convenient to define a quantity that allows us to calculate the work on a charge independent of the magnitude of the charge. Calculating the work directly may be difficult, since $W = \overrightarrow{\mathbf{F}} \cdot \vec{\mathbf{d}}$ and the direction and magnitude of $\overrightarrow{\mathbf{F}}$ can be complex for multiple charges, for odd-shaped objects, and along arbitrary paths. But we do know that because $\overrightarrow{\mathbf{F}} = q\overrightarrow{\mathbf{E}}$, the work, and hence ΔU, is proportional to the test charge q. To have a physical quantity that is independent of test charge, we define **electric potential** V (or simply potential, since electric is understood) to be the potential energy per unit charge:

Note:

Electric Potential

The electric potential energy per unit charge is

Equation:

$$V = \frac{U}{q}.$$

Since U is proportional to q, the dependence on q cancels. Thus, V does not depend on q. The change in potential energy ΔU is crucial, so we are concerned with the difference in potential or potential difference ΔV between two points, where

Equation:

$$\Delta V = V_B - V_A = \frac{\Delta U}{q}.$$

Note:
Electric Potential Difference
The **electric potential difference** between points A and B, $V_B - V_A$, is defined to be the change in potential energy of a charge q moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.
Equation:

$$1\ \text{V} = 1\ \text{J/C}$$

The familiar term **voltage** is the common name for electric potential difference. Keep in mind that whenever a voltage is quoted, it is understood to be the potential difference between two points. For example, every battery has two terminals, and its voltage is the potential difference between them. More fundamentally, the point you choose to be zero volts is arbitrary. This is analogous to the fact that gravitational potential energy has an arbitrary zero, such as sea level or perhaps a lecture hall floor. It is worthwhile to emphasize the distinction between potential difference and electrical potential energy.

Note:
Potential Difference and Electrical Potential Energy
The relationship between potential difference (or voltage) and electrical potential energy is given by
Equation:

$$\Delta V = \frac{\Delta U}{q} \text{ or } \Delta U = q\Delta V.$$

Voltage is not the same as energy. Voltage is the energy per unit charge. Thus, a motorcycle battery and a car battery can both have the same voltage (more precisely, the same potential difference between battery terminals), yet one stores much more energy than the other because $\Delta U = q\Delta V$. The car battery can move more charge than the motorcycle battery, although both are 12-V batteries.

Example:
Calculating Energy
You have a 12.0-V motorcycle battery that can move 5000 C of charge, and a 12.0-V car battery that can move 60,000 C of charge. How much energy does each deliver? (Assume that the numerical value of each charge is accurate to three significant figures.)
Strategy
To say we have a 12.0-V battery means that its terminals have a 12.0-V potential difference. When such a battery moves charge, it puts the charge through a potential difference of 12.0 V, and the charge is given a change in potential energy equal to $\Delta U = q\Delta V$. To find the energy output, we multiply the charge moved by the potential difference.
Solution
For the motorcycle battery, $q = 5000 \text{ C}$ and $\Delta V = 12.0 \text{ V}$. The total energy delivered by the motorcycle battery is
Equation:

$$\Delta U_{\text{cycle}} = (5000 \text{ C})(12.0 \text{ V}) = (5000 \text{ C})(12.0 \text{ J/C}) = 6.00 \times 10^4 \text{ J}.$$

Similarly, for the car battery, $q = 60{,}000 \text{ C}$ and
Equation:

$$\Delta U_{\text{car}} = (60{,}000 \text{ C})(12.0 \text{ V}) = 7.20 \times 10^5 \text{ J}.$$

Significance
Voltage and energy are related, but they are not the same thing. The voltages of the batteries are identical, but the energy supplied by each is quite different. A car battery has a much larger engine to start than a motorcycle. Note also that as a battery is discharged, some of its energy is used internally and its terminal voltage drops, such as when headlights dim because of a depleted car battery. The energy supplied by the battery is still calculated as in this example, but not all of the energy is available for external use.

Note:
Exercise:

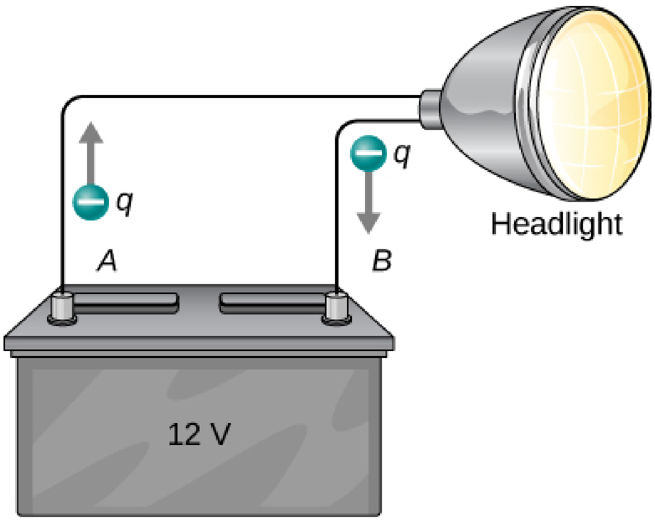
Problem:

Check Your Understanding How much energy does a 1.5-V AAA battery have that can move 100 C?

Solution:

$$\Delta U = q\Delta V = (100 \text{ C})(1.5 \text{ V}) = 150 \text{ J}$$

Note that the energies calculated in the previous example are absolute values. The change in potential energy for the battery is negative, since it loses energy. These batteries, like many electrical systems, actually move negative charge—electrons in particular. The batteries repel electrons from their negative terminals (A) through whatever circuitry is involved and attract them to their positive terminals (B), as shown in [link]. The change in potential is $\Delta V = V_B - V_A = +12\ \text{V}$ and the charge q is negative, so that $\Delta U = q\Delta V$ is negative, meaning the potential energy of the battery has decreased when q has moved from A to B.



A battery moves negative charge from its negative terminal through a headlight to its positive terminal. Appropriate combinations of chemicals in the battery separate charges so that the negative terminal has an excess of negative charge, which is repelled by it and attracted to the excess positive charge on the other terminal. In terms of potential, the positive terminal is at a higher voltage than the negative terminal. Inside the battery, both positive and negative charges move.

Example:
How Many Electrons Move through a Headlight Each Second?
When a 12.0-V car battery powers a single 30.0-W headlight, how many electrons pass through it each second?
Strategy

To find the number of electrons, we must first find the charge that moves in 1.00 s. The charge moved is related to voltage and energy through the equations $\Delta U = q\Delta V$. A 30.0-W lamp uses 30.0 joules per second. Since the battery loses energy, we have $\Delta U = -30\text{ J}$ and, since the electrons are going from the negative terminal to the positive, we see that $\Delta V = +12.0\text{ V}$.

Solution

To find the charge q moved, we solve the equation $\Delta U = q\Delta V$:

Equation:

$$q = \frac{\Delta U}{\Delta V}.$$

Entering the values for ΔU and ΔV, we get

Equation:

$$q = \frac{-30.0\text{ J}}{+12.0\text{ V}} = \frac{-30.0\text{ J}}{+12.0\text{ J/C}} = -2.50\text{ C}.$$

The number of electrons n_e is the total charge divided by the charge per electron. That is,

Equation:

$$n_e = \frac{-2.50\text{ C}}{-1.60 \times 10^{-19}\text{ C/e}^-} = 1.56 \times 10^{19}\text{ electrons.}$$

Significance

This is a very large number. It is no wonder that we do not ordinarily observe individual electrons with so many being present in ordinary systems. In fact, electricity had been in use for many decades before it was determined that the moving charges in many circumstances were negative. Positive charge moving in the opposite direction of negative charge often produces identical effects; this makes it difficult to determine which is moving or whether both are moving.

Note:

Exercise:

Problem:

Check Your Understanding How many electrons would go through a 24.0-W lamp?

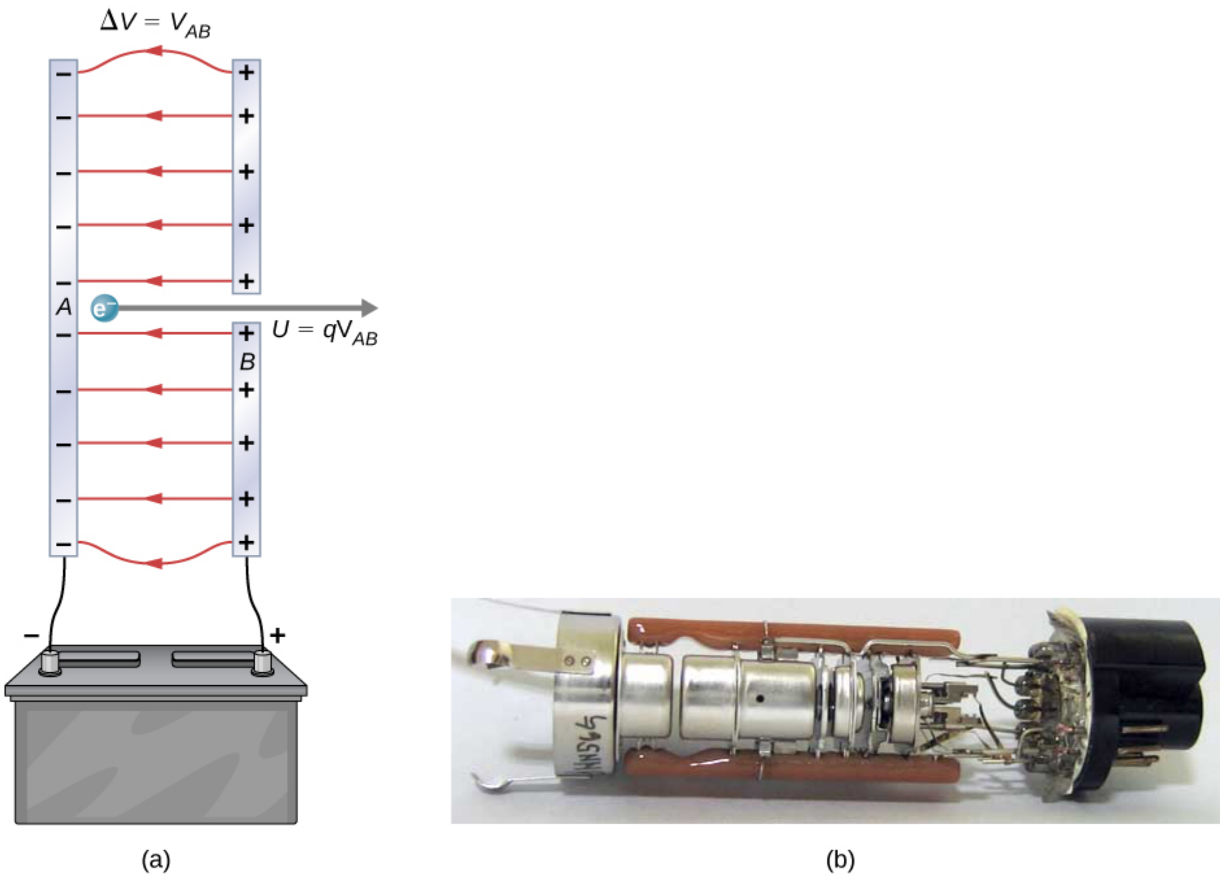
Solution:

–2.00 C, $n_e = 1.25 \times 10^{19}$ electrons

The Electron-Volt

The energy per electron is very small in macroscopic situations like that in the previous example—a tiny fraction of a joule. But on a submicroscopic scale, such energy per particle (electron, proton, or ion) can be of great importance. For example, even a tiny fraction of a joule can be great enough for these particles to destroy organic molecules and harm living tissue. The particle may do its damage by direct collision, or it may create harmful X-rays, which can also inflict damage. It is useful to have an energy unit related to submicroscopic effects.

[link] shows a situation related to the definition of such an energy unit. An electron is accelerated between two charged metal plates, as it might be in an old-model television tube or oscilloscope. The electron gains kinetic energy that is later converted into another form—light in the television tube, for example. (Note that in terms of energy, "downhill" for the electron is "uphill" for a positive charge.) Since energy is related to voltage by $\Delta U = q\Delta V$, we can think of the joule as a coulomb-volt.



A typical electron gun accelerates electrons using a potential difference between two separated metal plates. By conservation of energy, the kinetic energy has to equal the

change in potential energy, so $KE = qV$. The energy of the electron in electron-volts is numerically the same as the voltage between the plates. For example, a 5000-V potential difference produces 5000-eV electrons. The conceptual construct, namely two parallel plates with a hole in one, is shown in (a), while a real electron gun is shown in (b).

Note:
Electron-Volt
On the submicroscopic scale, it is more convenient to define an energy unit called the **electron-volt** (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,
Equation:

$$1\ \text{eV} = (1.60\ \times\ 10^{-19}\ \text{C})(1\ \text{V}) = (1.60\ \times\ 10^{-19}\ \text{C})(1\ \text{J/C}) = 1.60\ \times\ 10^{-19}\ \text{J}.$$

An electron accelerated through a potential difference of 1 V is given an energy of 1 eV. It follows that an electron accelerated through 50 V gains 50 eV. A potential difference of 100,000 V (100 kV) gives an electron an energy of 100,000 eV (100 keV), and so on. Similarly, an ion with a double positive charge accelerated through 100 V gains 200 eV of energy. These simple relationships between accelerating voltage and particle charges make the electron-volt a simple and convenient energy unit in such circumstances.

The electron-volt is commonly employed in submicroscopic processes—chemical valence energies and molecular and nuclear binding energies are among the quantities often expressed in electron-volts. For example, about 5 eV of energy is required to break up certain organic molecules. If a proton is accelerated from rest through a potential difference of 30 kV, it acquires an energy of 30 keV (30,000 eV) and can break up as many as 6000 of these molecules $(30{,}000\ \text{eV} \div 5\ \text{eV per molecule} = 6000\ \text{molecules})$. Nuclear decay energies are on the order of 1 MeV (1,000,000 eV) per event and can thus produce significant biological damage.

Conservation of Energy

The total energy of a system is conserved if there is no net addition (or subtraction) due to work or heat transfer. For conservative forces, such as the electrostatic force, conservation of energy states that mechanical energy is a constant.

Mechanical energy is the sum of the kinetic energy and potential energy of a system; that is, $K + U = \text{constant}$. A loss of U for a charged particle becomes an increase in its K. Conservation of energy is stated in equation form as

Equation:

$$K + U = \text{constant}$$

or

Equation:

$$K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}$$

where i and f stand for initial and final conditions. As we have found many times before, considering energy can give us insights and facilitate problem solving.

Example:

Electrical Potential Energy Converted into Kinetic Energy

Calculate the final speed of a free electron accelerated from rest through a potential difference of 100 V. (Assume that this numerical value is accurate to three significant figures.)

Strategy

We have a system with only conservative forces. Assuming the electron is accelerated in a vacuum, and neglecting the gravitational force (we will check on this assumption later), all of the electrical potential energy is converted into kinetic energy. We can identify the initial and final forms of energy to be $K_\text{i} = 0, K_\text{f} = \frac{1}{2}mv^2, U_\text{i} = qV, U_\text{f} = 0$.

Solution

Conservation of energy states that

Equation:

$$K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}.$$

Entering the forms identified above, we obtain

Equation:

$$qV = \frac{mv^2}{2}.$$

We solve this for *v*:

Equation:

$$v = \sqrt{\frac{2qV}{m}}.$$

Entering values for q, V, and m gives
Equation:

$$v = \sqrt{\frac{2(-1.60 \times 10^{-19}\,\text{C})(-100\,\text{J/C})}{9.11 \times 10^{-31}\,\text{kg}}} = 5.93 \times 10^{6}\,\text{m/s}.$$

Significance
Note that both the charge and the initial voltage are negative, as in [link]. From the discussion of electric charge and electric field, we know that electrostatic forces on small particles are generally very large compared with the gravitational force. The large final speed confirms that the gravitational force is indeed negligible here. The large speed also indicates how easy it is to accelerate electrons with small voltages because of their very small mass. Voltages much higher than the 100 V in this problem are typically used in electron guns. These higher voltages produce electron speeds so great that effects from special relativity must be taken into account and hence are reserved for a later chapter (Relativity). That is why we consider a low voltage (accurately) in this example.

Note:
Exercise:

Problem:

Check Your Understanding How would this example change with a positron? A positron is identical to an electron except the charge is positive.

Solution:

It would be going in the opposite direction, with no effect on the calculations as presented.

Voltage and Electric Field

So far, we have explored the relationship between voltage and energy. Now we want to explore the relationship between voltage and electric field. We will start with the general case for a non-uniform $\vec{\mathbf{E}}$ field. Recall that our general formula for the potential energy of a test charge q at point P relative to reference point R is
Equation:

$$U_P = -\int_R^P \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}.$$

When we substitute in the definition of electric field $(\vec{\mathbf{E}} = \vec{\mathbf{F}}/q)$, this becomes
Equation:

$$U_P = -q \int_R^P \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}.$$

Applying our definition of potential $(V = U/q)$ to this potential energy, we find that, in general,

Note:
Equation:

$$V_P = -\int_R^P \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}.$$

From our previous discussion of the potential energy of a charge in an electric field, the result is independent of the path chosen, and hence we can pick the integral path that is most convenient.

Consider the special case of a positive point charge q at the origin. To calculate the potential caused by q at a distance r from the origin relative to a reference of 0 at infinity (recall that we did the same for potential energy), let $P = r$ and $R = \infty$, with $d\vec{\mathbf{l}} = d\vec{\mathbf{r}} = \hat{\mathbf{r}}dr$ and use $\overrightarrow{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. When we evaluate the integral
Equation:

$$V_P = -\int_R^P \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

for this system, we have
Equation:

$$V_r = -\int_\infty^r \frac{kq}{r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}dr,$$

which simplifies to

Equation:

$$V_r = -\int_{\infty}^{r} \frac{kq}{r^2}\,dr = \frac{kq}{r} - \frac{kq}{\infty} = \frac{kq}{r}.$$

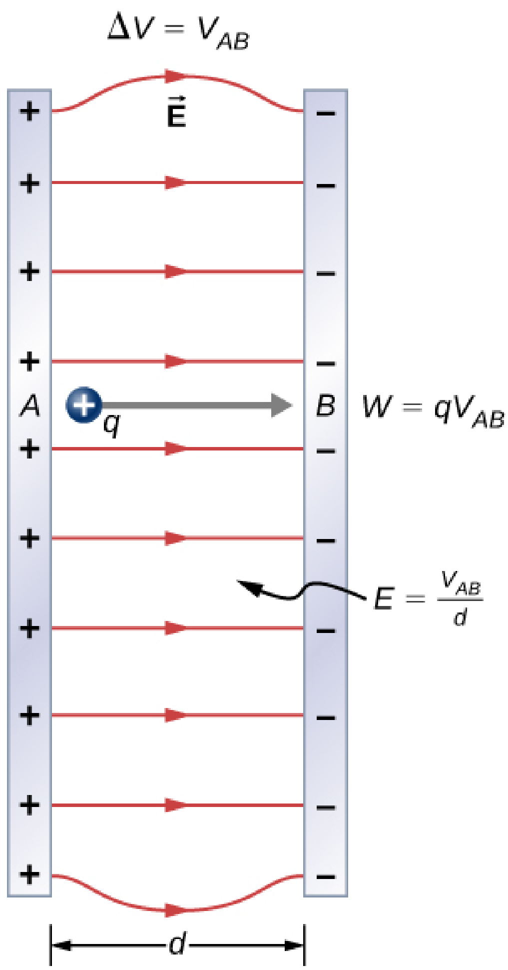
This result,

Equation:

$$V_r = \frac{kq}{r}$$

is the standard form of the potential of a point charge. This will be explored further in the next section.

To examine another interesting special case, suppose a uniform electric field $\vec{\mathbf{E}}$ is produced by placing a potential difference (or voltage) ΔV across two parallel metal plates, labeled *A* and *B* ([link]). Examining this situation will tell us what voltage is needed to produce a certain electric field strength. It will also reveal a more fundamental relationship between electric potential and electric field.



The relationship between V and E for parallel conducting plates is $E = V/d$. (Note that $\Delta V = V_{AB}$ in magnitude. For a charge that is moved from plate A at higher potential to plate B at lower potential, a minus sign needs to be included as follows: $-\Delta V = V_A - V_B = V_{AB}$.)

From a physicist's point of view, either ΔV or $\vec{\mathbf{E}}$ can be used to describe any interaction between charges. However, ΔV is a scalar quantity and has no direction, whereas $\vec{\mathbf{E}}$ is a vector quantity, having both magnitude and direction. (Note that the magnitude of the electric field, a scalar quantity, is represented by E.) The relationship between ΔV and $\vec{\mathbf{E}}$

is revealed by calculating the work done by the electric force in moving a charge from point A to point B. But, as noted earlier, arbitrary charge distributions require calculus. We therefore look at a uniform electric field as an interesting special case.

The work done by the electric field in [link] to move a positive charge q from A, the positive plate, higher potential, to B, the negative plate, lower potential, is
Equation:

$$W = -\Delta U = -q\Delta V.$$

The potential difference between points A and B is
Equation:

$$-\Delta V = -(V_B - V_A) = V_A - V_B = V_{AB}.$$

Entering this into the expression for work yields
Equation:

$$W = qV_{AB}.$$

Work is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = Fd\cos\theta$; here $\cos\theta = 1$, since the path is parallel to the field. Thus, $W = Fd$. Since $F = qE$, we see that $W = qEd$.

Substituting this expression for work into the previous equation gives
Equation:

$$qEd = qV_{AB}.$$

The charge cancels, so we obtain for the voltage between points A and B
Equation:

$$\left.\begin{array}{l} V_{AB} = Ed \\ E = \frac{V_{AB}}{d} \end{array}\right\}(\text{uniform } E\text{-field only})$$

where d is the distance from A to B, or the distance between the plates in [link]. Note that this equation implies that the units for electric field are volts per meter. We already know the units for electric field are newtons per coulomb; thus, the following relation among units is valid:
Equation:

$$1\,\mathrm{N/C} = 1\,\mathrm{V/m}.$$

Furthermore, we may extend this to the integral form. Substituting [link] into our definition for the potential difference between points A and B, we obtain

Equation:

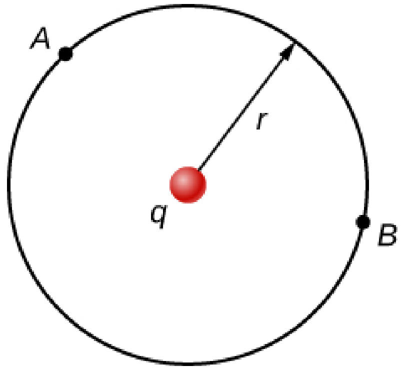
$$V_{BA} = V_B - V_A = -\int_R^B \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}} + \int_R^A \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

which simplifies to

Equation:

$$V_B - V_A = -\int_A^B \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}.$$

As a demonstration, from this we may calculate the potential difference between two points (A and B) equidistant from a point charge q at the origin, as shown in [link].



The arc for calculating the potential difference between two points that are equidistant from a point charge at the origin.

To do this, we integrate around an arc of the circle of constant radius r between A and B, which means we let $d\vec{\mathbf{l}} = r\widehat{\varphi}d\varphi$, while using $\overrightarrow{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. Thus,

Note:
Equation:

$$\Delta V_{BA} = V_B - V_A = -\int_A^B \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

for this system becomes
Equation:

$$V_B - V_A = -\int_A^B \frac{kq}{r^2} \hat{\mathbf{r}} \cdot r\widehat{\boldsymbol{\varphi}} d\varphi.$$

However, $\hat{\mathbf{r}} \cdot \widehat{\boldsymbol{\varphi}} = 0$ and therefore
Equation:

$$V_B - V_A = 0.$$

This result, that there is no difference in potential along a constant radius from a point charge, will come in handy when we map potentials.

Example:
What Is the Highest Voltage Possible between Two Plates?
Dry air can support a maximum electric field strength of about $3.0 \times 10^6 \text{ V/m}$. Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark that reduces the field. What, then, is the maximum voltage between two parallel conducting plates separated by 2.5 cm of dry air?
Strategy
We are given the maximum electric field *E* between the plates and the distance *d* between them. We can use the equation $V_{AB} = Ed$ to calculate the maximum voltage.
Solution
The potential difference or voltage between the plates is
Equation:

$$V_{AB} = Ed.$$

Entering the given values for *E* and *d* gives
Equation:

$$V_{AB} = (3.0 \times 10^6 \,\text{V/m})(0.025 \,\text{m}) = 7.5 \times 10^4 \,\text{V}$$

or

Equation:

$$V_{AB} = 75 \,\text{kV}.$$

(The answer is quoted to only two digits, since the maximum field strength is approximate.)

Significance

One of the implications of this result is that it takes about 75 kV to make a spark jump across a 2.5-cm (1-in.) gap, or 150 kV for a 5-cm spark. This limits the voltages that can exist between conductors, perhaps on a power transmission line. A smaller voltage can cause a spark if there are spines on the surface, since sharp points have larger field strengths than smooth surfaces. Humid air breaks down at a lower field strength, meaning that a smaller voltage will make a spark jump through humid air. The largest voltages can be built up with static electricity on dry days ([link]).

A spark chamber is used to trace the paths of high-energy particles. Ionization created by the particles as they pass through the gas between the plates allows a spark to jump. The sparks are perpendicular to the plates, following electric field lines between them. The potential difference between adjacent plates is not high enough to cause sparks without the ionization produced by particles from accelerator experiments (or cosmic rays). This form of detector is now archaic and no longer in use except for demonstration purposes. (credit b: modification of work by Jack Collins)

Example:

Field and Force inside an Electron Gun

An electron gun ([link]) has parallel plates separated by 4.00 cm and gives electrons 25.0 keV of energy. (a) What is the electric field strength between the plates? (b) What force would this field exert on a piece of plastic with a $0.500\text{-}\mu\text{C}$ charge that gets between the plates?

Strategy

Since the voltage and plate separation are given, the electric field strength can be calculated directly from the expression $E = \frac{V_{AB}}{d}$. Once we know the electric field strength, we can find the force on a charge by using $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. Since the electric field is in only one direction, we can write this equation in terms of the magnitudes, $F = qE$.

Solution

a. The expression for the magnitude of the electric field between two uniform metal plates is
   Equation:
   $$E = \frac{V_{AB}}{d}.$$
   Since the electron is a single charge and is given 25.0 keV of energy, the potential difference must be 25.0 kV. Entering this value for V_{AB} and the plate separation of 0.0400 m, we obtain
   Equation:
   $$E = \frac{25.0\,\text{kV}}{0.0400\,\text{m}} = 6.25 \times 10^5\ \text{V/m}.$$
b. The magnitude of the force on a charge in an electric field is obtained from the equation
   Equation:
   $$F = qE.$$
   Substituting known values gives
   Equation:
   $$F = (0.500 \times 10^{-6}\ \text{C})(6.25 \times 10^5\ \text{V/m}) = 0.313\,\text{N}.$$

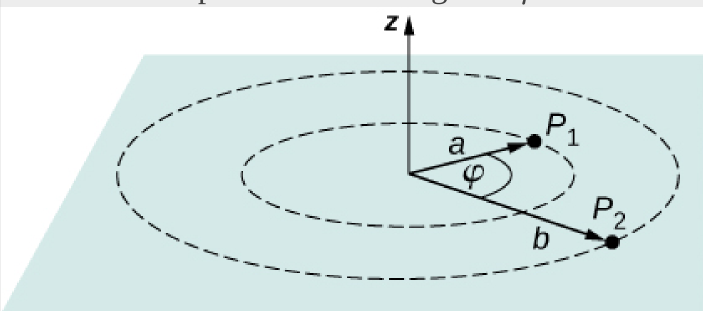
Significance

Note that the units are newtons, since $1\ \text{V/m} = 1\,\text{N/C}$. Because the electric field is uniform between the plates, the force on the charge is the same no matter where the charge is located between the plates.

Example:

Calculating Potential of a Point Charge

Given a point charge $q=+2.0\,\text{nC}$ at the origin, calculate the potential difference between point P_1 a distance $a=4.0\,\text{cm}$ from q, and P_2 a distance $b=12.0\,\text{cm}$ from q, where the two points have an angle of $\varphi=24^\circ$ between them ([link]).



Find the difference in potential between P_1 and P_2.

Strategy

Do this in two steps. The first step is to use $V_B-V_A=-\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{l}}$ and let $A=a=4.0\,\text{cm}$ and $B=b=12.0\,\text{cm}$, with $d\vec{\mathbf{l}}=d\vec{\mathbf{r}}=\hat{\mathbf{r}}dr$ and $\vec{\mathbf{E}}=\frac{kq}{r^2}\hat{\mathbf{r}}$. Then perform the integral. The second step is to integrate $V_B-V_A=-\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{l}}$ around an arc of constant radius r, which means we let $d\vec{\mathbf{l}}=r\hat{\varphi}d\varphi$ with limits $0\le\varphi\le 24^\circ$, still using $\vec{\mathbf{E}}=\frac{kq}{r^2}\hat{\mathbf{r}}$. Then add the two results together.

Solution

For the first part, $V_B-V_A=-\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{l}}$ for this system becomes $V_b-V_a=-\int_a^b \frac{kq}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}dr$ which computes to

Equation:

$$\begin{aligned}\Delta V &= -\int_a^b \frac{kq}{r^2}dr = kq\left[\frac{1}{a}-\frac{1}{b}\right]\\ &= \left(8.99\times10^9\,\text{Nm}^2/\text{C}^2\right)\left(2.0\times10^{-9}\,\text{C}\right)\left[\frac{1}{0.040\,\text{m}}-\frac{1}{0.12\,\text{m}}\right]=300\,\text{V}.\end{aligned}$$

For the second step, $V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ becomes $\Delta V = -\int_0^{24^\circ} \frac{kq}{r^2} \hat{\mathbf{r}} \cdot r\widehat{\boldsymbol{\varphi}} d\varphi$, but $\hat{\mathbf{r}} \cdot \widehat{\boldsymbol{\varphi}} = 0$ and therefore $\Delta V = 0$. Adding the two parts together, we get 300 V.

Significance

We have demonstrated the use of the integral form of the potential difference to obtain a numerical result. Notice that, in this particular system, we could have also used the formula for the potential due to a point charge at the two points and simply taken the difference.

Note:

Exercise:

Problem:

Check Your Understanding From the examples, how does the energy of a lightning strike vary with the height of the clouds from the ground? Consider the cloud-ground system to be two parallel plates.

Solution:

Given a fixed maximum electric field strength, the potential at which a strike occurs increases with increasing height above the ground. Hence, each electron will carry more energy. Determining if there is an effect on the total number of electrons lies in the future.

Before presenting problems involving electrostatics, we suggest a problem-solving strategy to follow for this topic.

Note:

Problem-Solving Strategy: Electrostatics

1. Examine the situation to determine if static electricity is involved; this may concern separated stationary charges, the forces among them, and the electric fields they create.
2. Identify the system of interest. This includes noting the number, locations, and types of charges involved.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Determine whether the Coulomb force is to be considered

directly—if so, it may be useful to draw a free-body diagram, using electric field lines.

4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). It is important to distinguish the Coulomb force F from the electric field E, for example.
5. Solve the appropriate equation for the quantity to be determined (the unknown) or draw the field lines as requested.
6. Examine the answer to see if it is reasonable: Does it make sense? Are units correct and the numbers involved reasonable?

Summary

- Electric potential is potential energy per unit charge.
- The potential difference between points A and B, $V_B - V_A$, that is, the change in potential of a charge q moved from A to B, is equal to the change in potential energy divided by the charge.
- Potential difference is commonly called voltage, represented by the symbol ΔV: $\Delta V = \frac{\Delta U}{q}$ or $\Delta U = q\Delta V$.
- An electron-volt is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form, $1 \text{ eV} = \left(1.60 \times 10^{-19} \text{ C}\right)(1 \text{ V}) = \left(1.60 \times 10^{-19} \text{ C}\right)(1 \text{ J/C}) = 1.60 \times 10^{-19} \text{ J}.$

Conceptual Questions

Exercise:

Problem:

Discuss how potential difference and electric field strength are related. Give an example.

Exercise:

Problem:

What is the strength of the electric field in a region where the electric potential is constant?

Solution:

The electric field strength is zero because electric potential differences are directly related to the field strength. If the potential difference is zero, then the field strength must also be zero.

Exercise:

Problem:

If a proton is released from rest in an electric field, will it move in the direction of increasing or decreasing potential? Also answer this question for an electron and a neutron. Explain why.

Exercise:

Problem:

Voltage is the common word for potential difference. Which term is more descriptive, voltage or potential difference?

Solution:

Potential difference is more descriptive because it indicates that it is the difference between the electric potential of two points.

Exercise:

Problem:

If the voltage between two points is zero, can a test charge be moved between them with zero net work being done? Can this necessarily be done without exerting a force? Explain.

Exercise:

Problem:

What is the relationship between voltage and energy? More precisely, what is the relationship between potential difference and electric potential energy?

Solution:

They are very similar, but potential difference is a feature of the system; when a charge is introduced to the system, it will have a potential energy which may be calculated by multiplying the magnitude of the charge by the potential difference.

Exercise:

Problem: Voltages are always measured between two points. Why?

Exercise:

Problem: How are units of volts and electron-volts related? How do they differ?

Solution:

An electron-volt is a volt multiplied by the charge of an electron. Volts measure potential difference, electron-volts are a unit of energy.

Exercise:

Problem:

Can a particle move in a direction of increasing electric potential, yet have its electric potential energy decrease? Explain

Problems

Exercise:

Problem:

Find the ratio of speeds of an electron and a negative hydrogen ion (one having an extra electron) accelerated through the same voltage, assuming non-relativistic final speeds. Take the mass of the hydrogen ion to be 1.67×10^{-27} kg.

Solution:

$$\begin{aligned}\frac{1}{2}m_e v_e^2 &= qV,\ \frac{1}{2}m_{\mathrm{H}}v_{\mathrm{H}}^2 = qV,\text{ so that}\\ \frac{m_e v_e^2}{m_{\mathrm{H}}v_{\mathrm{H}}^2} &= 1\text{ or }\frac{v_e}{v_{\mathrm{H}}} = 42.8\end{aligned}$$

Exercise:

Problem:

An evacuated tube uses an accelerating voltage of 40 kV to accelerate electrons to hit a copper plate and produce X-rays. Non-relativistically, what would be the maximum speed of these electrons?

Exercise:

Problem:

Show that units of V/m and N/C for electric field strength are indeed equivalent.

Solution:

$$1\text{ V} = 1\text{ J/C}; 1\text{ J} = 1\text{ N}\cdot\text{m} \rightarrow 1\text{ V/m} = 1\text{ N/C}$$

Exercise:

Problem:

What is the strength of the electric field between two parallel conducting plates separated by 1.00 cm and having a potential difference (voltage) between them of $1.50 \times 10^{4}\ \text{V}$?

Exercise:

Problem:

The electric field strength between two parallel conducting plates separated by 4.00 cm is $7.50 \times 10^{4}\ \text{V/m}$. (a) What is the potential difference between the plates? (b) The plate with the lowest potential is taken to be zero volts. What is the potential 1.00 cm from that plate and 3.00 cm from the other?

Solution:

a. $V_{AB} = 3.00\,\text{kV}$; b. $V_{AB} = 750\ \text{V}$

Exercise:

Problem:

The voltage across a membrane forming a cell wall is 80.0 mV and the membrane is 9.00 nm thick. What is the electric field strength? (The value is surprisingly large, but correct.) You may assume a uniform electric field.

Exercise:

Problem:

Two parallel conducting plates are separated by 10.0 cm, and one of them is taken to be at zero volts. (a) What is the electric field strength between them, if the potential 8.00 cm from the zero volt plate (and 2.00 cm from the other) is 450 V? (b) What is the voltage between the plates?

Solution:

a. $V_{AB} = Ed \rightarrow E = 5.63\,\text{kV/m}$;
b. $V_{AB} = 563\ \text{V}$

Exercise:

Problem:

Find the maximum potential difference between two parallel conducting plates separated by 0.500 cm of air, given the maximum sustainable electric field strength in air to be $3.0 \times 10^{6}\ \text{V/m}$.

Exercise:

Problem:

An electron is to be accelerated in a uniform electric field having a strength of $2.00 \times 10^6 \text{ V/m}$. (a) What energy in keV is given to the electron if it is accelerated through 0.400 m? (b) Over what distance would it have to be accelerated to increase its energy by 50.0 GeV?

Solution:

a. $$\begin{aligned}\Delta K &= q\Delta V \text{ and } V_{AB} = Ed, \text{ so that} \\ \Delta K &= 800 \,\text{keV};\end{aligned}$$

b. $d = 25.0 \,\text{km}$

Exercise:

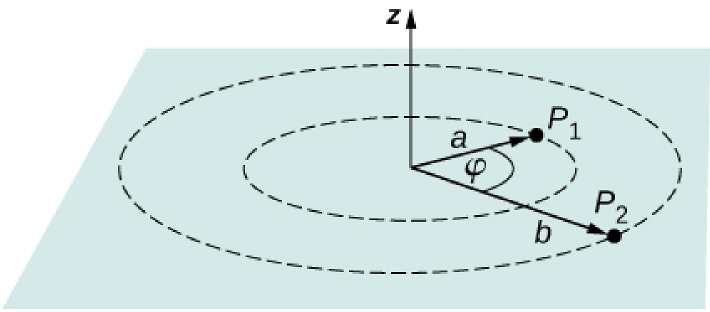
Problem:

Use the definition of potential difference in terms of electric field to deduce the formula for potential difference between $r = r_a$ and $r = r_b$ for a point charge located at the origin. Here *r* is the spherical radial coordinate.

Exercise:

Problem:

The electric field in a region is pointed away from the z-axis and the magnitude depends upon the distance *s* from the axis. The magnitude of the electric field is given as $E = \frac{\alpha}{s}$ where α is a constant. Find the potential difference between points P_1 and P_2, explicitly stating the path over which you conduct the integration for the line integral.



Solution:

One possibility is to stay at constant radius and go along the arc from P_1 to P_2, which will have zero potential due to the path being perpendicular to the electric field. Then integrate from *a* to *b*: $V_{ab} = \alpha \ln\left(\frac{b}{a}\right)$

Exercise:

Problem:

Singly charged gas ions are accelerated from rest through a voltage of 13.0 V. At what temperature will the average kinetic energy of gas molecules be the same as that given these ions?

Glossary

electric potential
: potential energy per unit charge

electric potential difference
: the change in potential energy of a charge *q* moved between two points, divided by the charge.

electron-volt
: energy given to a fundamental charge accelerated through a potential difference of one volt

voltage
: change in potential energy of a charge moved from one point to another, divided by the charge; units of potential difference are joules per coulomb, known as volt

Calculations of Electric Potential

By the end of this section, you will be able to:

- Calculate the potential due to a point charge
- Calculate the potential of a system of multiple point charges
- Describe an electric dipole
- Define dipole moment
- Calculate the potential of a continuous charge distribution

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (such as charge on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider.

We can use calculus to find the work needed to move a test charge *q* from a large distance away to a distance of *r* from a point charge *q*. Noting the connection between work and potential $W=-q\Delta V$, as in the last section, we can obtain the following result.

Note:

Electric Potential *V* of a Point Charge

The electric potential *V* of a point charge is given by

Equation:

$$V=\frac{kq}{r}\text{(point charge)}$$

where *k* is a constant equal to $8.99\ \times\ 10^{9}\ \mathrm{N}\cdot\mathrm{m}^{2}/\mathrm{C}^{2}$.

The potential at infinity is chosen to be zero. Thus, *V* for a point charge decreases with distance, whereas $\overrightarrow{\mathbf{E}}$ for a point charge decreases with distance squared:

Equation:

$$E=\frac{F}{q_t}=\frac{kq}{r^2}.$$

Recall that the electric potential *V* is a scalar and has no direction, whereas the electric field $\overrightarrow{\mathbf{E}}$ is a vector. To find the voltage due to a combination of point charges, you add the individual voltages as numbers. To find the total electric field, you must add the individual fields as vectors, taking magnitude and direction into account. This is consistent with the fact that *V* is closely associated with energy, a scalar, whereas $\overrightarrow{\mathbf{E}}$ is closely associated with force, a vector.

Example:

What Voltage Is Produced by a Small Charge on a Metal Sphere?

Charges in static electricity are typically in the nanocoulomb (nC) to microcoulomb $(\mu\mathrm{C})$ range. What is the voltage 5.00 cm away from the center of a 1-cm-diameter solid metal sphere that has a –3.00-nC static charge?

Strategy

As we discussed in Electric Charges and Fields, charge on a metal sphere spreads out uniformly and produces a field like that of a point charge located at its center. Thus, we can find the voltage using the equation $V=\frac{kq}{r}$.

Solution

Entering known values into the expression for the potential of a point charge, we obtain

Equation:

$$V=k\frac{q}{r}=\left(8.99\,\times\,10^{9}\,\mathrm{N\cdot m^{2}/C^{2}}\right)\left(\frac{-3.00\,\times\,10^{-9}\,\mathrm{C}}{5.00\,\times\,10^{-2}\,\mathrm{m}}\right)=-539\,\mathrm{V}.$$

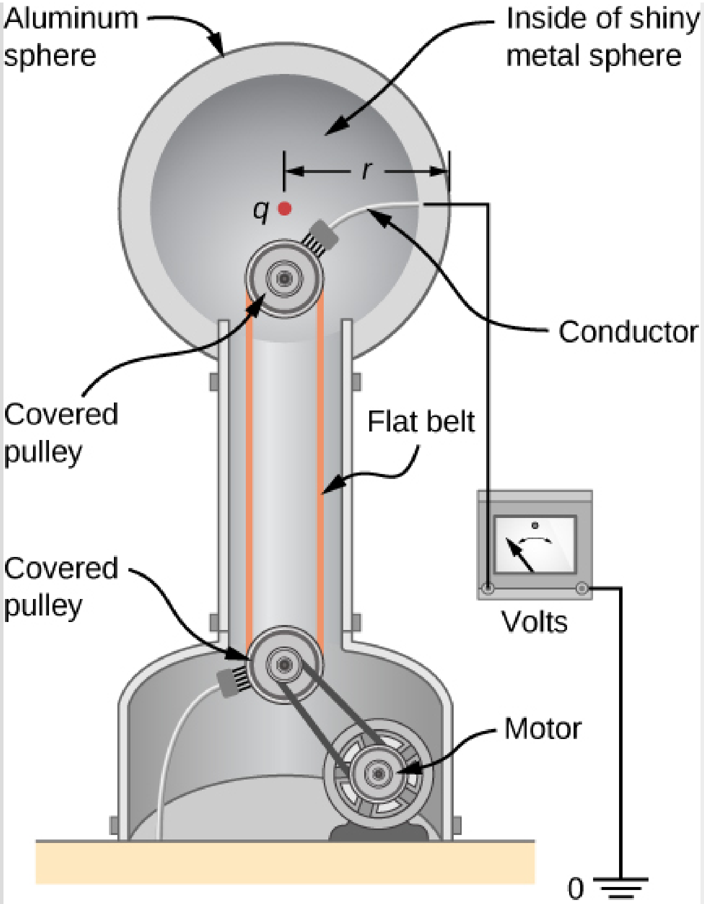
Significance

The negative value for voltage means a positive charge would be attracted from a larger distance, since the potential is lower (more negative) than at larger distances. Conversely, a negative charge would be repelled, as expected.

Example:

What Is the Excess Charge on a Van de Graaff Generator?

A demonstration Van de Graaff generator has a 25.0-cm-diameter metal sphere that produces a voltage of 100 kV near its surface ([link]). What excess charge resides on the sphere? (Assume that each numerical value here is shown with three significant figures.)



The voltage of this demonstration Van de Graaff generator is measured between the charged sphere and ground. Earth's potential is taken to be zero as a reference. The potential of the charged conducting sphere is the same as that of an equal point charge at its center.

Strategy

The potential on the surface is the same as that of a point charge at the center of the sphere, 12.5 cm away. (The radius of the sphere is 12.5 cm.) We can thus determine the excess charge using the equation

Equation:

$$V = \frac{kq}{r}.$$

Solution

Solving for q and entering known values gives

Equation:

$$q = \frac{rV}{k} = \frac{(0.125\ \text{m})\left(100 \times 10^3\ \text{V}\right)}{8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2} = 1.39 \times 10^{-6}\ \text{C} = 1.39\ \mu\text{C}.$$

Significance
This is a relatively small charge, but it produces a rather large voltage. We have another indication here that it is difficult to store isolated charges.

Note:
Exercise:

Problem:

Check Your Understanding What is the potential inside the metal sphere in [link]?

Solution:

$V = k\frac{q}{r} = \left(8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2\right)\left(\frac{-3.00 \times 10^{-9}\ \text{C}}{5.00 \times 10^{-3}\ \text{m}}\right) = -5390\ \text{V}$; recall that the electric field inside a conductor is zero. Hence, any path from a point on the surface to any point in the interior will have an integrand of zero when calculating the change in potential, and thus the potential in the interior of the sphere is identical to that on the surface.

The voltages in both of these examples could be measured with a meter that compares the measured potential with ground potential. Ground potential is often taken to be zero (instead of taking the potential at infinity to be zero). It is the potential difference between two points that is of importance, and very often there is a tacit assumption that some reference point, such as Earth or a very distant point, is at zero potential. As noted earlier, this is analogous to taking sea level as $h = 0$ when considering gravitational potential energy $U_g = mgh$.

Systems of Multiple Point Charges

Just as the electric field obeys a superposition principle, so does the electric potential. Consider a system consisting of N charges $q_1, q_2, \ldots, q_N$. What is the net electric potential V at a space point P from these charges? Each of these charges is a source charge that produces its own electric potential at point P, independent of whatever other changes may be doing. Let $V_1, V_2, \ldots, V_N$ be the electric potentials at P produced by the charges $q_1, q_2, \ldots, q_N$, respectively. Then, the net electric potential V_P at that point is equal to the sum of these individual electric potentials. You can easily show this by calculating the potential energy of a test charge when you bring the test charge from the reference point at infinity to point P:
Equation:

$$V_P = V_1 + V_2 + \cdots + V_N = \sum_1^N V_i.$$

Note that electric potential follows the same principle of superposition as electric field and electric potential energy. To show this more explicitly, note that a test charge q_i at the point P in space has distances of $r_1, r_2, \ldots, r_N$ from the N charges fixed in space above, as shown in [link]. Using our formula for the potential of a point charge for each of these (assumed to be point) charges, we find that

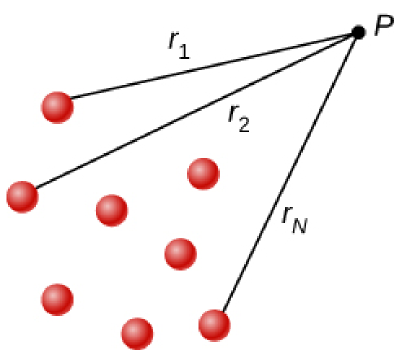
Note:
Equation:

$$V_P = \sum_1^N k \frac{q_i}{r_i} = k \sum_1^N \frac{q_i}{r_i}.$$

Therefore, the electric potential energy of the test charge is
Equation:

$$U_P = q_t V_P = q_t k \sum_1^N \frac{q_i}{r_i},$$

which is the same as the work to bring the test charge into the system, as found in the first section of the chapter.



Notation for direct distances from charges to a space point P.

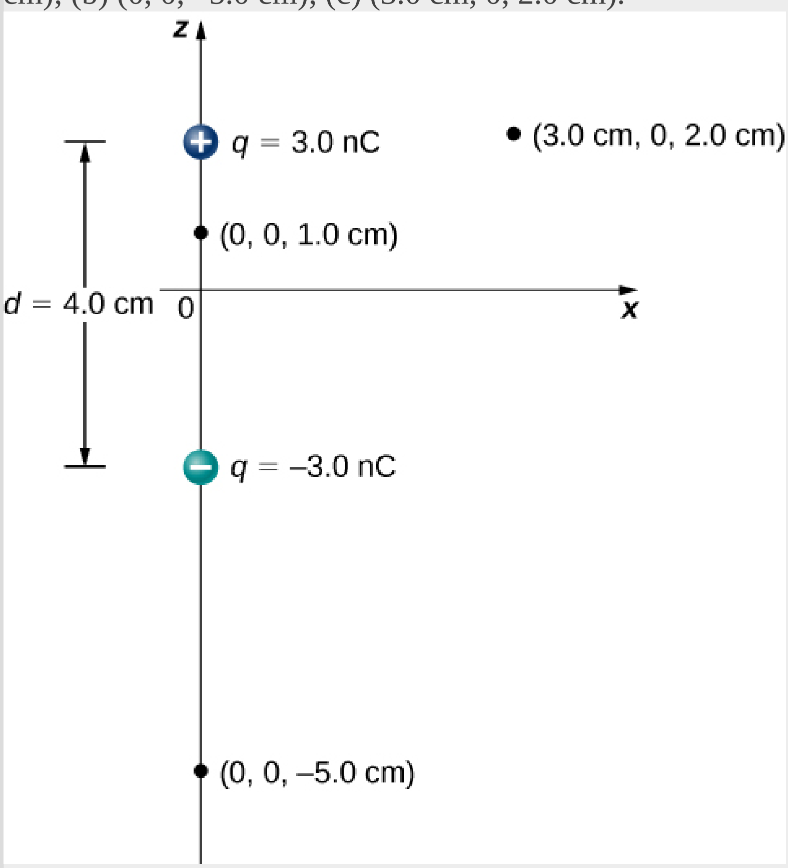
The Electric Dipole

An **electric dipole** is a system of two equal but opposite charges a fixed distance apart. This system is used to model many real-world systems, including atomic and molecular interactions. One of these systems is the water molecule, under certain circumstances. These circumstances are met inside a microwave oven, where electric fields with alternating directions make the water molecules change orientation. This vibration is the same as heat at the molecular level.

Example:

Electric Potential of a Dipole

Consider the dipole in [link] with the charge magnitude of $q = 3.0\ \text{nC}$ and separation distance $d = 4.0\ \text{cm}$. What is the potential at the following locations in space? (a) (0, 0, 1.0 cm); (b) (0, 0, –5.0 cm); (c) (3.0 cm, 0, 2.0 cm).



A general diagram of an electric dipole, and the notation for the distances from the individual charges

to a point P in space.

Strategy

Apply $V_P = k\sum_1^N \frac{q_i}{r_i}$ to each of these three points.

Solution

a. $V_P = k\sum_1^N \frac{q_i}{r_i} = (9.0\ \times\ 10^9\,\mathrm{N \cdot m^2/C^2})\left(\frac{3.0\,\mathrm{nC}}{0.010\,\mathrm{m}} - \frac{3.0\,\mathrm{nC}}{0.030\,\mathrm{m}}\right) = 1.8\ \times\ 10^3\ \mathrm{V}$

b. $V_P = k\sum_1^N \frac{q_i}{r_i} = (9.0\ \times\ 10^9\,\mathrm{N \cdot m^2/C^2})\left(\frac{3.0\,\mathrm{nC}}{0.070\,\mathrm{m}} - \frac{3.0\,\mathrm{nC}}{0.030\,\mathrm{m}}\right) = -5.1\ \times\ 10^2\ \mathrm{V}$

c. $V_P = k\sum_1^N \frac{q_i}{r_i} = (9.0\ \times\ 10^9\,\mathrm{N \cdot m^2/C^2})\left(\frac{3.0\,\mathrm{nC}}{0.030\,\mathrm{m}} - \frac{3.0\,\mathrm{nC}}{0.050\,\mathrm{m}}\right) = 3.6\ \times\ 10^2\ \mathrm{V}$

Significance

Note that evaluating potential is significantly simpler than electric field, due to potential being a scalar instead of a vector.

Note:

Exercise:

Problem: Check Your Understanding What is the potential on the *x*-axis? The *z*-axis?

Solution:

The *x*-axis the potential is zero, due to the equal and opposite charges the same distance from it. On the *z*-axis, we may superimpose the two potentials; we will find that for $z\ >>\ d$, again the potential goes to zero due to cancellation.

Now let us consider the special case when the distance of the point P from the dipole is much greater than the distance between the charges in the dipole, $r \gg d$; for example, when we are interested in the electric potential due to a polarized molecule such as a water molecule. This is not so far (infinity) that we can simply treat the potential as zero, but the distance is great enough that we can simplify our calculations relative to the previous example.

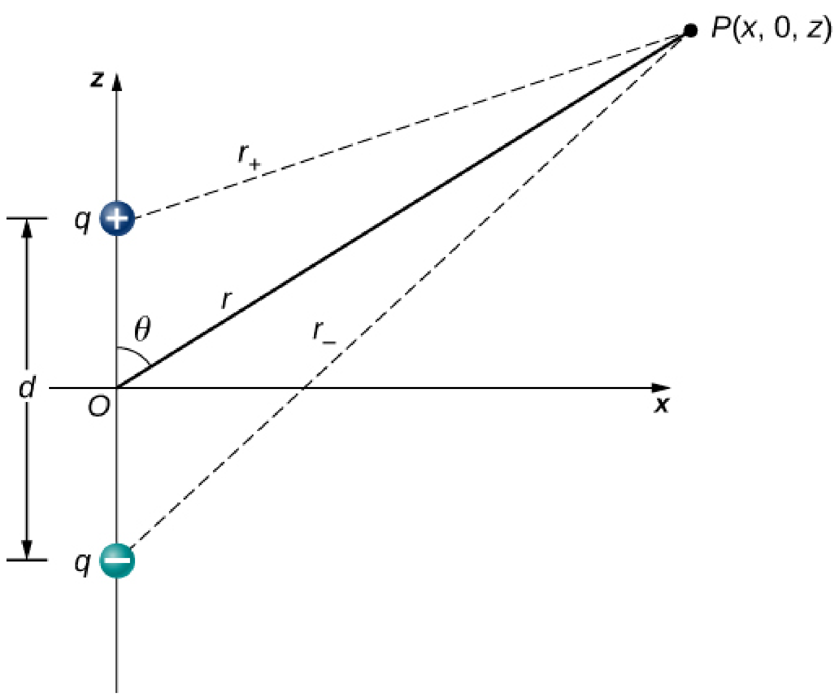
We start by noting that in [link] the potential is given by

Equation:

$$V_P = V_+ + V_- = k\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where
Equation:

$$r_\pm = \sqrt{x^2 + \left(z \mp \frac{d}{2}\right)^2}.$$



A general diagram of an electric dipole, and the notation for the distances from the individual charges to a point *P* in space.

This is still the exact formula. To take advantage of the fact that $r \gg d$, we rewrite the radii in terms of polar coordinates, with $x = r\sin\theta$ and $z = r\cos\theta$. This gives us
Equation:

$$r_\pm = \sqrt{r^2\sin^2\theta + \left(r\cos\theta \mp \frac{d}{2}\right)^2}.$$

We can simplify this expression by pulling r out of the root,
Equation:

$$r_{\pm} = r\sqrt{\sin^2\theta + \left(\cos\theta \mp \frac{d}{2r}\right)^2}$$

and then multiplying out the parentheses
Equation:

$$r_{\pm} = r\sqrt{\sin^2\theta + \cos^2\theta \mp \cos\theta\frac{d}{r} + \left(\frac{d}{2r}\right)^2} = r\sqrt{1 \mp \cos\theta\frac{d}{r} + \left(\frac{d}{2r}\right)^2}.$$

The last term in the root is small enough to be negligible (remember $r \gg d$, and hence $(d/r)^2$ is extremely small, effectively zero to the level we will probably be measuring), leaving us with
Equation:

$$r_{\pm} = r\sqrt{1 \mp \cos\theta\frac{d}{r}}.$$

Using the binomial approximation (a standard result from the mathematics of series, when α is small)
Equation:

$$\frac{1}{\sqrt{1 \mp \alpha}} \approx 1 \pm \frac{\alpha}{2}$$

and substituting this into our formula for V_P , we get
Equation:

$$V_P = k\left[\frac{q}{r}\left(1 + \frac{d\cos\theta}{2r}\right) - \frac{q}{r}\left(1 - \frac{d\cos\theta}{2r}\right)\right] = k\frac{qd\cos\theta}{r^2}.$$

This may be written more conveniently if we define a new quantity, the **electric dipole moment**,

Note:
Equation:

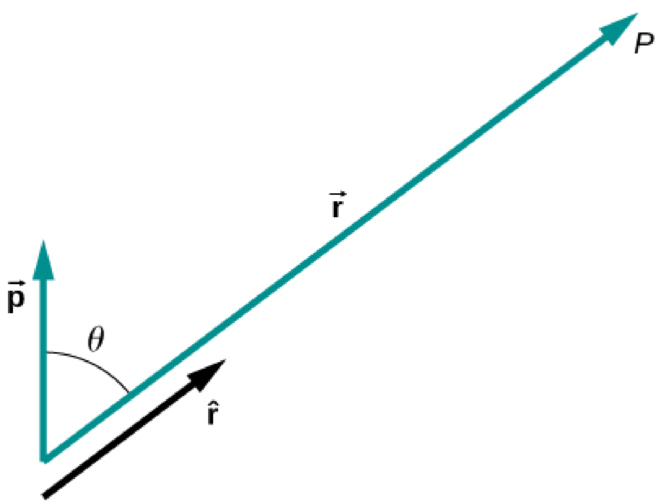
$$\vec{\mathbf{p}} = q\vec{\mathbf{d}},$$

where these vectors point from the negative to the positive charge. Note that this has magnitude *qd*. This quantity allows us to write the potential at point *P* due to a dipole at the origin as

Note:
Equation:

$$V_P = k\frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2}.$$

A diagram of the application of this formula is shown in [link].



The geometry for the application of the potential of a dipole.

There are also higher-order moments, for quadrupoles, octupoles, and so on. You will see these in future classes.

Potential of Continuous Charge Distributions

We have been working with point charges a great deal, but what about continuous charge distributions? Recall from [link] that

Equation:

$$V_P = k \sum \frac{q_i}{r_i}.$$

We may treat a continuous charge distribution as a collection of infinitesimally separated individual points. This yields the integral

Note:

Equation:

$$V_P = k \int \frac{dq}{r}$$

for the potential at a point P. Note that r is the distance from each individual point in the charge distribution to the point P. As we saw in Electric Charges and Fields, the infinitesimal charges are given by

Equation:

$$dq = \begin{cases} \lambda\, dl & \text{(one dimension)} \\ \sigma\, dA & \text{(two dimensions)} \\ \rho\, dV & \text{(three dimensions)} \end{cases}$$

where λ is linear charge density, σ is the charge per unit area, and ρ is the charge per unit volume.

Example:

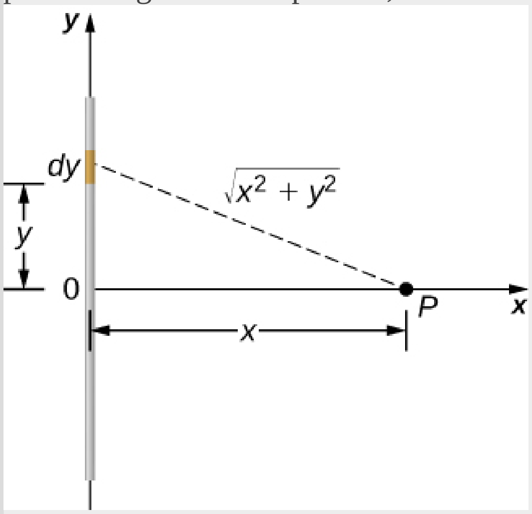
Potential of a Line of Charge

Find the electric potential of a uniformly charged, nonconducting wire with linear density λ (coulomb/meter) and length L at a point that lies on a line that divides the wire into two equal parts.

Strategy

To set up the problem, we choose Cartesian coordinates in such a way as to exploit the symmetry in the problem as much as possible. We place the origin at the center of the wire

and orient the *y*-axis along the wire so that the ends of the wire are at $y = \pm L/2$. The field point *P* is in the *xy*-plane and since the choice of axes is up to us, we choose the *x*-axis to pass through the field point *P*, as shown in [link].



We want to calculate the electric potential due to a line of charge.

Solution
Consider a small element of the charge distribution between *y* and $y + dy$. The charge in this cell is $dq = \lambda\, dy$ and the distance from the cell to the field point *P* is $\sqrt{x^2 + y^2}$. Therefore, the potential becomes

Equation:

$$\begin{aligned} V_P &= k\int \frac{dq}{r} = k\int_{-L/2}^{L/2} \frac{\lambda dy}{\sqrt{x^2+y^2}} = k\lambda\left[\ln\left(y+\sqrt{y^2+x^2}\right)\right]_{-L/2}^{L/2} \\ &= k\lambda\left[\ln\left(\left(\tfrac{L}{2}\right)+\sqrt{\left(\tfrac{L}{2}\right)^2+x^2}\right)-\ln\left(\left(-\tfrac{L}{2}\right)+\sqrt{\left(-\tfrac{L}{2}\right)^2+x^2}\right)\right] \\ &= k\lambda \ln\left[\tfrac{L+\sqrt{L^2+4x^2}}{-L+\sqrt{L^2+4x^2}}\right]. \end{aligned}$$

Significance
Note that this was simpler than the equivalent problem for electric field, due to the use of scalar quantities. Recall that we expect the zero level of the potential to be at infinity, when we have a finite charge. To examine this, we take the limit of the above potential as *x* approaches infinity; in this case, the terms inside the natural log approach one, and hence the potential approaches zero in this limit. Note that we could have done this problem equivalently in cylindrical coordinates; the only effect would be to substitute *r* for *x* and *z* for *y*.

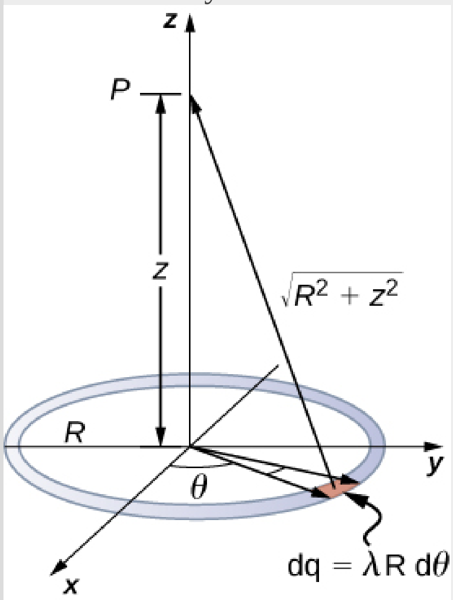
Example:

Potential Due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric potential at a point on the axis passing through the center of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use cylindrical coordinates shown in [link].



We want to calculate the electric potential due to a ring of charge.

Solution

A general element of the arc between θ and $\theta + d\theta$ is of length $Rd\theta$ and therefore contains a charge equal to $\lambda Rd\theta$. The element is at a distance of $\sqrt{z^2 + R^2}$ from P, and therefore the potential is

Equation:

$$V_P = k\int \frac{dq}{r} = k\int_0^{2\pi} \frac{\lambda R d\theta}{\sqrt{z^2+R^2}} = \frac{k\lambda R}{\sqrt{z^2+R^2}}\int_0^{2\pi} d\theta = \frac{2\pi k\lambda R}{\sqrt{z^2+R^2}} = k\frac{q_{\text{tot}}}{\sqrt{z^2+R^2}}.$$

Significance

This result is expected because every element of the ring is at the same distance from point P. The net potential at P is that of the total charge placed at the common distance, $\sqrt{z^2 + R^2}$.

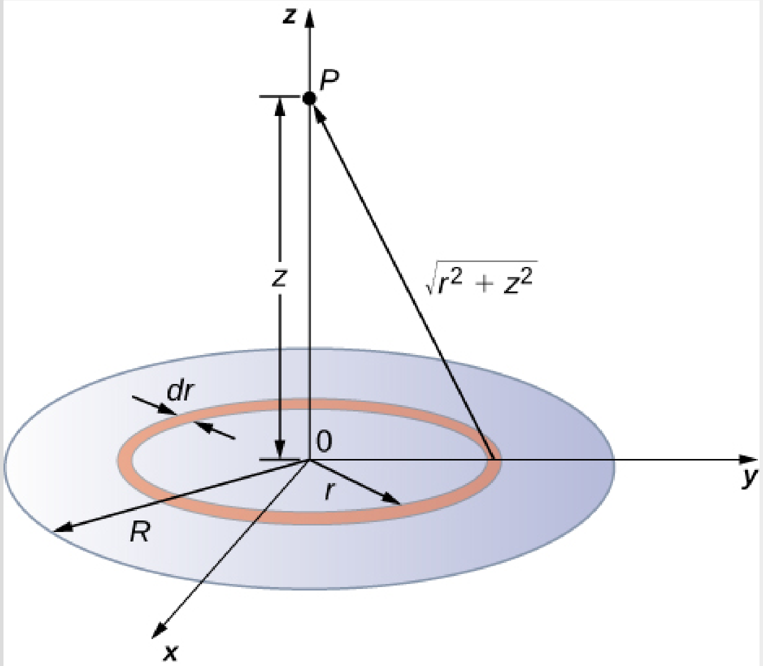
Example:

Potential Due to a Uniform Disk of Charge

A disk of radius R has a uniform charge density σ, with units of coulomb meter squared. Find the electric potential at any point on the axis passing through the center of the disk.

Strategy

We divide the disk into ring-shaped cells, and make use of the result for a ring worked out in the previous example, then integrate over r in addition to θ. This is shown in [link].



We want to calculate the electric potential due to a disk of charge.

Solution

An infinitesimal width cell between cylindrical coordinates r and $r + dr$ shown in [link] will be a ring of charges whose electric potential dV_P at the field point has the following expression

Equation:

$$dV_P = k\frac{dq}{\sqrt{z^2 + r^2}}$$

where

Equation:

$$dq = \sigma 2\pi r dr.$$

The superposition of potential of all the infinitesimal rings that make up the disk gives the net potential at point P. This is accomplished by integrating from $r = 0$ to $r = R$:

Equation:

$$\begin{aligned} V_P &= \int dV_P = k2\pi\sigma \int_0^R \frac{r\,dr}{\sqrt{z^2+r^2}}, \\ &= k2\pi\sigma\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right). \end{aligned}$$

Significance
The basic procedure for a disk is to first integrate around θ and then over r. This has been demonstrated for uniform (constant) charge density. Often, the charge density will vary with r, and then the last integral will give different results.

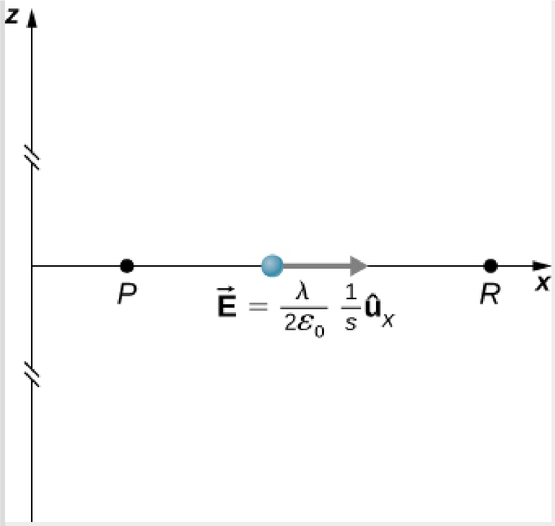
Example:
Potential Due to an Infinite Charged Wire
Find the electric potential due to an infinitely long uniformly charged wire.
Strategy
Since we have already worked out the potential of a finite wire of length L in [link], we might wonder if taking $L \to \infty$ in our previous result will work:
Equation:

$$V_P = \lim_{L\to\infty} k\lambda \ln\left(\frac{L+\sqrt{L^2+4x^2}}{-L+\sqrt{L^2+4x^2}}\right).$$

However, this limit does not exist because the argument of the logarithm becomes [2/0] as $L \to \infty$, so this way of finding V of an infinite wire does not work. The reason for this problem may be traced to the fact that the charges are not localized in some space but continue to infinity in the direction of the wire. Hence, our (unspoken) assumption that zero potential must be an infinite distance from the wire is no longer valid.
To avoid this difficulty in calculating limits, let us use the definition of potential by integrating over the electric field from the previous section, and the value of the electric field from this charge configuration from the previous chapter.
Solution
We use the integral
Equation:

$$V_P = -\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

where R is a finite distance from the line of charge, as shown in [link].



Points of interest for calculating the potential of an infinite line of charge.

With this setup, we use $\overrightarrow{\mathbf{E}}_P = 2k\lambda\frac{1}{s}\hat{\mathbf{s}}$ and $d\vec{\mathbf{l}} = d\vec{\mathbf{s}}$ to obtain

Equation:

$$V_P - V_R = -\int_R^P 2k\lambda\frac{1}{s}ds = -2k\lambda \ln\frac{s_P}{s_R}.$$

Now, if we define the reference potential $V_R = 0$ at $s_R = 1\ \text{m}$, this simplifies to

Equation:

$$V_P = -2k\lambda\ \ln\ s_P.$$

Note that this form of the potential is quite usable; it is 0 at 1 m and is undefined at infinity, which is why we could not use the latter as a reference.

Significance

Although calculating potential directly can be quite convenient, we just found a system for which this strategy does not work well. In such cases, going back to the definition of potential in terms of the electric field may offer a way forward.

Note:

Exercise:

Problem:

Check Your Understanding What is the potential on the axis of a nonuniform ring of charge, where the charge density is $\lambda(\theta) = \lambda \cos\theta$?

Solution:

It will be zero, as at all points on the axis, there are equal and opposite charges equidistant from the point of interest. Note that this distribution will, in fact, have a dipole moment.

Summary

- Electric potential is a scalar whereas electric field is a vector.
- Addition of voltages as numbers gives the voltage due to a combination of point charges, allowing us to use the principle of superposition: $V_P = k \sum_1^N \frac{q_i}{r_i}$.
- An electric dipole consists of two equal and opposite charges a fixed distance apart, with a dipole moment $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$.
- Continuous charge distributions may be calculated with $V_P = k \int \frac{dq}{r}$.

Conceptual Questions

Exercise:

Problem:

Compare the electric dipole moments of charges $\pm Q$ separated by a distance *d* and charges $\pm Q/2$ separated by a distance *d*/2.

Solution:

The second has 1/4 the dipole moment of the first.

Exercise:

Problem:

Would Gauss's law be helpful for determining the electric field of a dipole? Why?

Exercise:

Problem:

In what region of space is the potential due to a uniformly charged sphere the same as that of a point charge? In what region does it differ from that of a point charge?

Solution:

The region outside of the sphere will have a potential indistinguishable from a point charge; the interior of the sphere will have a different potential.

Exercise:

Problem:

Can the potential of a nonuniformly charged sphere be the same as that of a point charge? Explain.

Problems

Exercise:

Problem:

A 0.500-cm-diameter plastic sphere, used in a static electricity demonstration, has a uniformly distributed 40.0-pC charge on its surface. What is the potential near its surface?

Solution:

$V = 144\ \mathrm{V}$

Exercise:

Problem:

How far from a $1.00\text{-}\mu\mathrm{C}$ point charge is the potential 100 V? At what distance is it $2.00\ \times\ 10^2$ V?

Exercise:

Problem:

If the potential due to a point charge is $5.00\ \times\ 10^2$ V at a distance of 15.0 m, what are the sign and magnitude of the charge?

Solution:

$V = \frac{kQ}{r} \rightarrow Q = 8.33\ \times\ 10^{-7}\ \mathrm{C};$
The charge is positive because the potential is positive.

Exercise:

Problem:

In nuclear fission, a nucleus splits roughly in half. (a) What is the potential 2.00×10^{-14} m from a fragment that has 46 protons in it? (b) What is the potential energy in MeV of a similarly charged fragment at this distance?

Exercise:

Problem:

A research Van de Graaff generator has a 2.00-m-diameter metal sphere with a charge of 5.00 mC on it. Assume the potential energy is zero at a reference point infinitely far away from the Van de Graaff. (a) What is the potential near its surface? (b) At what distance from its center is the potential 1.00 MV? (c) An oxygen atom with three missing electrons is released near the Van de Graaff generator. What is its kinetic energy in MeV when the atom is at the distance found in part b?

Solution:

a. $V = 45.0\,\text{MV}$;
b. $V = \frac{kQ}{r} \rightarrow r = 45.0\,\text{m}$;
c. $\Delta U = 132\,\text{MeV}$

Exercise:

Problem:

An electrostatic paint sprayer has a 0.200-m-diameter metal sphere at a potential of 25.0 kV that repels paint droplets onto a grounded object.

(a) What charge is on the sphere? (b) What charge must a 0.100-mg drop of paint have to arrive at the object with a speed of 10.0 m/s?

Exercise:

Problem:

(a) What is the potential between two points situated 10 cm and 20 cm from a $3.0\text{-}\mu\text{C}$ point charge? (b) To what location should the point at 20 cm be moved to increase this potential difference by a factor of two?

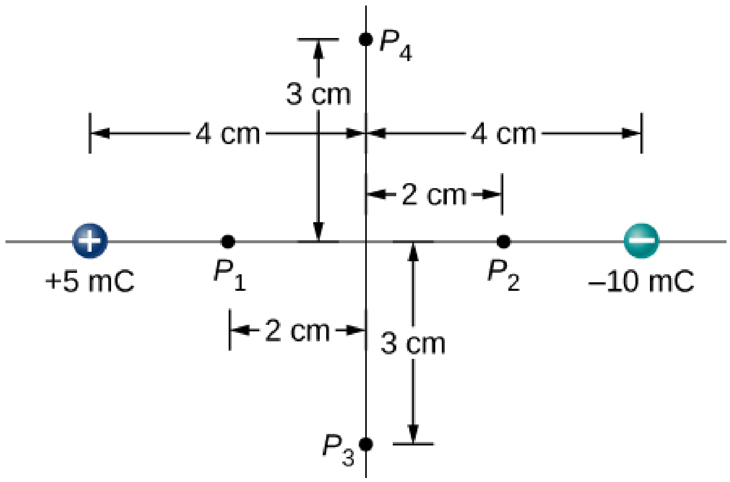
Solution:

$V = kQ/r$; a. Relative to origin, find the potential at each point and then calculate the difference.
$\Delta V = 135 \times 10^3$ V;
b. To double the potential difference, move the point from 20 cm to infinity; the potential at 20 cm is halfway between zero and that at 10 cm.

Exercise:

Problem:

Find the potential at points $P_1, P_2, P_3,$ and P_4 in the diagram due to the two given charges.



Exercise:

Problem:

Two charges $-2.0\ \mu\text{C}$ and $+2.0\ \mu\text{C}$ are separated by 4.0 cm on the *z*-axis symmetrically about origin, with the positive one uppermost. Two space points of interest P_1 and P_2 are located 3.0 cm and 30 cm from origin at an angle 30° with respect to the *z*-axis. Evaluate electric potentials at P_1 and P_2 in two ways: (a) Using the exact formula for point charges, and (b) using the approximate dipole potential formula.

Solution:

a. $V_{P1} = 7.4 \times 10^5\ \text{V}$
and $V_{P2} = 6.9 \times 10^3\ \text{V}$;
b. $V_{P1} = 6.9 \times 10^5\ \text{V}$ and $V_{P2} = 6.9 \times 10^3\ \text{V}$

Exercise:

Problem:

(a) Plot the potential of a uniformly charged 1-m rod with 1 C/m charge as a function of the perpendicular distance from the center. Draw your graph from $\text{s} = 0.1\ \text{m}$ to $\text{s} = 1.0\ \text{m}$. (b) On the same graph, plot the potential of a point charge with a 1-C charge at the origin. (c) Which potential is stronger near the rod? (d) What happens to the difference as the distance increases? Interpret your result.

Glossary

electric dipole
: system of two equal but opposite charges a fixed distance apart

electric dipole moment
: quantity defined as $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$ for all dipoles, where the vector points from the negative to positive charge

Determining Field from Potential

By the end of this section, you will be able to:

- Explain how to calculate the electric field in a system from the given potential
- Calculate the electric field in a given direction from a given potential
- Calculate the electric field throughout space from a given potential

Recall that we were able, in certain systems, to calculate the potential by integrating over the electric field. As you may already suspect, this means that we may calculate the electric field by taking derivatives of the potential, although going from a scalar to a vector quantity introduces some interesting wrinkles. We frequently need $\vec{\mathbf{E}}$ to calculate the force in a system; since it is often simpler to calculate the potential directly, there are systems in which it is useful to calculate *V* and then derive $\vec{\mathbf{E}}$ from it.

In general, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of $\vec{\mathbf{E}}$ and also in the direction of lower potential *V*. Furthermore, the magnitude of $\vec{\mathbf{E}}$ equals the rate of decrease of *V* with distance. The faster *V* decreases over distance, the greater the electric field. This gives us the following result.

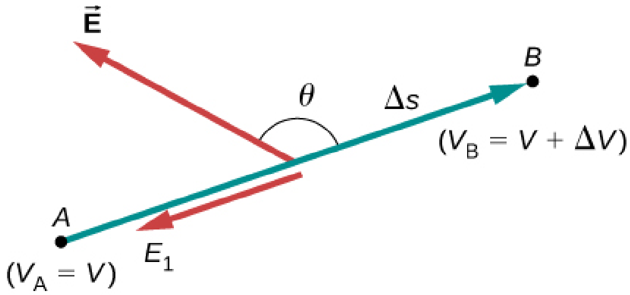
Note:
Relationship between Voltage and Uniform Electric Field
In equation form, the relationship between voltage and uniform electric field is
Equation:

$$E = -\frac{\Delta V}{\Delta s}$$

where Δs is the distance over which the change in potential ΔV takes place. The minus sign tells us that *E* points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

For continually changing potentials, ΔV and Δs become infinitesimals, and we need differential calculus to determine the electric field. As shown in [link], if we treat the distance Δs as very small so that the electric field is essentially constant over it, we find that
Equation:

$$E_s = -\frac{dV}{ds}.$$



The electric field component along the displacement Δs is given by $E = -\frac{\Delta V}{\Delta s}$. Note that A and B are assumed to be so close together that the field is constant along Δs.

Therefore, the electric field components in the Cartesian directions are given by

Note:
Equation:

$$E_x = -\frac{\partial V}{\partial x}, \; E_y = -\frac{\partial V}{\partial y}, \; E_z = -\frac{\partial V}{\partial z}.$$

This allows us to define the “grad” or “del” vector operator, which allows us to compute the gradient in one step. In Cartesian coordinates, it takes the form

Note:
Equation:

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}.$$

With this notation, we can calculate the electric field from the potential with

Note:
Equation:

$$\overrightarrow{\mathbf{E}} = -\vec{\nabla} V,$$

a process we call calculating the gradient of the potential.

If we have a system with either cylindrical or spherical symmetry, we only need to use the del operator in the appropriate coordinates:

Note:
Equation:

$$\text{Cylindrical: } \vec{\nabla} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\varphi} \frac{1}{r} \frac{\partial}{\partial \varphi} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

Note:
Equation:

$$\text{Spherical: } \vec{\nabla} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

Example:
Electric Field of a Point Charge
Calculate the electric field of a point charge from the potential.
Strategy
The potential is known to be $V = k\frac{q}{r}$, which has a spherical symmetry. Therefore, we use the spherical del operator in the formula $\overrightarrow{\mathbf{E}} = -\vec{\nabla} V$.
Solution
Performing this calculation gives us
Equation:

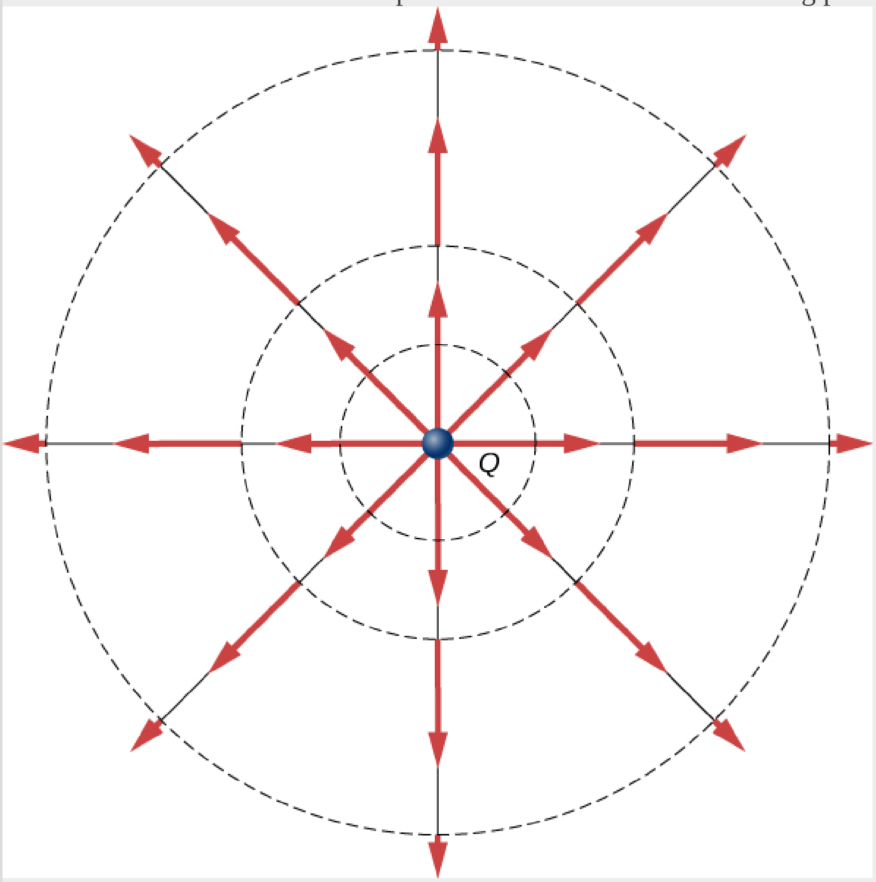
$$\overrightarrow{\mathbf{E}} = -\left(\hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}\right) k \frac{q}{r} = -kq \left(\hat{\mathbf{r}} \frac{\partial}{\partial r} \frac{1}{r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{1}{r} + \hat{\varphi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \frac{1}{r}\right).$$

This equation simplifies to
Equation:

$$\overrightarrow{\mathbf{E}} = -kq \left(\hat{\mathbf{r}} \frac{-1}{r^2} + \hat{\theta} 0 + \hat{\varphi} 0\right) = k \frac{q}{r^2} \hat{\mathbf{r}}$$

as expected.
Significance

We not only obtained the equation for the electric field of a point particle that we've seen before, we also have a demonstration that $\vec{\mathbf{E}}$ points in the direction of decreasing potential, as shown in [link].

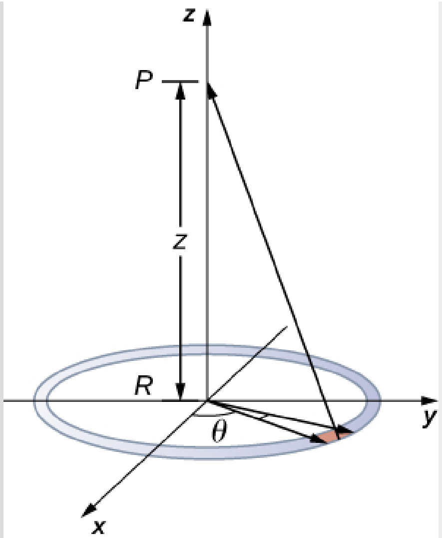


Electric field vectors inside and outside a uniformly charged sphere.

Example:

Electric Field of a Ring of Charge

Use the potential found in [link] to calculate the electric field along the axis of a ring of charge ([link]).



We want to calculate the electric field from the electric potential due to a ring charge.

Strategy
In this case, we are only interested in one dimension, the z-axis. Therefore, we use $E_z = -\frac{\partial V}{\partial z}$ with the potential $V = k\frac{q_{\text{tot}}}{\sqrt{z^2+R^2}}$ found previously.

Solution
Taking the derivative of the potential yields

Equation:

$$E_z = -\frac{\partial}{\partial z}\frac{kq_{\text{tot}}}{\sqrt{z^2+R^2}} = k\frac{q_{\text{tot}}z}{(z^2+R^2)^{3/2}}.$$

Significance
Again, this matches the equation for the electric field found previously. It also demonstrates a system in which using the full del operator is not necessary.

Note:
Exercise:

Problem:

Check Your Understanding Which coordinate system would you use to calculate the electric field of a dipole?

Solution:

Any, but cylindrical is closest to the symmetry of a dipole.

Summary

- Just as we may integrate over the electric field to calculate the potential, we may take the derivative of the potential to calculate the electric field.
- This may be done for individual components of the electric field, or we may calculate the entire electric field vector with the gradient operator.

Conceptual Questions

Exercise:

Problem:

If the electric field is zero throughout a region, must the electric potential also be zero in that region?

Solution:

No. It will be constant, but not necessarily zero.

Exercise:

Problem:

Explain why knowledge of $\vec{\mathbf{E}}(x, y, z)$ is not sufficient to determine *V*(*x*,*y*,*z*). What about the other way around?

Problems

Exercise:

Problem:

Throughout a region, equipotential surfaces are given by $z = \text{constant}$. The surfaces are equally spaced with $V = 100\text{ V}$ for $z = 0.00\text{ m}$, $V = 200\text{ V}$ for $z = 0.50\text{ m}$, $V = 300\text{ V}$ for $z = 1.00\text{ m}$. What is the electric field in this region?

Solution:

The problem is describing a uniform field, so $E = 200\text{ V/m}$ in the –*z*-direction.

Exercise:

Problem:

In a particular region, the electric potential is given by $V = -xy^2z + 4xy$. What is the electric field in this region?

Exercise:

Problem: Calculate the electric field of an infinite line charge, throughout space.

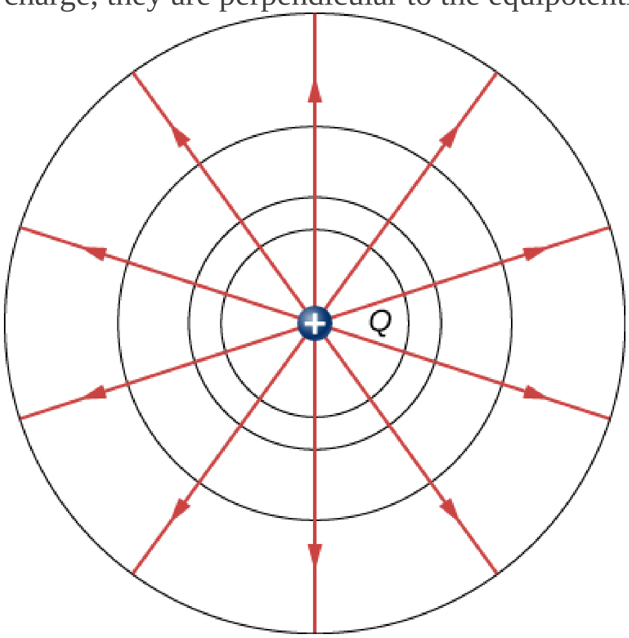
Solution:

Apply $\overrightarrow{\mathbf{E}} = -\vec{\nabla} V$ with $\vec{\nabla} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \widehat{\varphi} \frac{1}{r} \frac{\partial}{\partial \varphi} + \widehat{\mathbf{z}} \frac{\partial}{\partial z}$ to the potential calculated earlier, $V = -2k\lambda \ln s :\ \overrightarrow{\mathbf{E}} = 2k\lambda \frac{1}{r} \hat{\mathbf{r}}$ as expected.

Equipotential Surfaces and Conductors
By the end of this section, you will be able to:

- Define equipotential surfaces and equipotential lines
- Explain the relationship between equipotential lines and electric field lines
- Map equipotential lines for one or two point charges
- Describe the potential of a conductor
- Compare and contrast equipotential lines and elevation lines on topographic maps

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. This is not surprising, since the two concepts are related. Consider [link], which shows an isolated positive point charge and its electric field lines, which radiate out from a positive charge and terminate on negative charges. We use red arrows to represent the magnitude and direction of the electric field, and we use black lines to represent places where the electric potential is constant. These are called **equipotential surfaces** in three dimensions, or **equipotential lines** in two dimensions. The term *equipotential* is also used as a noun, referring to an equipotential line or surface. The potential for a point charge is the same anywhere on an imaginary sphere of radius r surrounding the charge. This is true because the potential for a point charge is given by $V = kq/r$ and thus has the same value at any point that is a given distance r from the charge. An equipotential sphere is a circle in the two-dimensional view of [link]. Because the electric field lines point radially away from the charge, they are perpendicular to the equipotential lines.



An isolated point charge Q with its electric field lines in red and equipotential lines in black. The potential is the same along each equipotential line, meaning that no work is required to move a charge anywhere along one of those lines. Work is needed to move a charge from one

is needed to move a charge from one equipotential line to another. Equipotential lines are perpendicular to electric field lines in every case. For a three-dimensional version, explore the first media link.

It is important to note that *equipotential lines are always perpendicular to electric field lines*. No work is required to move a charge along an equipotential, since $\Delta V = 0$. Thus, the work is

Equation:

$$W = -\Delta U = -q\Delta V = 0.$$

Work is zero if the direction of the force is perpendicular to the displacement. Force is in the same direction as *E*, so motion along an equipotential must be perpendicular to *E*. More precisely, work is related to the electric field by

Equation:

$$W = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{d}} = q\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{d}} = qEd\cos\theta = 0.$$

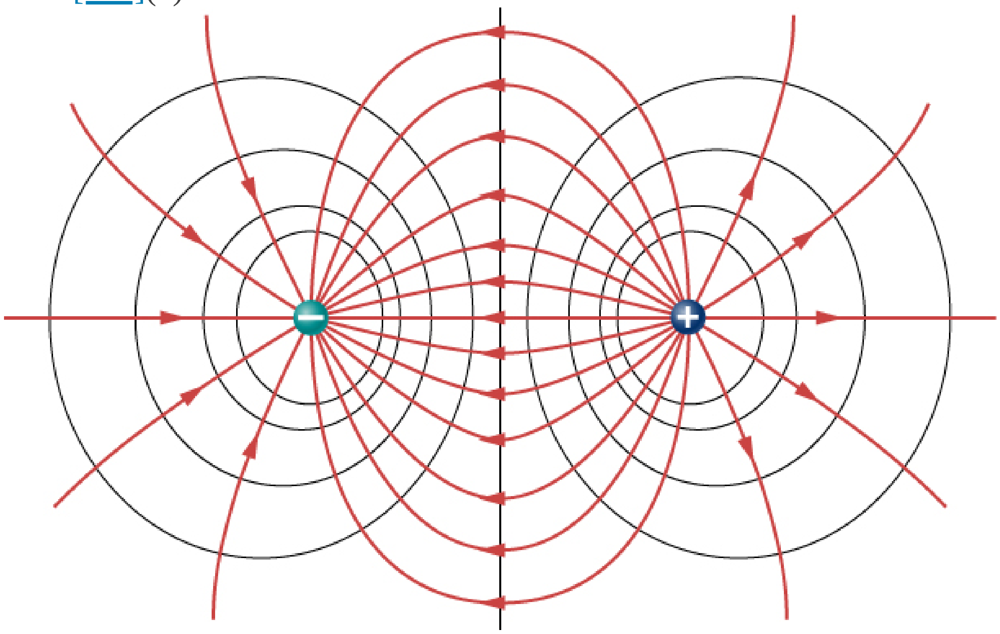
Note that in this equation, *E* and *F* symbolize the magnitudes of the electric field and force, respectively. Neither *q* nor *E* is zero; *d* is also not zero. So $\cos\theta$ must be 0, meaning θ must be 90º. In other words, motion along an equipotential is perpendicular to *E*.

One of the rules for static electric fields and conductors is that the electric field must be perpendicular to the surface of any conductor. This implies that a *conductor is an equipotential surface in static situations*. There can be no voltage difference across the surface of a conductor, or charges will flow. One of the uses of this fact is that a conductor can be fixed at what we consider zero volts by connecting it to the earth with a good conductor—a process called **grounding**. Grounding can be a useful safety tool. For example, grounding the metal case of an electrical appliance ensures that it is at zero volts relative to Earth.

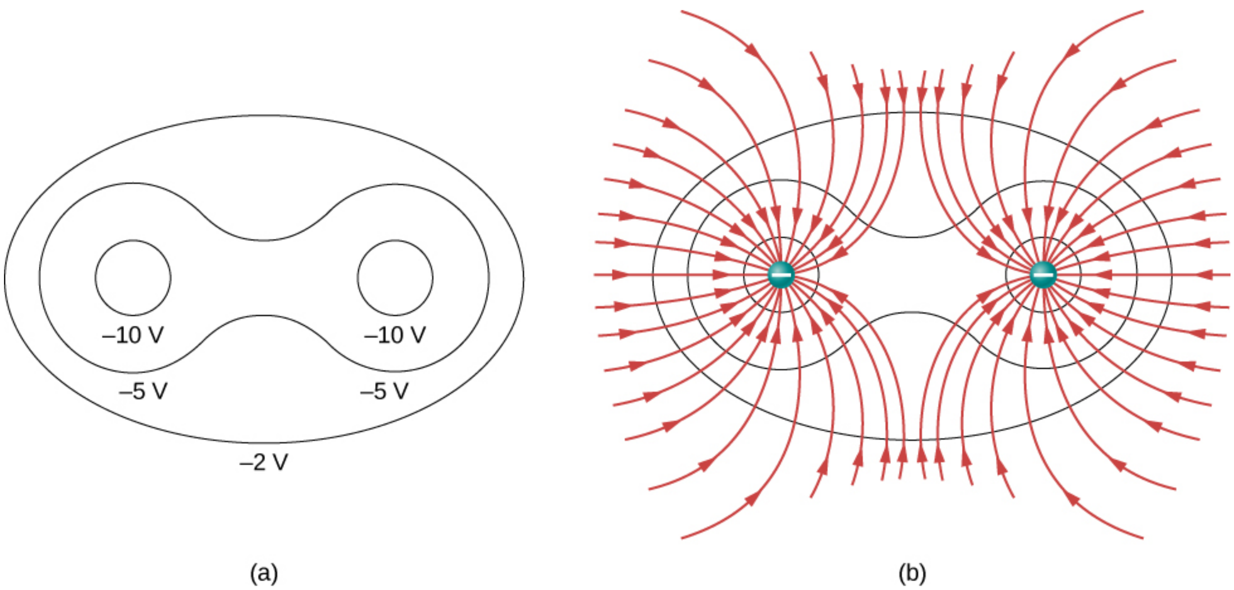
Because a conductor is an equipotential, it can replace any equipotential surface. For example, in [link], a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its center.

[link] shows the electric field and equipotential lines for two equal and opposite charges. Given the electric field lines, the equipotential lines can be drawn simply by making them perpendicular to the electric field lines. Conversely, given the equipotential lines, as in [link]

(a), the electric field lines can be drawn by making them perpendicular to the equipotentials, as in [link](b).



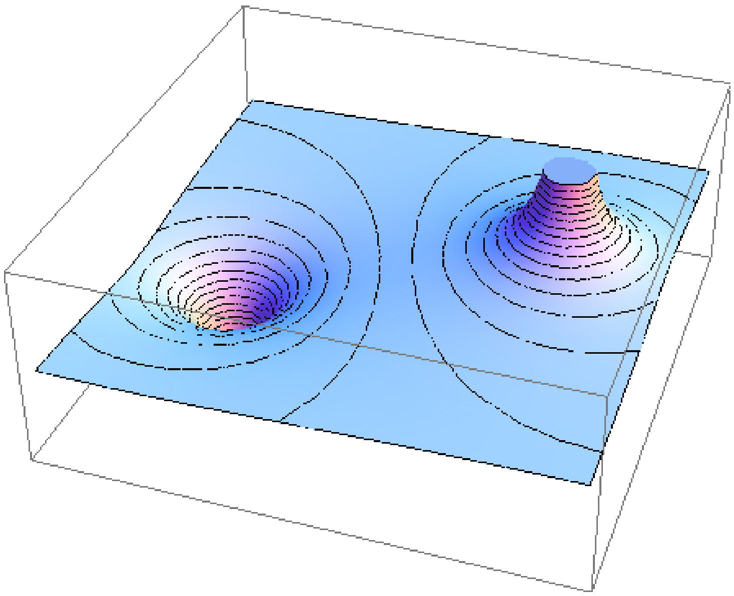
The electric field lines and equipotential lines for two equal but opposite charges. The equipotential lines can be drawn by making them perpendicular to the electric field lines, if those are known. Note that the potential is greatest (most positive) near the positive charge and least (most negative) near the negative charge. For a three-dimensional version, explore the first media link.



(a) These equipotential lines might be measured with a voltmeter in a laboratory

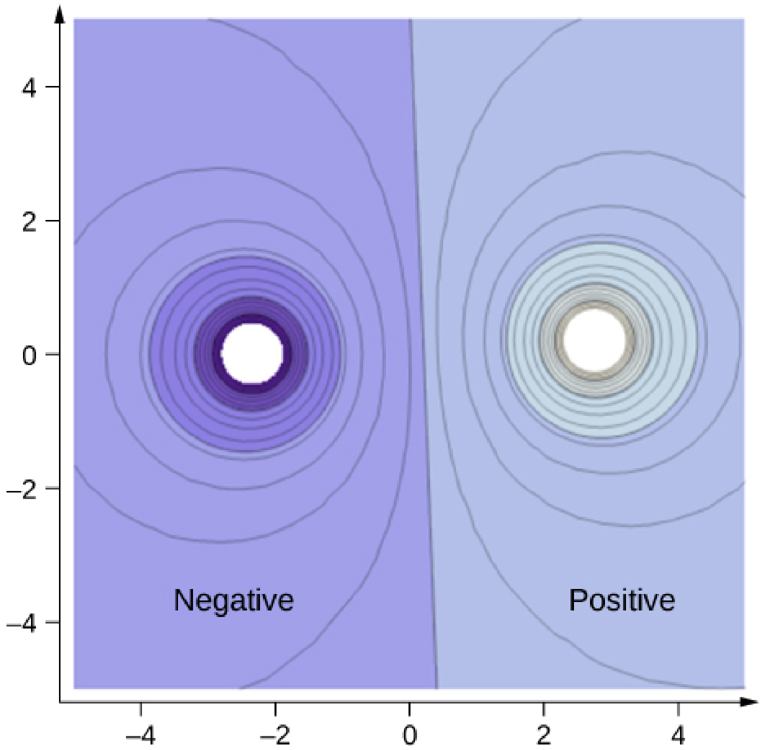
experiment. (b) The corresponding electric field lines are found by drawing them perpendicular to the equipotentials. Note that these fields are consistent with two equal negative charges. For a three-dimensional version, play with the first media link.

To improve your intuition, we show a three-dimensional variant of the potential in a system with two opposing charges. [link] displays a three-dimensional map of electric potential, where lines on the map are for equipotential surfaces. The hill is at the positive charge, and the trough is at the negative charge. The potential is zero far away from the charges. Note that the cut off at a particular potential implies that the charges are on conducting spheres with a finite radius.



Electric potential map of two opposite charges of equal magnitude on conducting spheres. The potential is negative near the negative charge and positive near the positive charge.

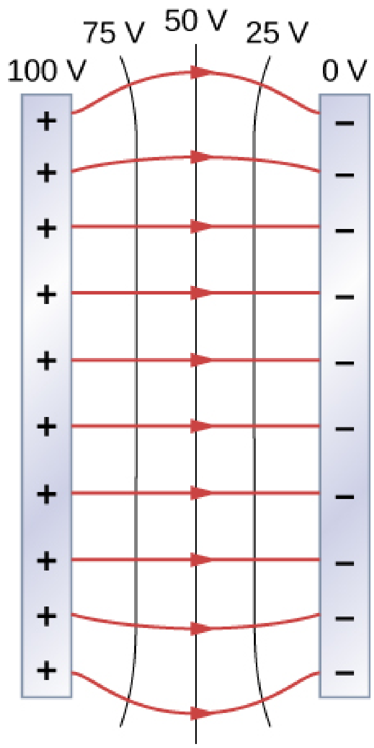
A two-dimensional map of the cross-sectional plane that contains both charges is shown in [link]. The line that is equidistant from the two opposite charges corresponds to zero potential, since at the points on the line, the positive potential from the positive charge cancels the negative potential from the negative charge. Equipotential lines in the cross-sectional plane are closed loops, which are not necessarily circles, since at each point, the net potential is the sum of the potentials from each charge.



A cross-section of the electric potential map of two opposite charges of equal magnitude. The potential is negative near the negative charge and positive near the positive charge.

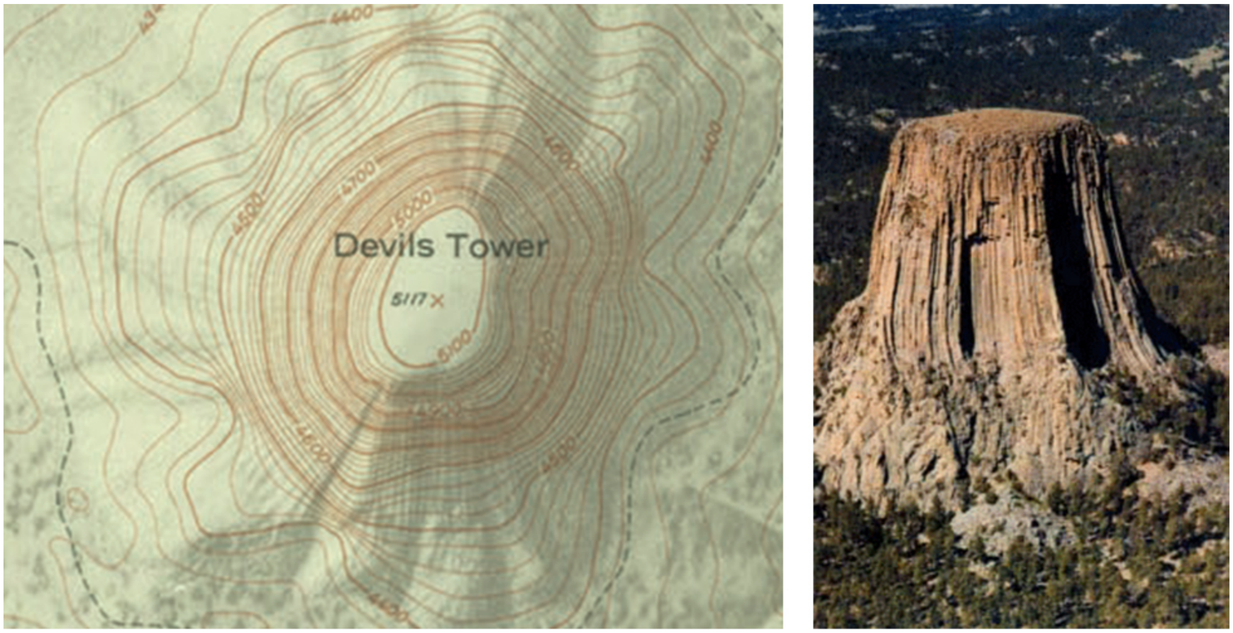
Note:
View this simulation to observe and modify the equipotential surfaces and electric fields for many standard charge configurations. There’s a lot to explore.

One of the most important cases is that of the familiar parallel conducting plates shown in [link]. Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.



The electric field and equipotential lines between two metal plates. Note that the electric field is perpendicular to the equipotentials and hence normal to the plates at their surface as well as in the center of the region between them.

Consider the parallel plates in [link]. These have equipotential lines that are parallel to the plates in the space between and evenly spaced. An example of this (with sample values) is given in [link]. We could draw a similar set of equipotential isolines for gravity on the hill shown in [link]. If the hill has any extent at the same slope, the isolines along that extent would be parallel to each other. Furthermore, in regions of constant slope, the isolines would be evenly spaced. An example of real topographic lines is shown in [link].



(a) (b)

A topographical map along a ridge has roughly parallel elevation lines, similar to the equipotential lines in [link]. (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Example:
Calculating Equipotential Lines
You have seen the equipotential lines of a point charge in [link]. How do we calculate them? For example, if we have a $+10\text{-nC}$ charge at the origin, what are the equipotential surfaces at which the potential is (a) 100 V, (b) 50 V, (c) 20 V, and (d) 10 V?

Strategy
Set the equation for the potential of a point charge equal to a constant and solve for the remaining variable(s). Then calculate values as needed.

Solution
In $V = k\frac{q}{r}$, let V be a constant. The only remaining variable is r; hence, $r = k\frac{q}{V} = \text{constant}$. Thus, the equipotential surfaces are spheres about the origin. Their locations are:

a. $r = k\frac{q}{V} = \left(8.99 \times 10^9\,\text{Nm}^2/\text{C}^2\right)\frac{(10 \times 10^{-9}\,\text{C})}{100\,\text{V}} = 0.90\,\text{m};$
b. $r = k\frac{q}{V} = \left(8.99 \times 10^9\,\text{Nm}^2/\text{C}^2\right)\frac{(10 \times 10^{-9}\,\text{C})}{50\,\text{V}} = 1.8\,\text{m};$

c. $r = k\frac{q}{V} = \left(8.99 \times 10^9 \, \text{Nm}^2/\text{C}^2\right) \frac{\left(10 \times 10^{-9} \, \text{C}\right)}{20 \, \text{V}} = 4.5 \, \text{m};$

d. $r = k\frac{q}{V} = \left(8.99 \times 10^9 \, \text{Nm}^2/\text{C}^2\right) \frac{\left(10 \times 10^{-9} \, \text{C}\right)}{10 \, \text{V}} = 9.0 \, \text{m}.$

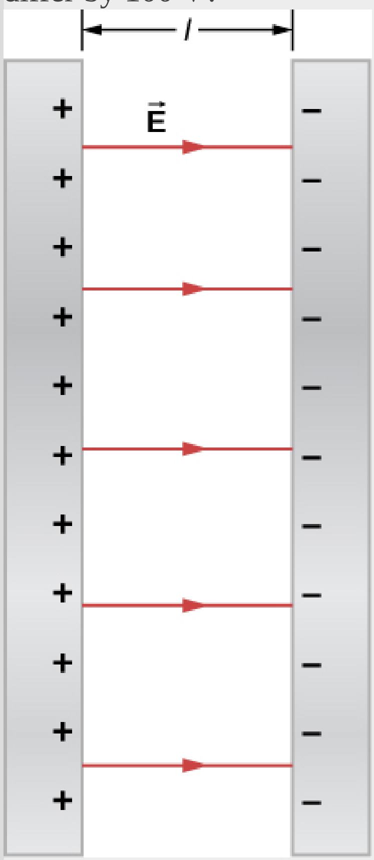
Significance

This means that equipotential surfaces around a point charge are spheres of constant radius, as shown earlier, with well-defined locations.

Example:

Potential Difference between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7} \, \text{C/m}^2$, as shown in [link]. The separation between the plates is $l = 6.50 \, \text{mm}$. (a) What is the electric field between the plates? (b) What is the potential difference between the plates? (c) What is the distance between equipotential planes which differ by 100 V?



The electric field between oppositely charged parallel plates.

A portion is released
at the positive plate.

Strategy
(a) Since the plates are described as “large” and the distance between them is not, we will approximate each of them as an infinite plane, and apply the result from Gauss’s law in the previous chapter.
(b) Use $\Delta V_{AB} = -\int_A^B \overrightarrow{\mathbf{E}} \cdot d\vec{\mathbf{l}}$.
(c) Since the electric field is constant, find the ratio of 100 V to the total potential difference; then calculate this fraction of the distance.
Solution

a. The electric field is directed from the positive to the negative plate as shown in the figure, and its magnitude is given by
   Equation:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{6.81 \times 10^{-7}\,\mathrm{C/m^2}}{8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2}} = 7.69 \times 10^4\,\mathrm{V/m}.$$

b. To find the potential difference ΔV between the plates, we use a path from the negative to the positive plate that is directed against the field. The displacement vector $d\vec{\mathbf{l}}$ and the electric field $\vec{\mathbf{E}}$ are antiparallel so $\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -E\,dl$. The potential difference between the positive plate and the negative plate is then
   Equation:

$$\Delta V = -\int E \cdot dl = E \int dl = El = (7.69 \times 10^4\,\mathrm{V/m})(6.50 \times 10^{-3}\,\mathrm{m}) = 500\,\mathrm{V}.$$

c. The total potential difference is 500 V, so 1/5 of the distance between the plates will be the distance between 100-V potential differences. The distance between the plates is 6.5 mm, so there will be 1.3 mm between 100-V potential differences.

Significance
You have now seen a numerical calculation of the locations of equipotentials between two charged parallel plates.

Note:
Exercise:

Problem:

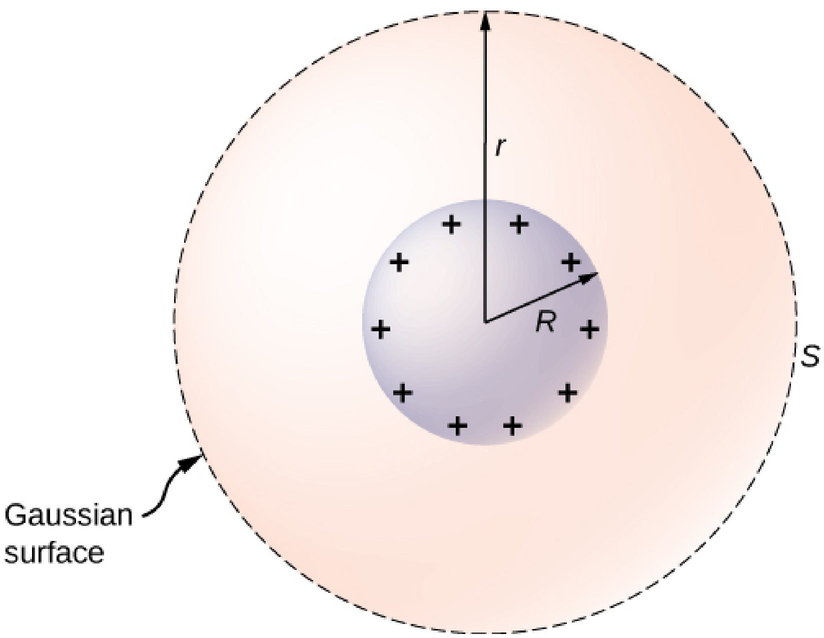
Check Your Understanding What are the equipotential surfaces for an infinite line charge?

Solution:

infinite cylinders of constant radius, with the line charge as the axis

Distribution of Charges on Conductors

In [link] with a point charge, we found that the equipotential surfaces were in the form of spheres, with the point charge at the center. Given that a conducting sphere in electrostatic equilibrium is a spherical equipotential surface, we should expect that we could replace one of the surfaces in [link] with a conducting sphere and have an identical solution outside the sphere. Inside will be rather different, however.



An isolated conducting sphere.

To investigate this, consider the isolated conducting sphere of [link] that has a radius R and an excess charge q. To find the electric field both inside and outside the sphere, note that the sphere is isolated, so its surface change distribution and the electric field of that distribution

are spherically symmetric. We can therefore represent the field as $\vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$. To calculate *E(r)*, we apply Gauss's law over a closed spherical surface *S* of radius *r* that is concentric with the conducting sphere. Since *r* is constant and $\widehat{\mathbf{n}} = \hat{\mathbf{r}}$ on the sphere,
Equation:

$$\oint_S \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}}\, da = E(r) \oint da = E(r)\, 4\pi r^2.$$

For $r \,<\, R$, *S* is within the conductor, so recall from our previous study of Gauss's law that $q_{\text{enc}} = 0$ and Gauss's law gives $E\,(r) = 0$, as expected inside a conductor at equilibrium. If $r \,>\, R$, *S* encloses the conductor so $q_{\text{enc}} = q$. From Gauss's law,
Equation:

$$E(r)\, 4\pi r^2 = \frac{q}{\varepsilon_0}.$$

The electric field of the sphere may therefore be written as
Equation:

$$\begin{array}{lcll} E & = & 0 & (r < R), \\ E & = & \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\hat{\mathbf{r}} & (r \geq R). \end{array}$$

As expected, in the region $r \geq R$, the electric field due to a charge *q* placed on an isolated conducting sphere of radius *R* is identical to the electric field of a point charge *q* located at the center of the sphere.

To find the electric potential inside and outside the sphere, note that for $r \geq R$, the potential must be the same as that of an isolated point charge *q* located at $r = 0$,
Equation:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r}\;(r \geq R)$$

simply due to the similarity of the electric field.

For $r < R, E = 0$, so *V*(*r*) is constant in this region. Since $V(R) = q/4\pi\varepsilon_0 R$,
Equation:

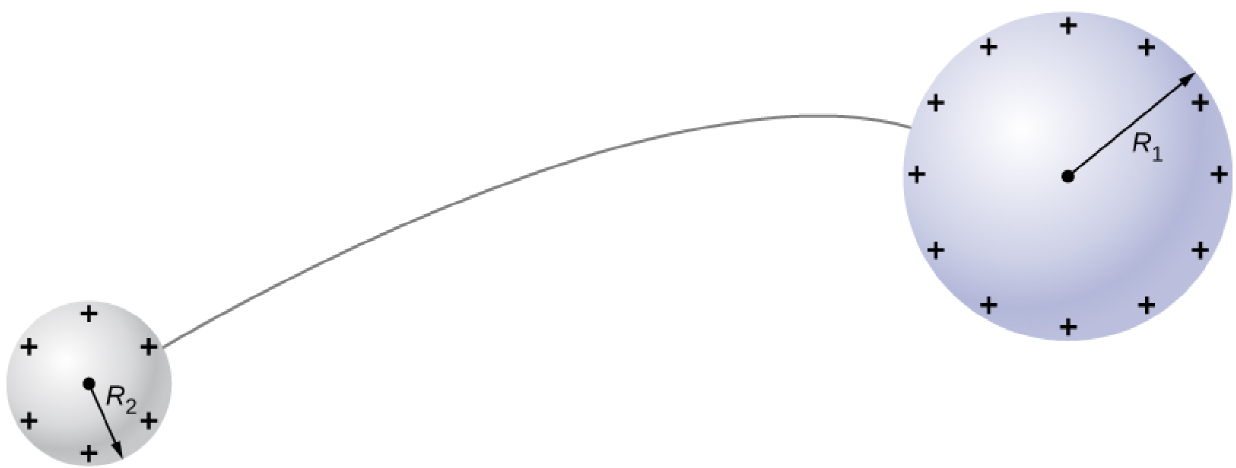
$$V(r) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{R}\;(r < R).$$

We will use this result to show that

Equation:

$$\sigma_1 R_1 = \sigma_2 R_2,$$

for two conducting spheres of radii R_1 and R_2, with surface charge densities σ_1 and σ_2 respectively, that are connected by a thin wire, as shown in [link]. The spheres are sufficiently separated so that each can be treated as if it were isolated (aside from the wire). Note that the connection by the wire means that this entire system must be an equipotential.



Two conducting spheres are connected by a thin conducting wire.

We have just seen that the electrical potential at the surface of an isolated, charged conducting sphere of radius *R* is

Equation:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}.$$

Now, the spheres are connected by a conductor and are therefore at the same potential; hence

Equation:

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_2},$$

and

Equation:

$$\frac{q_1}{R_1} = \frac{q_2}{R_2}.$$

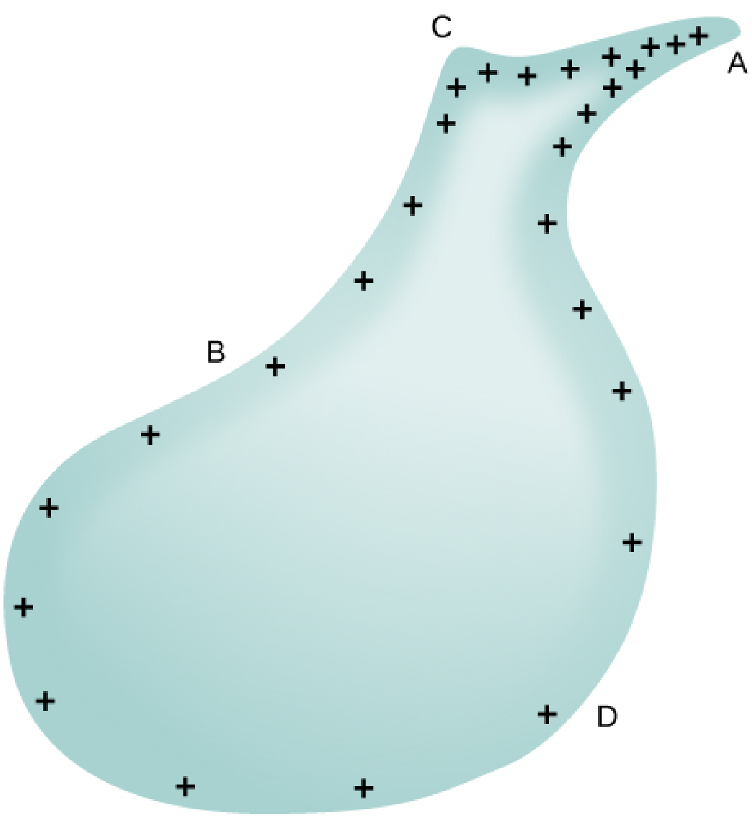
The net charge on a conducting sphere and its surface charge density are related by $q = \sigma(4\pi R^2)$. Substituting this equation into the previous one, we find

Equation:

$$\sigma_1 R_1 = \sigma_2 R_2.$$

Obviously, two spheres connected by a thin wire do not constitute a typical conductor with a variable radius of curvature. Nevertheless, this result does at least provide a qualitative idea of how charge density varies over the surface of a conductor. The equation indicates that where the radius of curvature is large (points *B* and *D* in [link]), σ and *E* are small.

Similarly, the charges tend to be denser where the curvature of the surface is greater, as demonstrated by the charge distribution on oddly shaped metal ([link]). The surface charge density is higher at locations with a small radius of curvature than at locations with a large radius of curvature.



The surface charge density and the electric field of a conductor are greater at regions with smaller radii of curvature.

A practical application of this phenomenon is the lightning rod, which is simply a grounded metal rod with a sharp end pointing upward. As positive charge accumulates in the ground due to a negatively charged cloud overhead, the electric field around the sharp point gets very large. When the field reaches a value of approximately $3.0 \times 10^6 \text{ N/C}$ (the *dielectric strength* of the air), the free ions in the air are accelerated to such high energies that their collisions with air molecules actually ionize the molecules. The resulting free electrons in the air then flow through the rod to Earth, thereby neutralizing some of the positive charge. This keeps the electric field between the cloud and the ground from getting large enough to produce a lightning bolt in the region around the rod.

An important application of electric fields and equipotential lines involves the heart. The heart relies on electrical signals to maintain its rhythm. The movement of electrical signals causes the chambers of the heart to contract and relax. When a person has a heart attack, the movement of these electrical signals may be disturbed. An artificial pacemaker and a defibrillator can be used to initiate the rhythm of electrical signals. The equipotential lines around the heart, the thoracic region, and the axis of the heart are useful ways of monitoring the structure and functions of the heart. An electrocardiogram (ECG) measures the small electric signals being generated during the activity of the heart.

Note:
Play around with this simulation to move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more.

Summary

- An equipotential surface is the collection of points in space that are all at the same potential. Equipotential lines are the two-dimensional representation of equipotential surfaces.
- Equipotential surfaces are always perpendicular to electric field lines.
- Conductors in static equilibrium are equipotential surfaces.
- Topographic maps may be thought of as showing gravitational equipotential lines.

Conceptual Questions

Exercise:

Problem:

If two points are at the same potential, are there any electric field lines connecting them?

Solution:

no

Exercise:

Problem:

Suppose you have a map of equipotential surfaces spaced 1.0 V apart. What do the distances between the surfaces in a particular region tell you about the strength of the $\vec{\mathbf{E}}$ in that region?

Exercise:

Problem: Is the electric potential necessarily constant over the surface of a conductor?

Solution:

No; it might not be at electrostatic equilibrium.

Exercise:

Problem:

Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all of the conduction electrons in a conductor are on the surface?

Exercise:

Problem: Can a positively charged conductor be at a negative potential? Explain.

Solution:

Yes. It depends on where the zero reference for potential is. (Though this might be unusual.)

Exercise:

Problem: Can equipotential surfaces intersect?

Problems

Exercise:

Problem:

Two very large metal plates are placed 2.0 cm apart, with a potential difference of 12 V between them. Consider one plate to be at 12 V, and the other at 0 V. (a) Sketch the equipotential surfaces for 0, 4, 8, and 12 V. (b) Next sketch in some electric field lines, and confirm that they are perpendicular to the equipotential lines.

Exercise:

Problem:

A very large sheet of insulating material has had an excess of electrons placed on it to a surface charge density of $-3.00\ \mathrm{nC/m^2}$. (a) As the distance from the sheet increases, does the potential increase or decrease? Can you explain why without any calculations? Does the location of your reference point matter? (b) What is the shape of the equipotential surfaces? (c) What is the spacing between surfaces that differ by 1.00 V?

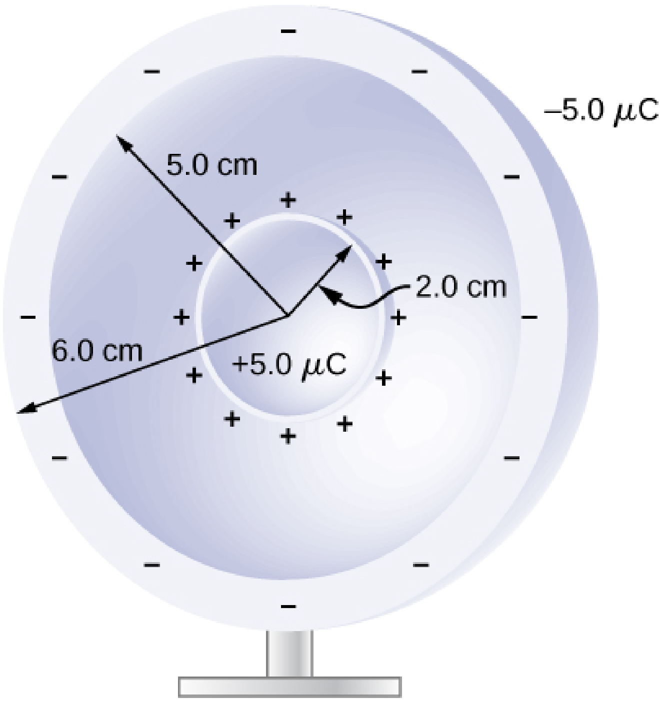
Solution:

a. increases; the constant (negative) electric field has this effect, the reference point only matters for magnitude; b. they are planes parallel to the sheet; c. 0.006 m/V

Exercise:

Problem:

A metallic sphere of radius 2.0 cm is charged with $+5.0\text{-}\mu\mathrm{C}$ charge, which spreads on the surface of the sphere uniformly. The metallic sphere stands on an insulated stand and is surrounded by a larger metallic spherical shell, of inner radius 5.0 cm and outer radius 6.0 cm. Now, a charge of $-5.0\text{-}\mu\mathrm{C}$ is placed on the inside of the spherical shell, which spreads out uniformly on the inside surface of the shell. If potential is zero at infinity, what is the potential of (a) the spherical shell, (b) the sphere, (c) the space between the two, (d) inside the sphere, and (e) outside the shell?



Exercise:

Problem:

Two large charged plates of charge density $\pm 30\ \mu\mathrm{C/m}^2$ face each other at a separation of 5.0 mm. (a) Find the electric potential everywhere. (b) An electron is released from rest at the negative plate; with what speed will it strike the positive plate?

Solution:

a. from the previous chapter, the electric field has magnitude $\frac{\sigma}{\varepsilon_0}$ in the region between the plates and zero outside; defining the negatively charged plate to be at the origin and zero potential, with the positively charged plate located at $+5\ \mathrm{mm}$ in the *z*-direction, $V = 1.7 \times 10^4$ V so the potential is 0 for $z < 0,\ 1.7 \times 10^4\ \mathrm{V}\left(\frac{z}{5\ \mathrm{mm}}\right)$ for $0 \leq z \leq 5\ \mathrm{mm}$, 1.7×10^4 V for $z > 5\ \mathrm{mm}$;
b. $qV = \frac{1}{2}mv^2 \rightarrow v = 7.7 \times 10^7\ \mathrm{m/s}$

Exercise:

Problem:

A long cylinder of aluminum of radius *R* meters is charged so that it has a uniform charge per unit length on its surface of λ.

(a) Find the electric field inside and outside the cylinder. (b) Find the electric potential inside and outside the cylinder. (c) Plot electric field and electric potential as a function of distance from the center of the rod.

Exercise:

Problem:

Two parallel plates 10 cm on a side are given equal and opposite charges of magnitude $5.0\ \times\ 10^{-9}\ \text{C}$. The plates are 1.5 mm apart. What is the potential difference between the plates?

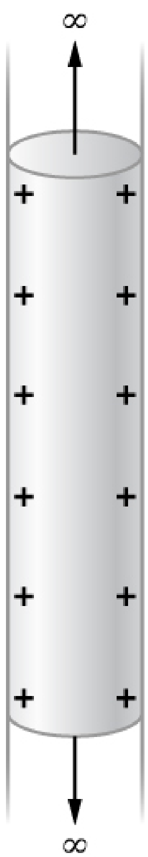
Solution:

$V = 85\ \text{V}$

Exercise:

Problem:

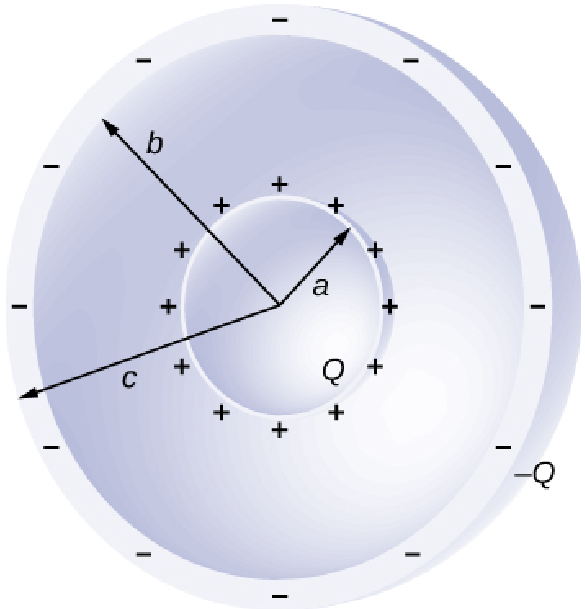
The surface charge density on a long straight metallic pipe is σ. What is the electric potential outside and inside the pipe? Assume the pipe has a diameter of $2a$.



Exercise:

Problem:

Concentric conducting spherical shells carry charges *Q* and –*Q*, respectively. The inner shell has negligible thickness. What is the potential difference between the shells?



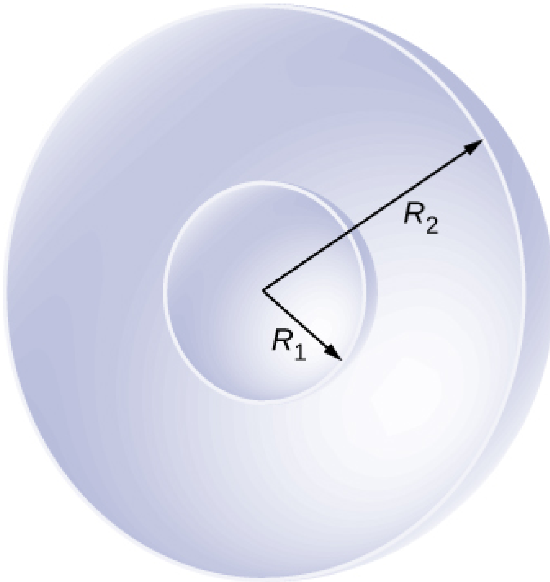
Solution:

In the region $a \le r \le b$, $\vec{\mathbf{E}} = \frac{kQ}{r^2}\hat{\mathbf{r}}$, and E is zero elsewhere; hence, the potential difference is $V = kQ\left(\frac{1}{a} - \frac{1}{b}\right)$.

Exercise:

Problem:

Shown below are two concentric spherical shells of negligible thicknesses and radii R_1 and R_2. The inner and outer shell carry net charges q_1 and q_2, respectively, where both q_1 and q_2 are positive. What is the electric potential in the regions (a) $r < R_1$, (b) $R_1 < r < R_2$, and (c) $r > R_2$?



Exercise:

Problem:

A solid cylindrical conductor of radius *a* is surrounded by a concentric cylindrical shell of inner radius *b*. The solid cylinder and the shell carry charges *Q* and –*Q*, respectively. Assuming that the length *L* of both conductors is much greater than *a* or *b*, what is the potential difference between the two conductors?

Solution:

From previous results $V_P - V_R = -2k\lambda \ln \frac{s_P}{s_R}$., note that *b* is a very convenient location to define the zero level of potential: $\Delta V = -2k\frac{Q}{L}\ln\frac{a}{b}$.

Glossary

equipotential line
: two-dimensional representation of an equipotential surface

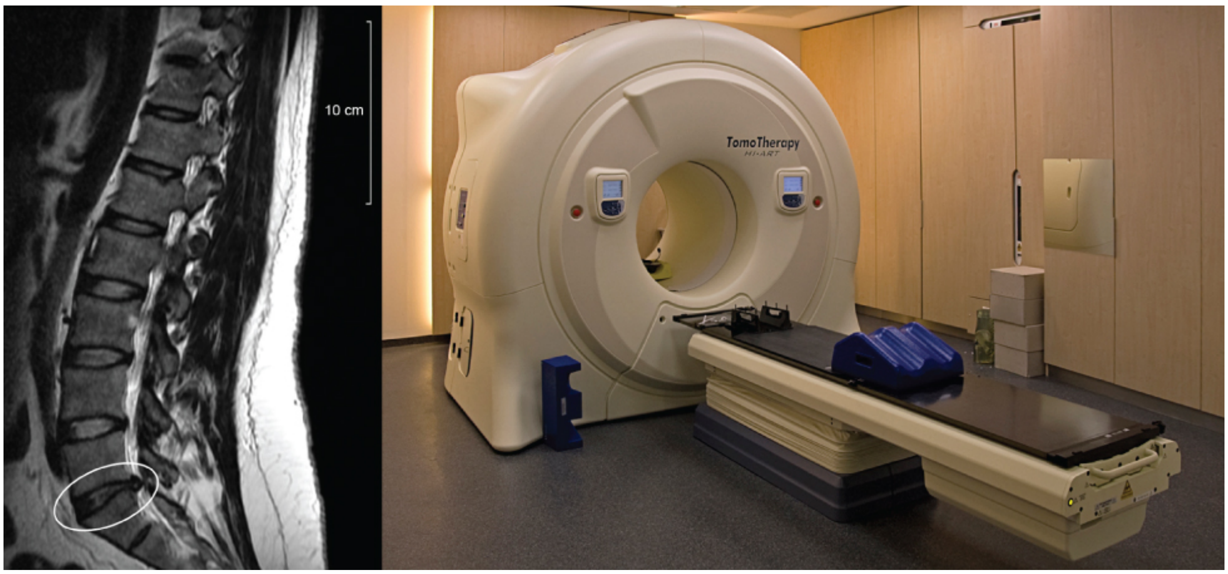
equipotential surface
: surface (usually in three dimensions) on which all points are at the same potential

grounding
: process of attaching a conductor to the earth to ensure that there is no potential difference between it and Earth

Introduction
class="introduction"

Magnetic resonance imaging (MRI) uses superconducting magnets and produces high-resolution images without the danger of radiation. The image on the left shows the spacing of vertebrae along a human spinal column, with the circle indicating where the vertebrae are too close due to a ruptured disc. On the right is a picture of the MRI instrument, which surrounds the patient on all sides. A large amount of electrical current is

required to operate the electromagnets (credit right: modification of work by “digital cat”/Flickr).



In this chapter, we study the electrical current through a material, where the electrical current is the rate of flow of charge. We also examine a characteristic of materials known as the resistance. Resistance is a measure of how much a material impedes the flow of charge, and it will be shown that the resistance depends on temperature. In general, a good conductor, such as copper, gold, or silver, has very low resistance. Some materials, called superconductors, have zero resistance at very low temperatures.

High currents are required for the operation of electromagnets. Superconductors can be used to make electromagnets that are 10 times stronger than the strongest conventional electromagnets. These superconducting magnets are used in the construction of magnetic resonance imaging (MRI) devices that can be used to make high-resolution images of the human body. The chapter-opening picture shows an MRI image of the vertebrae of a human subject and the MRI device itself.

Superconducting magnets have many other uses. For example, superconducting magnets are used in the Large Hadron Collider (LHC) to curve the path of protons in the ring.

Electrical Current
By the end of this section, you will be able to:

- Describe an electrical current
- Define the unit of electrical current
- Explain the direction of current flow

Up to now, we have considered primarily static charges. When charges did move, they were accelerated in response to an electrical field created by a voltage difference. The charges lost potential energy and gained kinetic energy as they traveled through a potential difference where the electrical field did work on the charge.

Although charges do not require a material to flow through, the majority of this chapter deals with understanding the movement of charges through a material. The rate at which the charges flow past a location—that is, the amount of charge per unit time—is known as the *electrical current*. When charges flow through a medium, the current depends on the voltage applied, the material through which the charges flow, and the state of the material. Of particular interest is the motion of charges in a conducting wire. In previous chapters, charges were accelerated due to the force provided by an electrical field, losing potential energy and gaining kinetic energy. In this chapter, we discuss the situation of the force provided by an electrical field in a conductor, where charges lose kinetic energy to the material reaching a constant velocity, known as the "*drift velocity*." This is analogous to an object falling through the atmosphere and losing kinetic energy to the air, reaching a constant terminal velocity.

If you have ever taken a course in first aid or safety, you may have heard that in the event of electric shock, it is the current, not the voltage, which is the important factor on the severity of the shock and the amount of damage to the human body. Current is measured in units called amperes; you may have noticed that circuit breakers in your home and fuses in your car are rated in amps (or amperes). But what is the ampere and what does it measure?

Defining Current and the Ampere

Electrical current is defined to be the rate at which charge flows. When there is a large current present, such as that used to run a refrigerator, a large amount of charge moves through the wire in a small amount of time. If the current is small, such as that used to operate a handheld calculator, a small amount of charge moves through the circuit over a long period of time.

Note:
Electrical Current
The average electrical current *I* is the rate at which charge flows,
Equation:

$$I_{\text{ave}} = \frac{\Delta Q}{\Delta t},$$

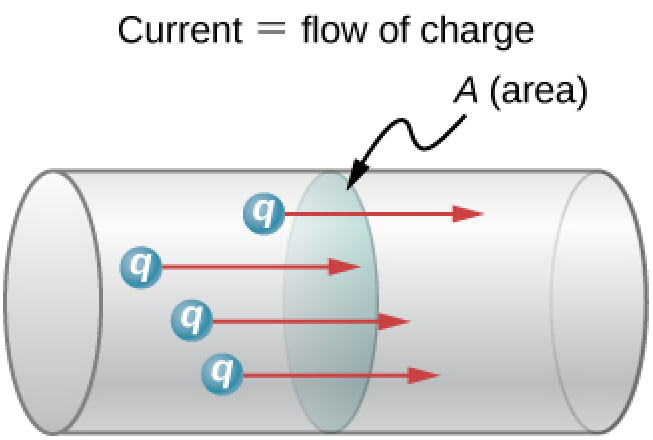
where ΔQ is the amount of net charge passing through a given cross-sectional area in time Δt ([link]). The SI unit for current is the **ampere** (A), named for the French physicist André-Marie Ampère (1775–1836). Since $I = \frac{\Delta Q}{\Delta t}$, we see that an ampere is defined as one coulomb of charge passing through a given area per second:
Equation:

$$1\text{A} \equiv 1\frac{\text{C}}{\text{s}}.$$

The instantaneous electrical current, or simply the **electrical current**, is the time derivative of the charge that flows and is found by taking the limit of the average electrical current as $\Delta t \to 0$:
Equation:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}.$$

Most electrical appliances are rated in amperes (or amps) required for proper operation, as are fuses and circuit breakers.



The rate of flow of charge is current. An ampere is the flow of one coulomb of charge through an area in one second. A current of one amp would result from 6.25×10^{18} electrons flowing through the area A each second.

Example:
Calculating the Average Current
The main purpose of a battery in a car or truck is to run the electric starter motor, which starts the engine. The operation of starting the vehicle requires a large current to be supplied by the battery. Once the engine starts, a device called an alternator takes over supplying the electric power required for running the vehicle and for charging the battery.
(a) What is the average current involved when a truck battery sets in motion 720 C of charge in 4.00 s while starting an engine? (b) How long does it take 1.00 C of charge to flow from the battery?

Strategy

We can use the definition of the average current in the equation $I=\frac{\Delta Q}{\Delta t}$ to find the average current in part (a), since charge and time are given. For part (b), once we know the average current, we can its definition $I=\frac{\Delta Q}{\Delta t}$ to find the time required for 1.00 C of charge to flow from the battery.

Solution

a. Entering the given values for charge and time into the definition of current gives

Equation:

$$I=\frac{\Delta Q}{\Delta t}=\frac{720\text{ C}}{4.00\text{ s}}=180\text{ C/s}=180\text{ A.}$$

b. Solving the relationship $I=\frac{\Delta Q}{\Delta t}$ for time Δt and entering the known values for charge and current gives

Equation:

$$\Delta t=\frac{\Delta Q}{I}=\frac{1.00\text{ C}}{180\text{ C/s}}=5.56\ \times\ 10^{-3}\text{s}=5.56\text{ ms.}$$

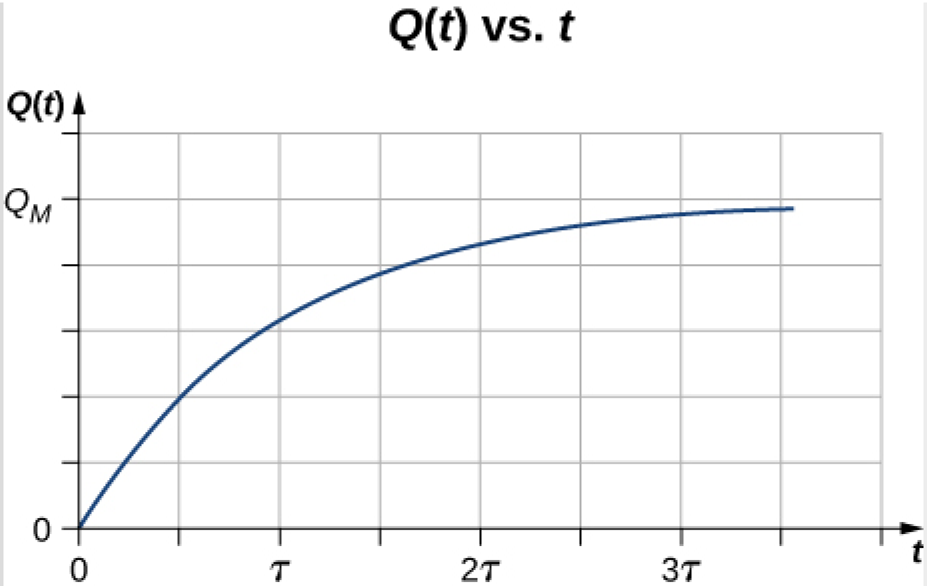
Significance

a. This large value for current illustrates the fact that a large charge is moved in a small amount of time. The currents in these “starter motors” are fairly large to overcome the inertia of the engine. b. A high current requires a short time to supply a large amount of charge. This large current is needed to supply the large amount of energy needed to start the engine.

Example:

Calculating Instantaneous Currents

Consider a charge moving through a cross-section of a wire where the charge is modeled as $Q\left(t\right)=Q_M\left(1-e^{-t/\tau}\right)$. Here, Q_M is the charge after a long period of time, as time approaches infinity, with units of coulombs, and τ is a time constant with units of seconds (see [link]). What is the current through the wire?



A graph of the charge moving through a cross-section of a wire over time.

Strategy

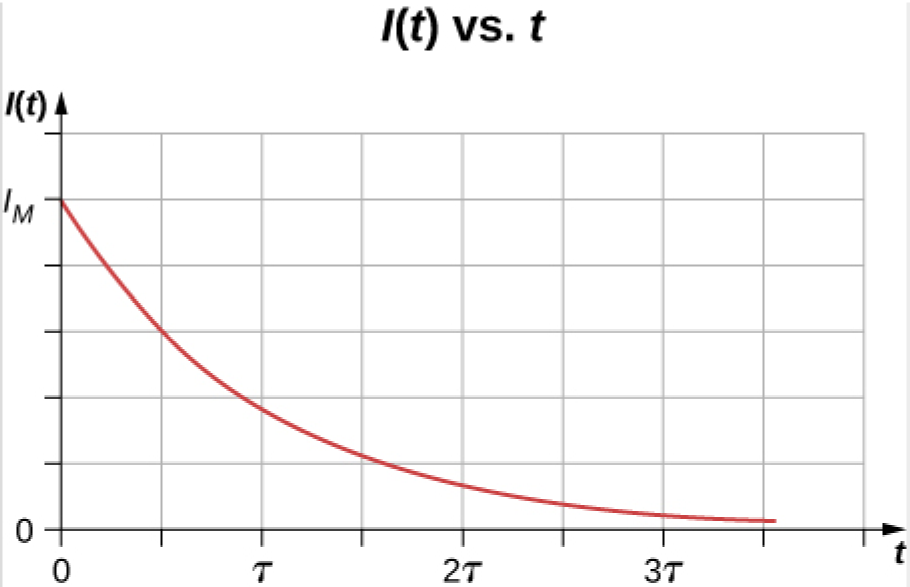
The current through the cross-section can be found from $I = \frac{dQ}{dt}$. Notice from the figure that the charge increases to Q_M and the derivative decreases, approaching zero, as time increases ([link]).

Solution

The derivative can be found using $\frac{d}{dx}e^u = e^u \frac{du}{dx}$.

Equation:

$$I = \frac{dQ}{dt} = \frac{d}{dt}\left[Q_M\left(1 - e^{-t/\tau}\right)\right] = \frac{Q_M}{\tau}e^{-t/\tau}.$$



A graph of the current flowing through the wire over time.

Significance
The current through the wire in question decreases exponentially, as shown in [link]. In later chapters, it will be shown that a time-dependent current appears when a capacitor charges or discharges through a resistor. Recall that a capacitor is a device that stores charge. You will learn about the resistor in Model of Conduction in Metals.

Note:
Exercise:

Problem:

Check Your Understanding Handheld calculators often use small solar cells to supply the energy required to complete the calculations needed to complete your next physics exam. The current needed to run your calculator can be as small as 0.30 mA. How long would it take for 1.00 C of charge to flow from the solar cells? Can solar cells be used, instead of batteries, to start traditional internal combustion engines presently used in most cars and trucks?

Solution:

The time for 1.00 C of charge to flow would be $\Delta t = \frac{\Delta Q}{I} = \frac{1.00\,\text{C}}{0.300 \times 10^{-3}\,\text{C/s}} = 3.33\ \times\ 10^{3}\text{s}$, slightly less than an hour. This is quite different from the 5.55 ms for the truck battery. The calculator takes a very small amount of energy to operate, unlike the truck's starter motor. There are several reasons that vehicles use batteries and not solar cells. Aside from the obvious fact that a light source to run the solar cells for a car or truck is not always available, the large amount of current needed to start the engine cannot easily be supplied by present-day solar cells. Solar cells can possibly be used to charge the batteries. Charging the battery requires a small amount of energy when compared to the energy required to run the engine and the other accessories such as the heater and air conditioner. Present day solar-powered cars are powered by solar panels, which may power an electric motor, instead of an internal combustion engine.

Note:
Exercise:

Problem:

Check Your Understanding Circuit breakers in a home are rated in amperes, normally in a range from 10 amps to 30 amps, and are used to protect the residents from harm and their appliances from damage due to large currents. A single 15-amp circuit breaker may be used to protect several outlets in the living room, whereas a single 20-amp circuit breaker may be used to protect the refrigerator in the kitchen. What can you deduce from this about current used by the various appliances?

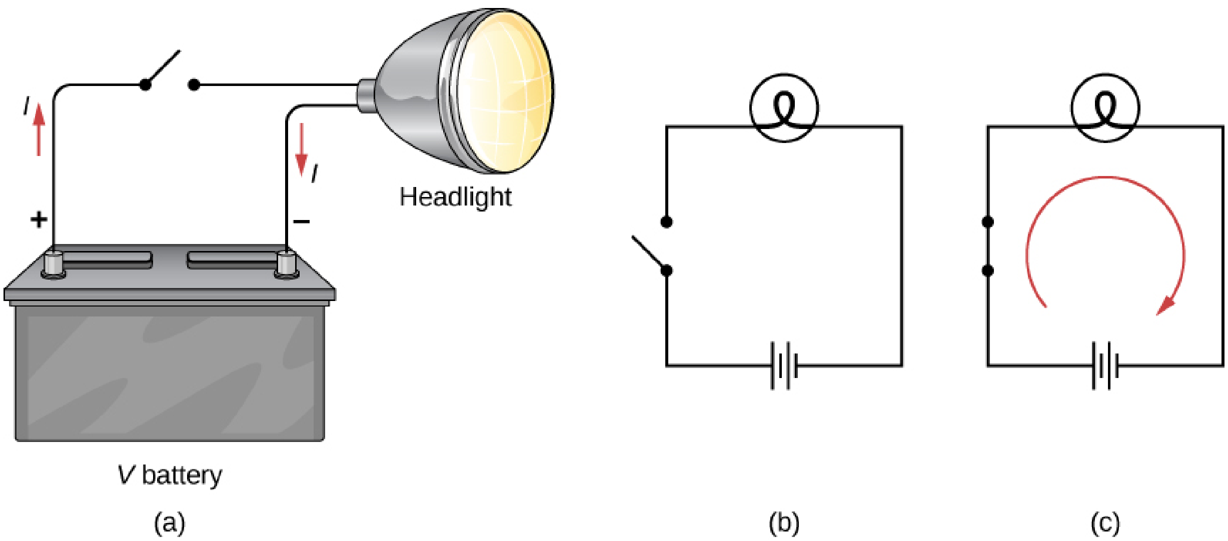
Solution:

The total current needed by all the appliances in the living room (a few lamps, a television, and your laptop) draw less current and require less power than the refrigerator.

Current in a Circuit

In the previous paragraphs, we defined the current as the charge that flows through a cross-sectional area per unit time. In order for charge to flow through an appliance, such as the headlight shown in [link], there must be a complete path (or **circuit**) from the positive terminal to the negative terminal. Consider a simple circuit of a car battery, a switch, a headlight lamp, and wires that provide a current path between the components. In order for the lamp to light, there must be a complete path for current flow. In other words, a charge must be able to leave the positive terminal of the battery, travel through the component, and back to the negative terminal of the battery. The switch is there to control the circuit. Part (a) of the figure shows the simple circuit of a car battery, a switch, a conducting path, and a headlight lamp. Also shown is the **schematic** of the circuit [part (b)]. A schematic is a graphical representation of a circuit and is very useful in visualizing the main features of a circuit. Schematics use standardized symbols to represent the components in a circuits and solid lines to represent the wires connecting the components. The battery is shown as a

series of long and short lines, representing the historic voltaic pile. The lamp is shown as a circle with a loop inside, representing the filament of an incandescent bulb. The switch is shown as two points with a conducting bar to connect the two points and the wires connecting the components are shown as solid lines. The schematic in part (c) shows the direction of current flow when the switch is closed.

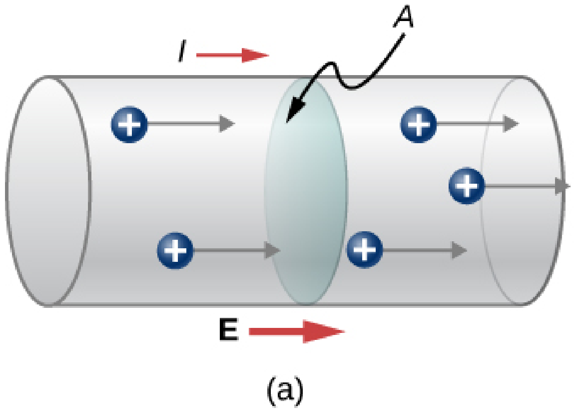


(a) A simple electric circuit of a headlight (lamp), a battery, and a switch. When the switch is closed, an uninterrupted path for current to flow through is supplied by conducting wires connecting a load to the terminals of a battery. (b) In this schematic, the battery is represented by parallel lines, which resemble plates in the original design of a battery. The longer lines indicate the positive terminal. The conducting wires are shown as solid lines. The switch is shown, in the open position, as two terminals with a line representing a conducting bar that can make contact between the two terminals. The lamp is represented by a circle encompassing a filament, as would be seen in an incandescent light bulb. (c) When the switch is closed, the circuit is complete and current flows from the positive terminal to the negative terminal of the battery.

When the switch is closed in [link](c), there is a complete path for charges to flow, from the positive terminal of the battery, through the switch, then through the headlight and back to the negative terminal of the battery. Note that the direction of current flow is from positive to negative. The direction of **conventional current** is always represented in the direction that positive charge would flow, from the positive terminal to the negative terminal.

The conventional current flows from the positive terminal to the negative terminal, but depending on the actual situation, positive charges, negative charges, or both may move. In metal wires, for example, current is carried by electrons—that is, negative charges move. In ionic solutions, such as salt water, both positive and negative charges move. This is also true in nerve cells. A Van de Graaff generator, used for nuclear research, can produce a current of pure positive charges, such as protons. In the Tevatron Accelerator at Fermilab, before it was shut down in 2011, beams of protons and antiprotons traveling in opposite directions were collided. The protons are positive and therefore their current is in the same direction as they travel. The antiprotons are negativity charged and thus their current is in the opposite direction that the actual particles travel.

A closer look at the current flowing through a wire is shown in [link]. The figure illustrates the movement of charged particles that compose a current. The fact that conventional current is taken to be in the direction that positive charge would flow can be traced back to American scientist and statesman Benjamin Franklin in the 1700s. Having no knowledge of the particles that make up the atom (namely the proton, electron, and neutron), Franklin believed that electrical current flowed from a material that had more of an “electrical fluid” and to a material that had less of this “electrical fluid.” He coined the term *positive* for the material that had more of this electrical fluid and *negative* for the material that lacked the electrical fluid. He surmised that current would flow from the material with more electrical fluid—the positive material—to the negative material, which has less electrical fluid. Franklin called this direction of current a positive current flow. This was pretty advanced thinking for a man who knew nothing about the atom.



Current *I* is the rate at which charge moves through an area *A*, such as the cross-section of a wire. Conventional current is defined to move in the direction of the electrical field. (a) Positive charges move in the direction of the electrical field, which is the same direction as conventional current. (b) Negative charges move in the direction opposite to the electrical field. Conventional current is in the direction opposite to the movement of negative charge. The flow of electrons is sometimes referred to as electronic flow.

We now know that a material is positive if it has a greater number of protons than electrons, and it is negative if it has a greater number of electrons than protons. In a conducting metal, the current flow is due primarily to electrons flowing from the negative material to the positive material, but for historical reasons, we consider the positive current flow and the current is shown to flow from the positive terminal of the battery to the negative terminal.

It is important to realize that an electrical field is present in conductors and is responsible for producing the current ([link]). In previous chapters, we considered the static electrical case, where charges in a conductor quickly redistribute themselves on the surface of the conductor in order to cancel out the external electrical field and restore equilibrium. In the case of an electrical circuit, the charges are prevented from ever reaching equilibrium by an external source of electric potential, such as a battery. The energy

needed to move the charge is supplied by the electric potential from the battery.

Although the electrical field is responsible for the motion of the charges in the conductor, the work done on the charges by the electrical field does not increase the kinetic energy of the charges. We will show that the electrical field is responsible for keeping the electric charges moving at a "drift velocity."

Summary

- The average electrical current I_{ave} is the rate at which charge flows, given by $I_{\text{ave}} = \frac{\Delta Q}{\Delta t}$, where ΔQ is the amount of charge passing through an area in time Δt.
- The instantaneous electrical current, or simply the current *I*, is the rate at which charge flows. Taking the limit as the change in time approaches zero, we have $I = \frac{dQ}{dt}$, where $\frac{dQ}{dt}$ is the time derivative of the charge.
- The direction of conventional current is taken as the direction in which positive charge moves. In a simple direct-current (DC) circuit, this will be from the positive terminal of the battery to the negative terminal.
- The SI unit for current is the ampere, or simply the amp (A), where $1\text{ A} = 1\text{ C/s}$.
- Current consists of the flow of free charges, such as electrons, protons, and ions.

Conceptual Questions

Exercise:

Problem:

Can a wire carry a current and still be neutral—that is, have a total charge of zero? Explain.

Solution:

If a wire is carrying a current, charges enter the wire from the voltage source's positive terminal and leave at the negative terminal, so the total charge remains zero while the current flows through it.

Exercise:

Problem:

Car batteries are rated in ampere-hours $(\mathrm{A} \cdot \mathrm{h})$. To what physical quantity do ampere-hours correspond (voltage, current, charge, energy, power,…)?

Exercise:

Problem:

When working with high-power electric circuits, it is advised that whenever possible, you work "one-handed" or "keep one hand in your pocket." Why is this a sensible suggestion?

Solution:

Using one hand will reduce the possibility of "completing the circuit" and having current run through your body, especially current running through your heart.

Problems

Exercise:

Problem:

A Van de Graaff generator is one of the original particle accelerators and can be used to accelerate charged particles like protons or electrons. You may have seen it used to make human hair stand on end or produce large sparks. One application of the Van de Graaff generator is to create X-rays by bombarding a hard metal target with the beam. Consider a beam of protons at 1.00 keV and a current of 5.00 mA produced by the generator. (a) What is the speed of the protons? (b) How many protons are produced each second?

Solution:

a. $v = 4.38 \times 10^5 \frac{\text{m}}{\text{s}}$;
b. $\Delta q = 5.00 \times 10^{-3}\text{C}, \quad \text{no. of protons} = 3.13 \times 10^{16}$

Exercise:

Problem:

A cathode ray tube (CRT) is a device that produces a focused beam of electrons in a vacuum. The electrons strike a phosphor-coated glass screen at the end of the tube, which produces a bright spot of light. The position of the bright spot of light on the screen can be adjusted by deflecting the electrons with electrical fields, magnetic fields, or both. Although the CRT tube was once commonly found in televisions, computer displays, and oscilloscopes, newer appliances use a liquid crystal display (LCD) or plasma screen. You still may come across a CRT in your study of science. Consider a CRT with an electron beam average current of $25.00\mu \text{ A}$. How many electrons strike the screen every minute?

Exercise:

Problem:

How many electrons flow through a point in a wire in 3.00 s if there is a constant current of $I = 4.00 \text{ A}$?

Solution:

$I = \frac{\Delta Q}{\Delta t}, \quad \Delta Q = 12.00 \text{ C}$
no. of electrons = 7.5×10^{19}

Exercise:

Problem:

A conductor carries a current that is decreasing exponentially with time. The current is modeled as $I = I_0 e^{-t/\tau}$, where $I_0 = 3.00\text{ A}$ is the current at time $t = 0.00\text{ s}$ and $\tau = 0.50\text{ s}$ is the time constant. How much charge flows through the conductor between $t = 0.00\text{ s}$ and $t = 3\tau$?

Exercise:

Problem:

The quantity of charge through a conductor is modeled as $Q = 4.00\frac{\text{C}}{\text{s}^4}t^4 - 1.00\frac{\text{C}}{\text{s}}t + 6.00\text{ mC}$.

What is the current at time $t = 3.00\text{ s}$?

Solution:

$$I\left(t\right) = 0.016\frac{\text{C}}{\text{s}^4}t^3 - 0.001\frac{\text{C}}{\text{s}}$$
$$I\left(3.00\text{ s}\right) = 0.431\text{ A}$$

Exercise:

Problem:

The current through a conductor is modeled as $I\left(t\right) = I_m \sin\left(2\pi\left[60\text{ Hz}\right]t\right)$. Write an equation for the charge as a function of time.

Exercise:

Problem:

The charge on a capacitor in a circuit is modeled as $Q\left(t\right) = Q_{\text{max}}\cos\left(\omega t + \phi\right)$. What is the current through the circuit as a function of time?

Solution:

$$I(t) = -I_{\max} \sin\left(\omega t + \phi\right)$$

Glossary

ampere (amp)
: SI unit for current; $1\ \mathrm{A} = 1\ \mathrm{C/s}$

circuit
: complete path that an electrical current travels along

conventional current
: current that flows through a circuit from the positive terminal of a battery through the circuit to the negative terminal of the battery

electrical current
: rate at which charge flows, $I = \frac{dQ}{dt}$

schematic
: graphical representation of a circuit using standardized symbols for components and solid lines for the wire connecting the components

Model of Conduction in Metals
By the end of this section, you will be able to:

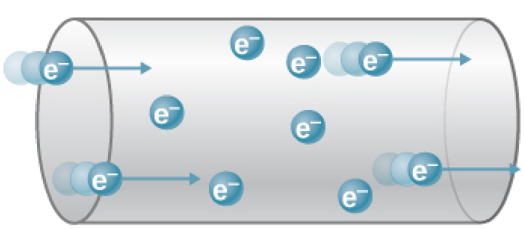
- Define the drift velocity of charges moving through a metal
- Define the vector current density
- Describe the operation of an incandescent lamp

When electrons move through a conducting wire, they do not move at a constant velocity, that is, the electrons do not move in a straight line at a constant speed. Rather, they interact with and collide with atoms and other free electrons in the conductor. Thus, the electrons move in a zig-zag fashion and drift through the wire. We should also note that even though it is convenient to discuss the direction of current, current is a scalar quantity. When discussing the velocity of charges in a current, it is more appropriate to discuss the current density. We will come back to this idea at the end of this section.

Drift Velocity

Electrical signals move very rapidly. Telephone conversations carried by currents in wires cover large distances without noticeable delays. Lights come on as soon as a light switch is moved to the 'on' position. Most electrical signals carried by currents travel at speeds on the order of $10^8\ \mathrm{m/s}$, a significant fraction of the speed of light. Interestingly, the individual charges that make up the current move much slower on average, typically drifting at speeds on the order of $10^{-4}\mathrm{m/s}$. How do we reconcile these two speeds, and what does it tell us about standard conductors?

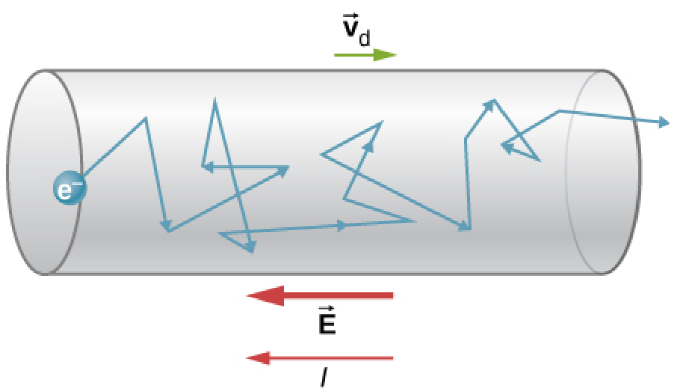
The high speed of electrical signals results from the fact that the force between charges acts rapidly at a distance. Thus, when a free charge is forced into a wire, as in [link], the incoming charge pushes other charges ahead of it due to the repulsive force between like charges. These moving charges push on charges farther down the line. The density of charge in a system cannot easily be increased, so the signal is passed on rapidly. The resulting electrical shock wave moves through the system at nearly the speed of light. To be precise, this fast-moving signal, or shock wave, is a rapidly propagating change in the electrical field.



When charged particles are forced into this volume of a conductor, an equal number are quickly forced to leave. The repulsion between like charges makes it difficult to increase the number of charges in a volume. Thus, as one charge enters, another

leaves almost immediately, carrying the signal rapidly forward.

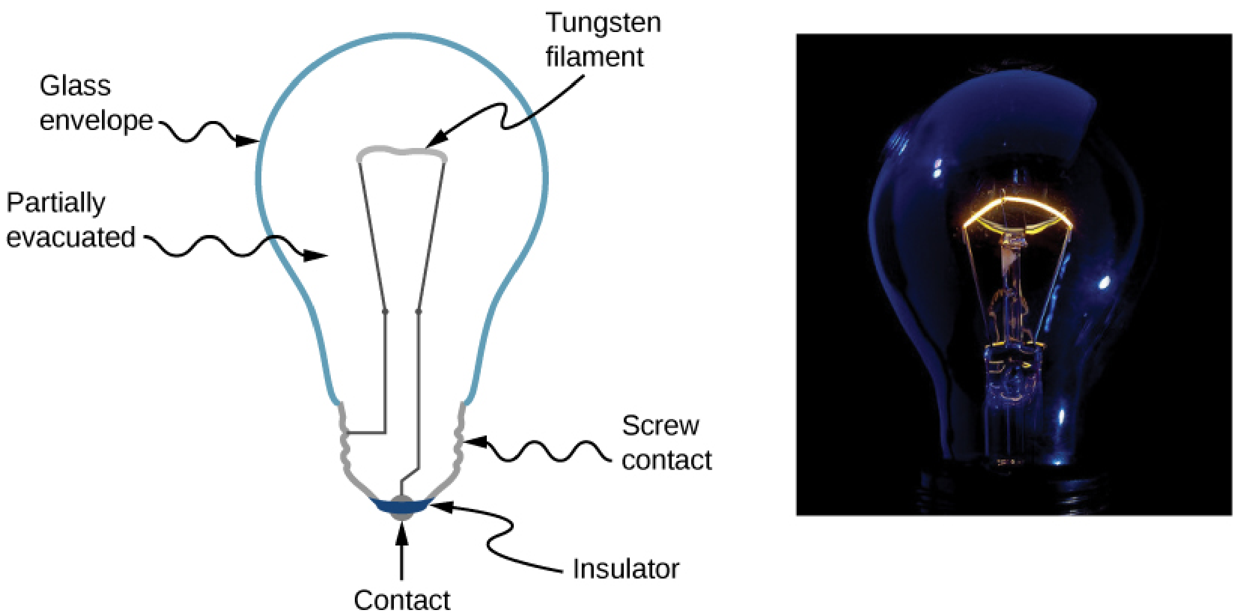
Good conductors have large numbers of free charges. In metals, the free charges are free electrons. (In fact, good electrical conductors are often good heat conductors too, because large numbers of free electrons can transport thermal energy as well as carry electrical current.) [link] shows how free electrons move through an ordinary conductor. The distance that an individual electron can move between collisions with atoms or other electrons is quite small. The electron paths thus appear nearly random, like the motion of atoms in a gas. But there is an electrical field in the conductor that causes the electrons to drift in the direction shown (opposite to the field, since they are negative). The **drift velocity** $\vec{\mathbf{v}}_d$ is the average velocity of the free charges. Drift velocity is quite small, since there are so many free charges. If we have an estimate of the density of free electrons in a conductor, we can calculate the drift velocity for a given current. The larger the density, the lower the velocity required for a given current.



Free electrons moving in a conductor make many collisions with other electrons and other particles. A typical path of one electron is shown. The average velocity of the free charges is called the drift velocity $\vec{\mathbf{v}}_d$ and for electrons, it is in the direction opposite to the electrical field. The collisions normally transfer energy to the conductor, requiring a constant supply of energy to maintain a steady current.

Free-electron collisions transfer energy to the atoms of the conductor. The electrical field does work in moving the electrons through a distance, but that work does not increase the kinetic energy (nor speed) of the electrons. The work is transferred to the conductor’s atoms, often increasing temperature. Thus, a continuous power input is required to keep a current flowing. (An exception is superconductors, for reasons we shall explore in a later chapter. Superconductors can have a steady current without a continual supply of energy—a great energy savings.) For a conductor that is not a

superconductor, the supply of energy can be useful, as in an incandescent light bulb filament ([link]). The supply of energy is necessary to increase the temperature of the tungsten filament, so that the filament glows.



The incandescent lamp is a simple design. A tungsten filament is placed in a partially evacuated glass envelope. One end of the filament is attached to the screw base, which is made out of a conducting material. The second end of the filament is attached to a second contact in the base of the bulb. The two contacts are separated by an insulating material. Current flows through the filament, and the temperature of the filament becomes large enough to cause the filament to glow and produce light. However, these bulbs are not very energy efficient, as evident from the heat coming from the bulb. In the year 2012, the United States, along with many other countries, began to phase out incandescent lamps in favor of more energy-efficient lamps, such as light-emitting diode (LED) lamps and compact fluorescent lamps (CFL) (credit right: modification of work by Serge Saint).

We can obtain an expression for the relationship between current and drift velocity by considering the number of free charges in a segment of wire, as illustrated in [link]. The number of free charges per unit volume, or the number density of free charges, is given the symbol n where $n=\frac{\text{number of charges}}{\text{volume}}$. The value of n depends on the material. The shaded segment has a volume $Av_{\text{d}}dt$, so that the number of free charges in the volume is $nAv_{\text{d}}dt$. The charge dQ in this segment is thus $qnAv_{\text{d}}dt$, where q is the amount of charge on each carrier. (The magnitude of the charge of electrons is $q=1.60\,\times\,10^{-19}$ C.) Current is charge moved per unit time; thus, if all the original charges move out of this segment in time dt, the current is
Equation:

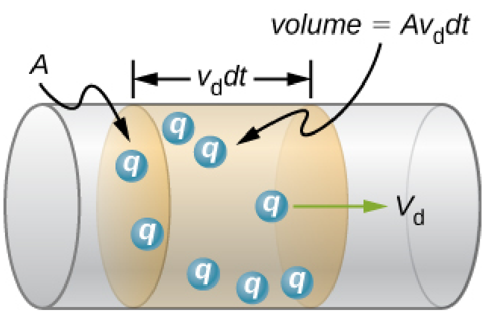
$$I=\frac{dQ}{dt}=qnAv_{\text{d}}.$$

Rearranging terms gives

Note:
Equation:

$$v_{\mathrm{d}} = \frac{I}{nqA}$$

where v_{d} is the drift velocity, n is the free charge density, A is the cross-sectional area of the wire, and I is the current through the wire. The carriers of the current each have charge q and move with a drift velocity of magnitude v_{d}.

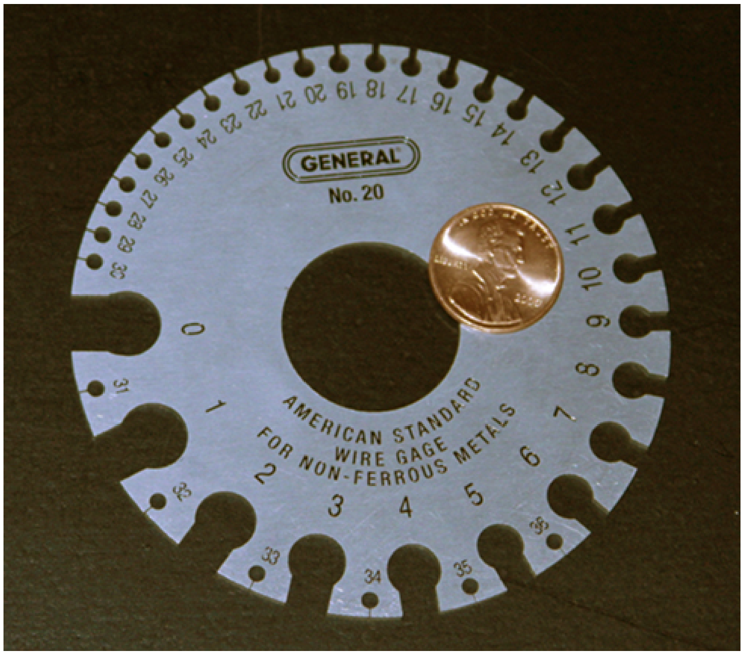


All the charges in the shaded volume of this wire move out in a time *dt*, having a drift velocity of magnitude v_{d}.

Note that simple drift velocity is not the entire story. The speed of an electron is sometimes much greater than its drift velocity. In addition, not all of the electrons in a conductor can move freely, and those that do move might move somewhat faster or slower than the drift velocity. So what do we mean by free electrons?

Atoms in a metallic conductor are packed in the form of a lattice structure. Some electrons are far enough away from the atomic nuclei that they do not experience the attraction of the nuclei as strongly as the inner electrons do. These are the free electrons. They are not bound to a single atom but can instead move freely among the atoms in a “sea” of electrons. When an electrical field is applied, these free electrons respond by accelerating. As they move, they collide with the atoms in the lattice and with other electrons, generating thermal energy, and the conductor gets warmer. In an insulator, the organization of the atoms and the structure do not allow for such free electrons.

As you know, electric power is usually supplied to equipment and appliances through round wires made of a conducting material (copper, aluminum, silver, or gold) that are stranded or solid. The diameter of the wire determines the current-carrying capacity—the larger the diameter, the greater the current-carrying capacity. Even though the current-carrying capacity is determined by the diameter, wire is not normally characterized by the diameter directly. Instead, wire is commonly sold in a unit known as “gauge.” Wires are manufactured by passing the material through circular forms called “drawing dies.” In order to make thinner wires, manufacturers draw the wires through multiple dies of successively thinner diameter. Historically, the gauge of the wire was related to the number of drawing processes required to manufacture the wire. For this reason, the larger the gauge, the smaller the diameter. In the United States, the American Wire Gauge (AWG) was developed to standardize the system. Household wiring commonly consists of 10-gauge (2.588-mm diameter) to 14-gauge (1.628-mm diameter) wire. A device used to measure the gauge of wire is shown in [link].



A device for measuring the gauge of electrical wire. As you can see, higher gauge numbers indicate thinner wires. (credit: Joseph J. Trout)

Example:
Calculating Drift Velocity in a Common Wire
Calculate the drift velocity of electrons in a copper wire with a diameter of 2.053 mm (12-gauge) carrying a 20.0-A current, given that there is one free electron per copper atom. (Household wiring often contains 12-gauge copper wire, and the maximum current allowed in such wire is usually 20.0 A.) The density of copper is $8.80 \times 10^3 \text{ kg/m}^3$ and the atomic mass of copper is 63.54 g/mol.
Strategy

We can calculate the drift velocity using the equation $I = nqAv_{\text{d}}$. The current is $I = 20.00\text{ A}$ and $q = 1.60 \times 10^{-19}\text{C}$ is the charge of an electron. We can calculate the area of a cross-section of the wire using the formula $A = \pi r^2$, where *r* is one-half the diameter. The given diameter is 2.053 mm, so *r* is 1.0265 mm. We are given the density of copper, $8.80 \times 10^3\text{ kg/m}^3$, and the atomic mass of copper is 63.54 g/mol. We can use these two quantities along with Avogadro's number, $6.02 \times 10^{23}\text{ atoms/mol}$, to determine *n*, the number of free electrons per cubic meter.

Solution

First, we calculate the density of free electrons in copper. There is one free electron per copper atom. Therefore, the number of free electrons is the same as the number of copper atoms per m^3. We can now find *n* as follows:

Equation:

$$\begin{aligned} n &= \frac{1\,e^-}{\text{atom}} \times \frac{6.02 \times 10^{23}\text{atoms}}{\text{mol}} \times \frac{1\text{ mol}}{63.54\text{ g}} \times \frac{1000\text{ g}}{\text{kg}} \times \frac{8.80 \times 10^3\text{kg}}{1\text{ m}^3} \\ &= 8.34 \times 10^{28} e^-/\text{m}^3. \end{aligned}$$

The cross-sectional area of the wire is

Equation:

$$A = \pi r^2 = \pi \left(\frac{2.05 \times 10^{-3}\text{m}}{2} \right)^2 = 3.30 \times 10^{-6}\text{m}^2.$$

Rearranging $I = nqAv_{\text{d}}$ to isolate drift velocity gives

Equation:

$$v_{\text{d}} = \frac{I}{nqA} = \frac{20.00\text{ A}}{(8.34 \times 10^{28}/\text{m}^3)(-1.60 \times 10^{-19}\text{C})(3.30 \times 10^{-6}\text{m}^2)} = -4.54 \times 10^{-4}\text{ m/s}.$$

Significance

The minus sign indicates that the negative charges are moving in the direction opposite to conventional current. The small value for drift velocity (on the order of 10^{-4} m/s) confirms that the signal moves on the order of 10^{12} times faster (about 10^8 m/s) than the charges that carry it.

Note:

Exercise:

Problem:

Check Your Understanding In [link], the drift velocity was calculated for a 2.053-mm diameter (12-gauge) copper wire carrying a 20-amp current. Would the drift velocity change for a 1.628-mm diameter (14-gauge) wire carrying the same 20-amp current?

Solution:

The diameter of the 14-gauge wire is smaller than the diameter of the 12-gauge wire. Since the drift velocity is inversely proportional to the cross-sectional area, the drift velocity in the 14-gauge wire is larger than the drift velocity in the 12-gauge wire carrying the same current. The number of electrons per cubic meter will remain constant.

Current Density

Although it is often convenient to attach a negative or positive sign to indicate the overall direction of motion of the charges, current is a scalar quantity, $I = \frac{dQ}{dt}$. It is often necessary to discuss the details of the motion of the charge, instead of discussing the overall motion of the charges. In such cases, it is necessary to discuss the current density, $\vec{\mathbf{J}}$, a vector quantity. The **current density** is the flow of charge through an infinitesimal area, divided by the area. The current density must take into account the local magnitude and direction of the charge flow, which varies from point to point. The unit of current density is ampere per meter squared, and the direction is defined as the direction of net flow of positive charges through the area.

The relationship between the current and the current density can be seen in [link]. The differential current flow through the area $d\vec{\mathbf{A}}$ is found as

Equation:

$$dI = \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J dA \cos \theta,$$

where θ is the angle between the area and the current density. The total current passing through area $d\vec{\mathbf{A}}$ can be found by integrating over the area,

Note:

Equation:

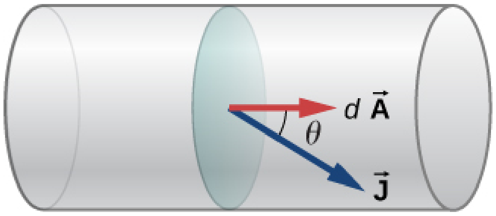
$$I = \iint_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}.$$

Consider the magnitude of the current density, which is the current divided by the area:

Equation:

$$J = \frac{I}{A} = \frac{n\left|q\right|Av_{\text{d}}}{A} = n\left|q\right|v_{\text{d}}.$$

Thus, the current density is $\vec{\mathbf{J}} = nq\vec{\mathbf{v}}_{\text{d}}$. If q is positive, $\vec{\mathbf{v}}_{\text{d}}$ is in the same direction as the electrical field $\vec{\mathbf{E}}$. If q is negative, $\vec{\mathbf{v}}_{\text{d}}$ is in the opposite direction of $\vec{\mathbf{E}}$. Either way, the direction of the current density $\vec{\mathbf{J}}$ is in the direction of the electrical field $\vec{\mathbf{E}}$.



The current density $\vec{\mathbf{J}}$ is defined as the current passing through an infinitesimal cross-sectional area divided by the area. The direction of the current density is the direction of the net flow of positive charges and the magnitude is equal to the current divided by the infinitesimal area.

Example:

Calculating the Current Density in a Wire

The current supplied to a lamp with a 100-W light bulb is 0.87 amps. The lamp is wired using a copper wire with diameter 2.588 mm (10-gauge). Find the magnitude of the current density.

Strategy

The current density is the current moving through an infinitesimal cross-sectional area divided by the area. We can calculate the magnitude of the current density using $J = \frac{I}{A}$. The current is given as 0.87 A. The cross-sectional area can be calculated to be $A = 5.26\,\text{mm}^2$.

Solution

Calculate the current density using the given current $I = 0.87\text{ A}$ and the area, found to be $A = 5.26\,\text{mm}^2$.

Equation:

$$J = \frac{I}{A} = \frac{0.87\text{ A}}{5.26 \times 10^{-6}\text{m}^2} = 1.65 \times 10^5 \frac{\text{A}}{\text{m}^2}.$$

Significance

The current density in a conducting wire depends on the current through the conducting wire and the cross-sectional area of the wire. For a given current, as the diameter of the wire increases, the charge density decreases.

Note:

Exercise:

Problem:

Check Your Understanding The current density is proportional to the current and inversely proportional to the area. If the current density in a conducting wire increases, what would happen to the drift velocity of the charges in the wire?

Solution:

The current density in a conducting wire increases due to an increase in current. The drift velocity is inversely proportional to the current $\left(v_d = \frac{I}{nqA}\right)$, so the drift velocity would decrease.

What is the significance of the current density? The current density is proportional to the current, and the current is the number of charges that pass through a cross-sectional area per second. The charges move through the conductor, accelerated by the electric force provided by the electrical field. The electrical field is created when a voltage is applied across the conductor. In Ohm's Law, we will use this relationship between the current density and the electrical field to examine the relationship between the current through a conductor and the voltage applied.

Summary

- The current through a conductor depends mainly on the motion of free electrons.
- When an electrical field is applied to a conductor, the free electrons in a conductor do not move through a conductor at a constant speed and direction; instead, the motion is almost random due to collisions with atoms and other free electrons.
- Even though the electrons move in a nearly random fashion, when an electrical field is applied to the conductor, the overall velocity of the electrons can be defined in terms of a drift velocity.
- The current density is a vector quantity defined as the current through an infinitesimal area divided by the area.
- The current can be found from the current density, $I = \iint_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$.
- An incandescent light bulb is a filament of wire enclosed in a glass bulb that is partially evacuated. Current runs through the filament, where the electrical energy is converted to light and heat.

Conceptual Questions

Exercise:

Problem:

Incandescent light bulbs are being replaced with more efficient LED and CFL light bulbs. Is there any obvious evidence that incandescent light bulbs might not be that energy efficient? Is energy converted into anything but visible light?

Exercise:

Problem:

It was stated that the motion of an electron appears nearly random when an electrical field is applied to the conductor. What makes the motion nearly random and differentiates it from the random motion of molecules in a gas?

Solution:

Even though the electrons collide with atoms and other electrons in the wire, they travel from the negative terminal to the positive terminal, so they drift in one direction. Gas molecules travel in completely random directions.

Exercise:

Problem:

Electric circuits are sometimes explained using a conceptual model of water flowing through a pipe. In this conceptual model, the voltage source is represented as a pump that pumps water through pipes and the pipes connect components in the circuit. Is a conceptual model of water flowing through a pipe an adequate representation of the circuit? How are electrons and wires similar to water molecules and pipes? How are they different?

Exercise:

Problem: An incandescent light bulb is partially evacuated. Why do you suppose that is?

Solution:

In the early years of light bulbs, the bulbs are partially evacuated to reduce the amount of heat conducted through the air to the glass envelope. Dissipating the heat would cool the filament, increasing the amount of energy needed to produce light from the filament. It also protects the glass from the heat produced from the hot filament. If the glass heats, it expands, and as it cools, it contacts. This expansion and contraction could cause the glass to become brittle and crack, reducing the life of the bulbs. Many bulbs are now partially filled with an inert gas. It is also useful to remove the oxygen to reduce the possibility of the filament actually burning. When the original filaments were replaced with more efficient tungsten filaments, atoms from the tungsten would evaporate off the filament at such high temperatures. The atoms collide with the atoms of the inert gas and land back on the filament.

Problems

Exercise:

Problem:

An aluminum wire 1.628 mm in diameter (14-gauge) carries a current of 3.00 amps. (a) What is the absolute value of the charge density in the wire? (b) What is the drift velocity of the electrons? (c) What would be the drift velocity if the same gauge copper were used instead of aluminum? The density of copper is $8.96\ \mathrm{g/cm^3}$ and the density of aluminum is $2.70\ \mathrm{g/cm^3}$. The molar mass of aluminum is 26.98 g/mol and the molar mass of copper is 63.5 g/mol. Assume each atom of metal contributes one free electron.

Exercise:

Problem:

The current of an electron beam has a measured current of $I = 50.00\ \mu\mathrm{A}$ with a radius of 1.00 mm. What is the magnitude of the current density of the beam?

Solution:

$|J| = 15.92\ \mathrm{A/m^2}$

Exercise:

Problem:

A high-energy proton accelerator produces a proton beam with a radius of $r = 0.90\ \mathrm{mm}$. The beam current is $I = 9.00\ \mu\mathrm{A}$ and is constant. The charge density of the beam is $n = 6.00\ \times\ 10^{11}$ protons per cubic meter. (a) What is the current density of the beam? (b) What is the drift velocity of the beam? (c) How much time does it take for $1.00\ \times\ 10^{10}$ protons to be emitted by the accelerator?

Exercise:

Problem:

Consider a wire of a circular cross-section with a radius of $R = 3.00\ \mathrm{mm}$. The magnitude of the current density is modeled as $J = cr^2 = 5.00\ \times\ 10^6 \frac{\mathrm{A}}{\mathrm{m}^4} r^2$. What is the current through the inner section of the wire from the center to $r = 0.5R$?

Solution:

$I = 3.98 \times 10^{-5}\ \mathrm{A}$

Exercise:

Problem:

A cylindrical wire has a current density from the center of the wire's cross section as $J(r) = Cr^2$ where r is in meters, J is in amps per square meter, and $C = 10^3\ \mathrm{A/m^4}$. This current density continues to the end of the wire at a radius of 1.0 mm. Calculate the current just outside of this wire.

Exercise:

Problem:

The current supplied to an air conditioner unit is 4.00 amps. The air conditioner is wired using a 10-gauge (diameter 2.588 mm) wire. The charge density is $n = 8.48 \times 10^{28} \frac{\text{electrons}}{\text{m}^3}$. Find the magnitude of (a) current density and (b) the drift velocity.

Solution:

a. $|J| = 7.60 \times 10^5 \frac{\text{A}}{\text{m}^2}$; b. $v_\text{d} = 5.60 \times 10^{-5} \frac{\text{m}}{\text{s}}$

Glossary

current density
: flow of charge through a cross-sectional area divided by the area

drift velocity
: velocity of a charge as it moves nearly randomly through a conductor, experiencing multiple collisions, averaged over a length of a conductor, whose magnitude is the length of conductor traveled divided by the time it takes for the charges to travel the length

Resistivity and Resistance

By the end of this section, you will be able to:

- Differentiate between resistance and resistivity
- Define the term conductivity
- Describe the electrical component known as a resistor
- State the relationship between resistance of a resistor and its length, cross-sectional area, and resistivity
- State the relationship between resistivity and temperature

What drives current? We can think of various devices—such as batteries, generators, wall outlets, and so on—that are necessary to maintain a current. All such devices create a potential difference and are referred to as voltage sources. When a voltage source is connected to a conductor, it applies a potential difference *V* that creates an electrical field. The electrical field, in turn, exerts force on free charges, causing current. The amount of current depends not only on the magnitude of the voltage, but also on the characteristics of the material that the current is flowing through. The material can resist the flow of the charges, and the measure of how much a material resists the flow of charges is known as the *resistivity*. This resistivity is crudely analogous to the friction between two materials that resists motion.

Resistivity

When a voltage is applied to a conductor, an electrical field $\vec{\mathbf{E}}$ is created, and charges in the conductor feel a force due to the electrical field. The current density $\vec{\mathbf{J}}$ that results depends on the electrical field and the properties of the material. This dependence can be very complex. In some materials, including metals at a given temperature, the current density is approximately proportional to the electrical field. In these cases, the current density can be modeled as

Equation:

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}},$$

where σ is the **electrical conductivity**. The electrical conductivity is analogous to thermal conductivity and is a measure of a material’s ability to conduct or transmit electricity. Conductors have a higher electrical conductivity than insulators. Since the electrical conductivity is $\sigma = J/E$, the units are

Equation:

$$\sigma = \frac{[J]}{[E]} = \frac{\mathrm{A/m^2}}{\mathrm{V/m}} = \frac{\mathrm{A}}{\mathrm{V \cdot m}}.$$

Here, we define a unit named the **ohm** with the Greek symbol uppercase omega, Ω. The unit is named after Georg Simon Ohm, whom we will discuss later in this chapter. The Ω is used to avoid confusion with the number 0. One ohm equals one volt per amp: $1\,\Omega\ = 1\,\mathrm{V/A}$. The units of electrical conductivity are therefore $(\Omega\ \cdot \mathrm{m})^{-1}$.

Conductivity is an intrinsic property of a material. Another intrinsic property of a material is the **resistivity**, or electrical resistivity. The resistivity of a material is a measure of how strongly a material opposes the flow of electrical current. The symbol for resistivity is the lowercase Greek letter rho, ρ, and resistivity is the reciprocal of electrical conductivity:

Equation:

$$\rho = \frac{1}{\sigma}.$$

The unit of resistivity in SI units is the ohm-meter $(\Omega \cdot \text{m})$. We can define the resistivity in terms of the electrical field and the current density,

Note:

Equation:

$$\rho = \frac{E}{J}.$$

The greater the resistivity, the larger the field needed to produce a given current density. The lower the resistivity, the larger the current density produced by a given electrical field. Good conductors have a high conductivity and low resistivity. Good insulators have a low conductivity and a high resistivity. [link] lists resistivity and conductivity values for various materials.

Material	**Conductivity,** σ $(\Omega \cdot \text{m})^{-1}$	**Resistivity,** ρ $(\Omega \cdot \text{m})$	**Temperature Coefficient,** α $(^\circ\text{C})^{-1}$
Conductors			
Silver	6.29×10^{7}	1.59×10^{-8}	0.0038
Copper	5.95×10^{7}	1.68×10^{-8}	0.0039
Gold	4.10×10^{7}	2.44×10^{-8}	0.0034
Aluminum	3.77×10^{7}	2.65×10^{-8}	0.0039

Material	**Conductivity, σ** $(\Omega \cdot \mathrm{m})^{-1}$	**Resistivity, ρ** $(\Omega \cdot \mathrm{m})$	**Temperature Coefficient, α** $(^\circ\mathrm{C})^{-1}$
Tungsten	1.79×10^7	5.60×10^{-8}	0.0045
Iron	1.03×10^7	9.71×10^{-8}	0.0065
Platinum	0.94×10^7	10.60×10^{-8}	0.0039
Steel	0.50×10^7	20.00×10^{-8}	
Lead	0.45×10^7	22.00×10^{-8}	
Manganin (Cu, Mn, Ni alloy)	0.21×10^7	48.20×10^{-8}	0.000002
Constantan (Cu, Ni alloy)	0.20×10^7	49.00×10^{-8}	0.00003
Mercury	0.10×10^7	98.00×10^{-8}	0.0009
Nichrome (Ni, Fe, Cr alloy)	0.10×10^7	100.00×10^{-8}	0.0004
Semiconductors[1]			
Carbon (pure)	2.86×10^4	3.50×10^{-5}	−0.0005
Carbon	$(2.86 - 1.67) \times 10^{-6}$	$(3.5 - 60) \times 10^{-5}$	−0.0005
Germanium (pure)		600×10^{-3}	−0.048
Germanium		$(1 - 600) \times 10^{-3}$	−0.050
Silicon (pure)		2300	−0.075
Silicon		$0.1 - 2300$	−0.07
Insulators			
Amber	2.00×10^{-15}	5×10^{14}	
Glass	$10^{-9} - 10^{-14}$	$10^9 - 10^{14}$	

Material	Conductivity, σ $(\Omega \cdot \mathrm{m})^{-1}$	Resistivity, ρ $(\Omega \cdot \mathrm{m})$	Temperature Coefficient, α $(^{\circ}\mathrm{C})^{-1}$
Lucite	$<10^{-13}$	$>10^{13}$	
Mica	$10^{-11} - 10^{-15}$	$10^{11} - 10^{15}$	
Quartz (fused)	1.33×10^{-18}	75×10^{16}	
Rubber (hard)	$10^{-13} - 10^{-16}$	$10^{13} - 10^{16}$	
Sulfur	10^{-15}	10^{15}	
Teflon$^{\mathrm{TM}}$	$<10^{-13}$	$>10^{13}$	
Wood	$10^{-8} - 10^{-11}$	$10^{8} - 10^{11}$	

Resistivities and Conductivities of Various Materials at 20 °C[1] Values depend strongly on amounts and types of impurities.

The materials listed in the table are separated into categories of conductors, semiconductors, and insulators, based on broad groupings of resistivity. Conductors have the smallest resistivity, and insulators have the largest; semiconductors have intermediate resistivity. Conductors have varying but large, free charge densities, whereas most charges in insulators are bound to atoms and are not free to move. Semiconductors are intermediate, having far fewer free charges than conductors, but having properties that make the number of free charges depend strongly on the type and amount of impurities in the semiconductor. These unique properties of semiconductors are put to use in modern electronics, as we will explore in later chapters.

Note:
Exercise:

Problem:

Check Your Understanding Copper wires use routinely used for extension cords and house wiring for several reasons. Copper has the highest electrical conductivity rating, and therefore the lowest resistivity rating, of all nonprecious metals. Also important is the tensile strength, where the tensile strength is a measure of the force required to pull an object to the point where it breaks. The tensile strength of a material is the maximum amount of tensile stress it can take before breaking. Copper has a high tensile strength, $2 \times 10^8 \frac{\text{N}}{\text{m}^2}$. A third important characteristic is ductility. Ductility is a measure of a material's ability to be drawn into wires and a measure of the flexibility of the material, and copper has a high ductility. Summarizing, for a conductor to be a suitable candidate for making wire, there are at least three important characteristics: low resistivity, high tensile strength, and high ductility. What other materials are used for wiring and what are the advantages and disadvantages?

Solution:

Silver, gold, and aluminum are all used for making wires. All four materials have a high conductivity, silver having the highest. All four can easily be drawn into wires and have a high tensile strength, though not as high as copper. The obvious disadvantage of gold and silver is the cost, but silver and gold wires are used for special applications, such as speaker wires. Gold does not oxidize, making better connections between components. Aluminum wires do have their drawbacks. Aluminum has a higher resistivity than copper, so a larger diameter is needed to match the resistance per length of copper wires, but aluminum is cheaper than copper, so this is not a major drawback. Aluminum wires do not have as high of a ductility and tensile strength as copper, but the ductility and tensile strength is within acceptable levels. There are a few concerns that must be addressed in using aluminum and care must be used when making connections. Aluminum has a higher rate of thermal expansion than copper, which can lead to loose connections and a possible fire hazard. The oxidation of aluminum does not conduct and can cause problems. Special techniques must be used when using aluminum wires and components, such as electrical outlets, must be designed to accept aluminum wires.

Note:
View this interactive simulation to see what the effects of the cross-sectional area, the length, and the resistivity of a wire are on the resistance of a conductor. Adjust the variables using slide bars and see if the resistance becomes smaller or larger.

Temperature Dependence of Resistivity

Looking back at [link], you will see a column labeled "Temperature Coefficient." The resistivity of some materials has a strong temperature dependence. In some materials, such as copper, the resistivity increases with increasing temperature. In fact, in most conducting metals, the resistivity increases with increasing temperature. The increasing temperature causes increased vibrations of

the atoms in the lattice structure of the metals, which impede the motion of the electrons. In other materials, such as carbon, the resistivity decreases with increasing temperature. In many materials, the dependence is approximately linear and can be modeled using a linear equation:

Note:
Equation:

$$\rho \approx \rho_0 \left[1 + \alpha \left(T - T_0\right)\right],$$

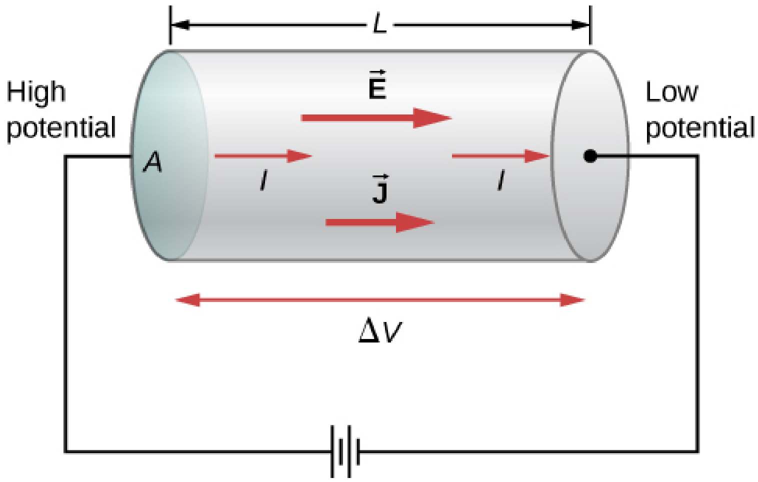
where ρ is the resistivity of the material at temperature T, α is the temperature coefficient of the material, and ρ_0 is the resistivity at T_0, usually taken as $T_0 = 20.00\ ^\circ\text{C}$.

Note also that the temperature coefficient α is negative for the semiconductors listed in [link], meaning that their resistivity decreases with increasing temperature. They become better conductors at higher temperature, because increased thermal agitation increases the number of free charges available to carry current. This property of decreasing ρ with temperature is also related to the type and amount of impurities present in the semiconductors.

Resistance

We now consider the resistance of a wire or component. The resistance is a measure of how difficult it is to pass current through a wire or component. Resistance depends on the resistivity. The resistivity is a characteristic of the material used to fabricate a wire or other electrical component, whereas the resistance is a characteristic of the wire or component.

To calculate the resistance, consider a section of conducting wire with cross-sectional area A, length L, and resistivity ρ. A battery is connected across the conductor, providing a potential difference ΔV across it ([link]). The potential difference produces an electrical field that is proportional to the current density, according to $\vec{\mathbf{E}} = \rho \vec{\mathbf{J}}$.



A potential provided by a battery is applied to a segment of a conductor with a cross-sectional area A and a length L.

The magnitude of the electrical field across the segment of the conductor is equal to the voltage divided by the length, $E = V/L$, and the magnitude of the current density is equal to the current divided by the cross-sectional area, $J = I/A$. Using this information and recalling that the electrical field is proportional to the resistivity and the current density, we can see that the voltage is proportional to the current:

Equation:

$$\begin{aligned} E &= \rho J \\ \frac{V}{L} &= \rho \frac{I}{A} \\ V &= \left(\rho \frac{L}{A}\right) I. \end{aligned}$$

Note:

Resistance

The ratio of the voltage to the current is defined as the **resistance** R:

Equation:

$$R \equiv \frac{V}{I}.$$

The resistance of a cylindrical segment of a conductor is equal to the resistivity of the material times the length divided by the area:

Equation:

$$R \equiv \frac{V}{I} = \rho \frac{L}{A}.$$

The unit of resistance is the ohm, Ω. For a given voltage, the higher the resistance, the lower the current.

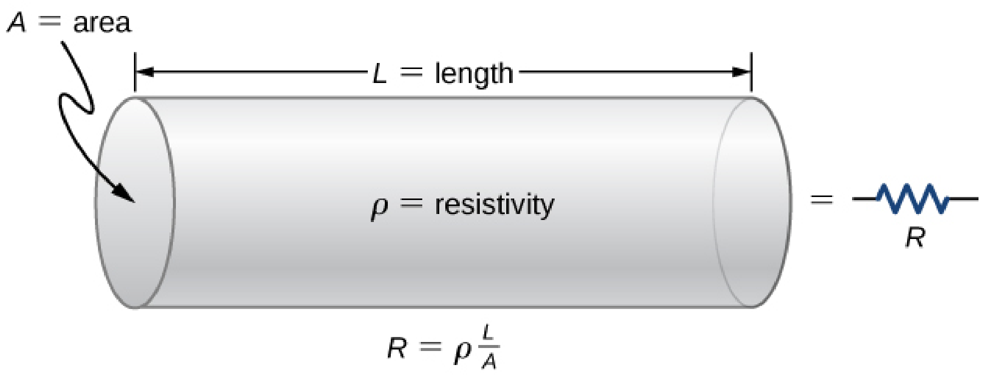
Resistors

A common component in electronic circuits is the resistor. The resistor can be used to reduce current flow or provide a voltage drop. [link] shows the symbols used for a resistor in schematic diagrams of a circuit. Two commonly used standards for circuit diagrams are provided by the American National Standard Institute (ANSI, pronounced "AN-see") and the International Electrotechnical Commission (IEC). Both systems are commonly used. We use the ANSI standard in this text for its visual recognition, but we note that for larger, more complex circuits, the IEC standard may have a cleaner presentation, making it easier to read.

Symbols for a resistor used in circuit diagrams. (a) The ANSI symbol; (b) the IEC symbol.

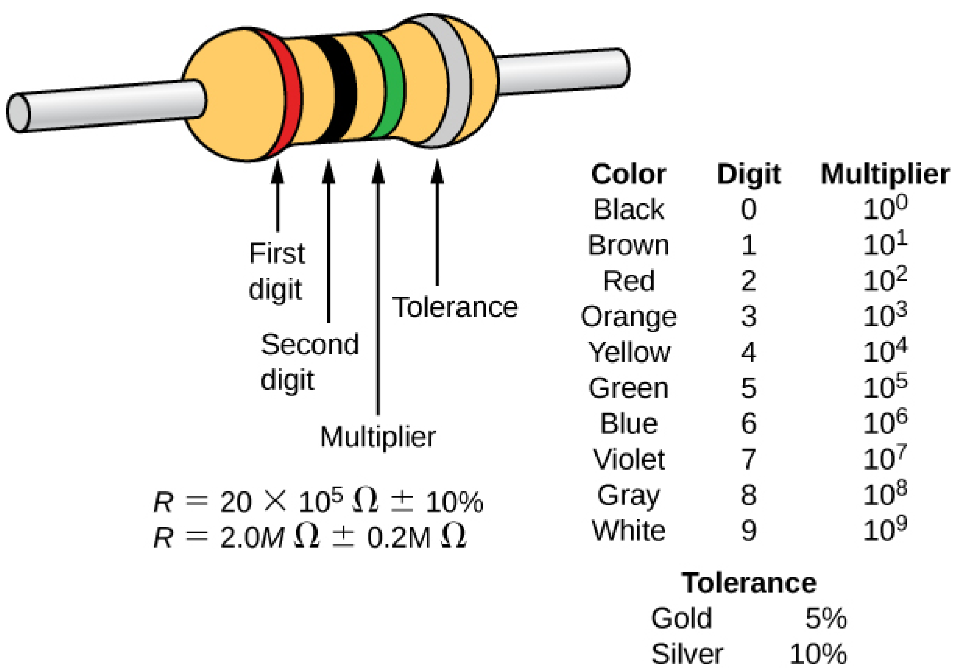
Material and shape dependence of resistance

A resistor can be modeled as a cylinder with a cross-sectional area A and a length L, made of a material with a resistivity ρ ([link]). The resistance of the resistor is $R = \rho \frac{L}{A}$.



A model of a resistor as a uniform cylinder of length *L* and cross-sectional area *A*. Its resistance to the flow of current is analogous to the resistance posed by a pipe to fluid flow. The longer the cylinder, the greater its resistance. The larger its cross-sectional area *A*, the smaller its resistance.

The most common material used to make a resistor is carbon. A carbon track is wrapped around a ceramic core, and two copper leads are attached. A second type of resistor is the metal film resistor, which also has a ceramic core. The track is made from a metal oxide material, which has semiconductive properties similar to carbon. Again, copper leads are inserted into the ends of the resistor. The resistor is then painted and marked for identification. A resistor has four colored bands, as shown in [link].



Many resistors resemble the figure shown above. The four bands are used to identify the resistor. The first two colored bands represent the first two digits of the resistance of the resistor. The third color is the multiplier. The fourth color represents the tolerance of the resistor. The resistor shown has a resistance of $20 \times 10^5 \, \Omega \pm 10\%$.

Resistances range over many orders of magnitude. Some ceramic insulators, such as those used to support power lines, have resistances of $10^{12} \, \Omega$ or more. A dry person may have a hand-to-foot resistance of $10^5 \, \Omega$, whereas the resistance of the human heart is about $10^3 \, \Omega$. A meter-long piece of large-diameter copper wire may have a resistance of $10^{-5} \, \Omega$, and superconductors have no resistance at all at low temperatures. As we have seen, resistance is related to the shape of an object and the material of which it is composed.

Example:

Current Density, Resistance, and Electrical field for a Current-Carrying Wire

Calculate the current density, resistance, and electrical field of a 5-m length of copper wire with a diameter of 2.053 mm (12-gauge) carrying a current of $I = 10 \, \text{mA}$.

Strategy

We can calculate the current density by first finding the cross-sectional area of the wire, which is $A = 3.31 \, \text{mm}^2$, and the definition of current density $J = \frac{I}{A}$. The resistance can be found using the length of the wire $L = 5.00 \, \text{m}$, the area, and the resistivity of copper $\rho = 1.68 \times 10^{-8} \, \Omega \cdot \text{m}$, where $R = \rho \frac{L}{A}$. The resistivity and current density can be used to find the electrical field.

Solution

First, we calculate the current density:

Equation:

$$J = \frac{I}{A} = \frac{10 \times 10^{-3} \text{A}}{3.31 \times 10^{-6} \text{m}^2} = 3.02 \times 10^3 \frac{\text{A}}{\text{m}^2}.$$

The resistance of the wire is

Equation:

$$R = \rho \frac{L}{A} = \left(1.68 \times 10^{-8} \, \Omega \cdot \text{m}\right) \frac{5.00 \, \text{m}}{3.31 \times 10^{-6} \text{m}^2} = 0.025 \, \Omega \, .$$

Finally, we can find the electrical field:

Equation:

$$E = \rho J = 1.68 \times 10^{-8} \, \Omega \cdot \text{m} \left(3.02 \times 10^3 \frac{\text{A}}{\text{m}^2}\right) = 5.07 \times 10^{-5} \frac{\text{V}}{\text{m}}.$$

Significance

From these results, it is not surprising that copper is used for wires for carrying current because the resistance is quite small. Note that the current density and electrical field are independent of the length of the wire, but the voltage depends on the length.

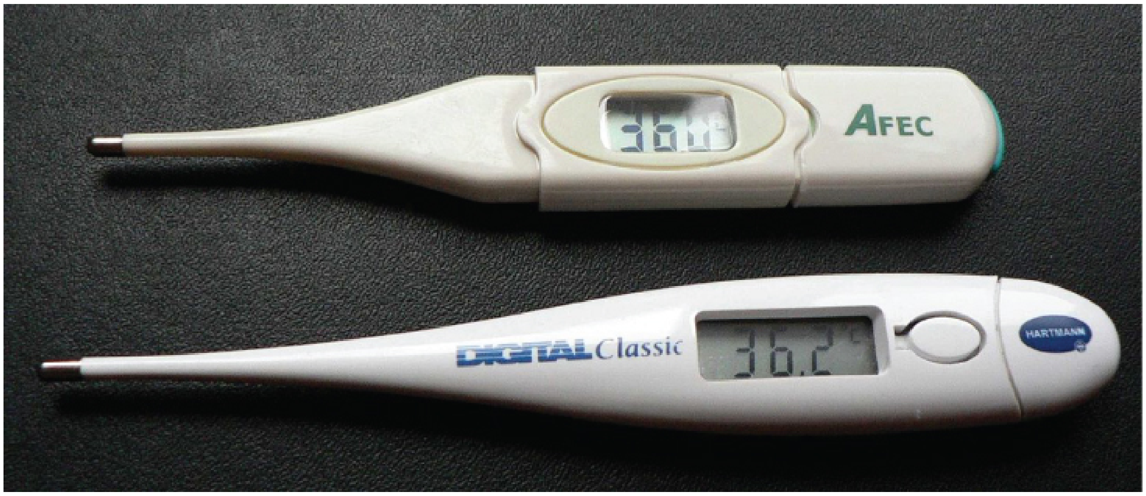
The resistance of an object also depends on temperature, since R_0 is directly proportional to ρ. For a cylinder, we know $R = \rho\frac{L}{A}$, so if L and A do not change greatly with temperature, R has the same temperature dependence as ρ. (Examination of the coefficients of linear expansion shows them to be about two orders of magnitude less than typical temperature coefficients of resistivity, so the effect of temperature on L and A is about two orders of magnitude less than on ρ.) Thus,

Note:
Equation:

$$R = R_0(1 + \alpha\Delta T)$$

is the temperature dependence of the resistance of an object, where R_0 is the original resistance (usually taken to be $20.00\ ^\circ\mathrm{C}$) and R is the resistance after a temperature change ΔT. The color code gives the resistance of the resistor at a temperature of $T = 20.00\ ^\circ\mathrm{C}$.

Numerous thermometers are based on the effect of temperature on resistance ([link]). One of the most common thermometers is based on the thermistor, a semiconductor crystal with a strong temperature dependence, the resistance of which is measured to obtain its temperature. The device is small, so that it quickly comes into thermal equilibrium with the part of a person it touches.



These familiar thermometers are based on the automated measurement of a thermistor’s temperature-dependent resistance.

Example:

Calculating Resistance

Although caution must be used in applying $\rho = \rho_0(1 + \alpha\Delta T)$ and $R = R_0(1 + \alpha\Delta T)$ for temperature changes greater than $100\ ^\circ\mathrm{C}$, for tungsten, the equations work reasonably well for very large temperature changes. A tungsten filament at $20\ ^\circ\mathrm{C}$ has a resistance of $0.350\,\Omega$. What would the resistance be if the temperature is increased to $2850\ ^\circ\mathrm{C}$?

Strategy

This is a straightforward application of $R = R_0(1 + \alpha\Delta T)$, since the original resistance of the filament is given as $R_0 = 0.350\,\Omega$ and the temperature change is $\Delta T = 2830\ ^\circ\mathrm{C}$.

Solution

The resistance of the hotter filament *R* is obtained by entering known values into the above equation:

Equation:

$$R = R_0\left(1 + \alpha\Delta T\right) = \left(0.350\,\Omega\right)\left[1 + \left(\frac{4.5\ \times\ 10^{-3}}{^\circ\mathrm{C}}\right)\left(2830\ ^\circ\mathrm{C}\right)\right] = 4.8\,\Omega\,.$$

Significance

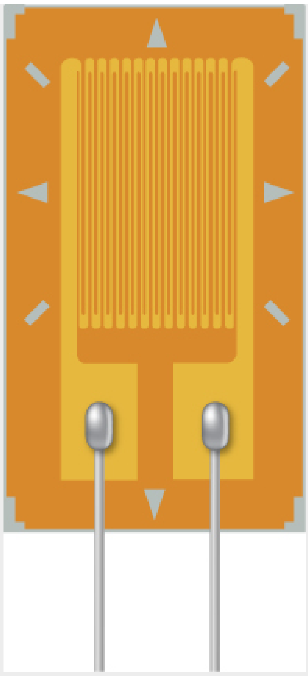
Notice that the resistance changes by more than a factor of 10 as the filament warms to the high temperature and the current through the filament depends on the resistance of the filament and the voltage applied. If the filament is used in an incandescent light bulb, the initial current through the filament when the bulb is first energized will be higher than the current after the filament reaches the operating temperature.

Note:

Exercise:

Problem:

Check Your Understanding A strain gauge is an electrical device to measure strain, as shown below. It consists of a flexible, insulating backing that supports a conduction foil pattern. The resistance of the foil changes as the backing is stretched. How does the strain gauge resistance change? Is the strain gauge affected by temperature changes?



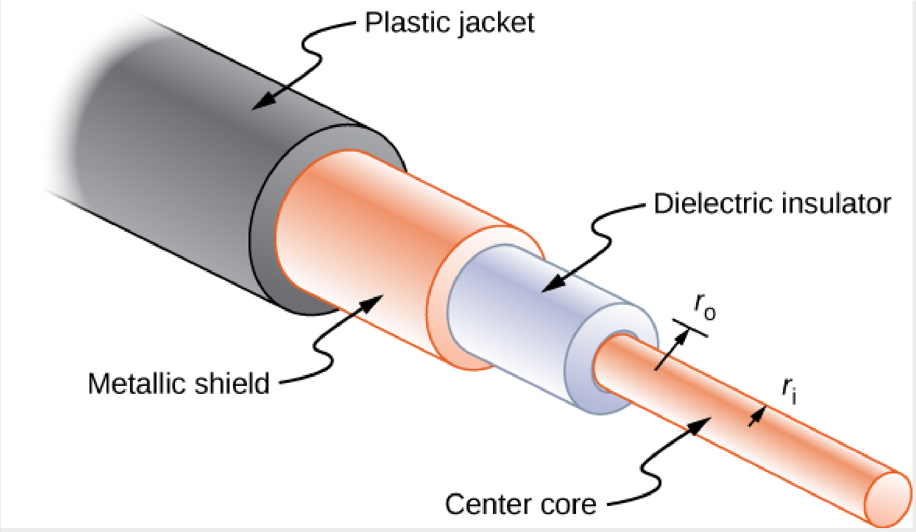
Solution:

The foil pattern stretches as the backing stretches, and the foil tracks become longer and thinner. Since the resistance is calculated as $R = \rho \frac{L}{A}$, the resistance increases as the foil tracks are stretched. When the temperature changes, so does the resistivity of the foil tracks, changing the resistance. One way to combat this is to use two strain gauges, one used as a reference and the other used to measure the strain. The two strain gauges are kept at a constant temperature

Example:

The Resistance of Coaxial Cable

Long cables can sometimes act like antennas, picking up electronic noise, which are signals from other equipment and appliances. Coaxial cables are used for many applications that require this noise to be eliminated. For example, they can be found in the home in cable TV connections or other audiovisual connections. Coaxial cables consist of an inner conductor of radius r_{i} surrounded by a second, outer concentric conductor with radius r_{o} ([link]). The space between the two is normally filled with an insulator such as polyethylene plastic. A small amount of radial leakage current occurs between the two conductors. Determine the resistance of a coaxial cable of length L.



Coaxial cables consist of two concentric conductors separated by insulation. They are often used in cable TV or other audiovisual connections.

Strategy

We cannot use the equation $R = \rho \frac{L}{A}$ directly. Instead, we look at concentric cylindrical shells, with thickness *dr*, and integrate.

Solution

We first find an expression for *dR* and then integrate from r_{i} to r_{o},

Equation:

$$\begin{aligned} dR &= \frac{\rho}{A}dr = \frac{\rho}{2\pi rL}dr, \\ R &= \int_{r_{\text{i}}}^{r_{\text{o}}} dR = \int_{r_{\text{i}}}^{r_{\text{o}}} \frac{\rho}{2\pi rL}dr = \frac{\rho}{2\pi L}\int_{r_{\text{i}}}^{r_{\text{o}}} \frac{1}{r}dr = \frac{\rho}{2\pi L}\ln\frac{r_{\text{o}}}{r_{\text{i}}}. \end{aligned}$$

Significance

The resistance of a coaxial cable depends on its length, the inner and outer radii, and the resistivity of the material separating the two conductors. Since this resistance is not infinite, a small leakage current occurs between the two conductors. This leakage current leads to the attenuation (or weakening) of the signal being sent through the cable.

Note:

Exercise:

Problem:

Check Your Understanding The resistance between the two conductors of a coaxial cable depends on the resistivity of the material separating the two conductors, the length of the cable and the inner and outer radius of the two conductor. If you are designing a coaxial cable, how does the resistance between the two conductors depend on these variables?

Solution:

The longer the length, the smaller the resistance. The greater the resistivity, the higher the resistance. The larger the difference between the outer radius and the inner radius, that is, the greater the ratio between the two, the greater the resistance. If you are attempting to maximize the resistance, the choice of the values for these variables will depend on the application. For example, if the cable must be flexible, the choice of materials may be limited.

Note:
View this simulation to see how the voltage applied and the resistance of the material the current flows through affects the current through the material. You can visualize the collisions of the electrons and the atoms of the material effect the temperature of the material.

Summary

- Resistance has units of ohms (Ω), related to volts and amperes by $1\,\Omega\ =\ 1\,\text{V/A}$.
- The resistance R of a cylinder of length L and cross-sectional area A is $R = \frac{\rho L}{A}$, where ρ is the resistivity of the material.
- Values of ρ in [link] show that materials fall into three groups—conductors, semiconductors, and insulators.
- Temperature affects resistivity; for relatively small temperature changes ΔT, resistivity is $\rho = \rho_0\,(1 + \alpha\Delta T)$, where ρ_0 is the original resistivity and α is the temperature coefficient of resistivity.
- The resistance R of an object also varies with temperature: $R = R_0\,(1 + \alpha\Delta T)$, where R_0 is the original resistance, and R is the resistance after the temperature change.

Conceptual Questions

Exercise:

Problem:

The IR drop across a resistor means that there is a change in potential or voltage across the resistor. Is there any change in current as it passes through a resistor? Explain.

Exercise:

Problem:

Do impurities in semiconducting materials listed in [link] supply free charges? (*Hint*: Examine the range of resistivity for each and determine whether the pure semiconductor has the higher or lower conductivity.)

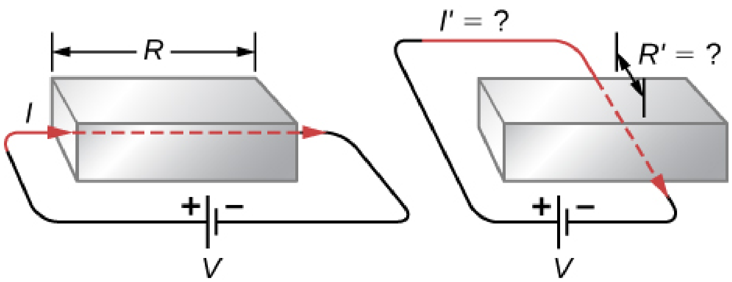
Solution:

In carbon, resistivity increases with the amount of impurities, meaning fewer free charges. In silicon and germanium, impurities decrease resistivity, meaning more free electrons.

Exercise:

Problem:

Does the resistance of an object depend on the path current takes through it? Consider, for example, a rectangular bar—is its resistance the same along its length as across its width?



Exercise:

Problem:

If aluminum and copper wires of the same length have the same resistance, which has the larger diameter? Why?

Solution:

Copper has a lower resistivity than aluminum, so if length is the same, copper must have the smaller diameter.

Problems

Exercise:

Problem:

What current flows through the bulb of a 3.00-V flashlight when its hot resistance is $3.60\ \Omega$?

Exercise:

Problem:

Calculate the effective resistance of a pocket calculator that has a 1.35-V battery and through which 0.200 mA flows.

Solution:

$R = 6.750\,\mathrm{k}\,\Omega$

Exercise:

Problem:

How many volts are supplied to operate an indicator light on a DVD player that has a resistance of $140\ \Omega$, given that 25.0 mA passes through it?

Exercise:

Problem:

What is the resistance of a 20.0-m-long piece of 12-gauge copper wire having a 2.053-mm diameter?

Solution:

$R = 0.10\ \Omega$

Exercise:

Problem:

The diameter of 0-gauge copper wire is 8.252 mm. Find the resistance of a 1.00-km length of such wire used for power transmission.

Exercise:

Problem:

If the 0.100-mm-diameter tungsten filament in a light bulb is to have a resistance of $0.200\ \Omega$ at $20.0\ ^\circ\mathrm{C}$, how long should it be?

Solution:

$R = \rho \frac{L}{A}$

$L = 3\ \mathrm{cm}$

Exercise:

Problem:

A lead rod has a length of 30.00 cm and a resistance of $5.00\ \mu\Omega$. What is the radius of the rod?

Exercise:

Problem:

Find the ratio of the diameter of aluminum to copper wire, if they have the same resistance per unit length (as they might in household wiring).

Solution:

$$\frac{R_{\text{Al}}/L_{\text{Al}}}{R_{\text{Cu}}/L_{\text{Cu}}}=\frac{\rho_{\text{Al}}\frac{1}{\pi\left(\frac{D_{\text{Al}}}{2}\right)^2}}{\rho_{\text{Cu}}\frac{1}{\pi\left(\frac{D_{\text{Cu}}}{2}\right)^2}}=\frac{\rho_{\text{Al}}}{\rho_{\text{Cu}}}\left(\frac{D_{\text{Cu}}}{D_{\text{Al}}}\right)^2=1,\quad \frac{D_{\text{Al}}}{D_{\text{Cu}}}=\sqrt{\frac{\rho_{\text{Al}}}{\rho_{Cu}}}$$

Exercise:

Problem:

What current flows through a 2.54-cm-diameter rod of pure silicon that is 20.0 cm long, when 1.00×10^3 V is applied to it? (Such a rod may be used to make nuclear-particle detectors, for example.)

Exercise:

Problem:

(a) To what temperature must you raise a copper wire, originally at $20.0\ ^\circ\text{C}$, to double its resistance, neglecting any changes in dimensions? (b) Does this happen in household wiring under ordinary circumstances?

Solution:

a. $R=R_0\left(1+\alpha\Delta T\right),\quad 2=1+\alpha\Delta T,\quad \Delta T=256.4\ ^\circ\text{C},\quad T=276.4\ ^\circ\text{C};$
b. Under normal conditions, no it should not occur.

Exercise:

Problem:

A resistor made of nichrome wire is used in an application where its resistance cannot change more than 1.00% from its value at $20.0\ ^\circ\text{C}$. Over what temperature range can it be used?

Exercise:

Problem:

Of what material is a resistor made if its resistance is 40.0% greater at $100.0\ ^\circ\text{C}$ than at $20.0\ ^\circ\text{C}$?

Solution:

$$\begin{aligned}R&=R_0\left(1+\alpha\Delta T\right)\\ \alpha&=0.006\ ^\circ\text{C}^{-1}\end{aligned}$$, iron

Exercise:

Problem:

An electronic device designed to operate at any temperature in the range from $-10.0\ ^\circ\text{C}$ to $55.0\ ^\circ\text{C}$ contains pure carbon resistors. By what factor does their resistance increase over this range?

Exercise:

Problem:

(a) Of what material is a wire made, if it is 25.0 m long with a diameter of 0.100 mm and has a resistance of $77.7\ \Omega$ at $20.0\ ^\circ\text{C}$? (b) What is its resistance at $150.0\ ^\circ\text{C}$?

Solution:

a. $R = \rho \frac{L}{A}, \quad \rho = 2.44\ \times\ 10^{-8}\ \Omega\ \cdot \text{m}$, gold;
$R = \rho \frac{L}{A}(1 + \alpha \Delta T)$

b. $R = 2.44\ \times\ 10^{-8}\ \Omega\ \cdot \text{m} \left(\frac{25\ \text{m}}{\pi \left(\frac{0.100 \times 10^{-3} \text{m}}{2} \right)^2} \right) \left(1 + 0.0034\ ^\circ\text{C}^{-1}\ (150\ ^\circ\text{C} - 20\ ^\circ\text{C})\right)$

$R = 112\ \Omega$

Exercise:

Problem:

Assuming a constant temperature coefficient of resistivity, what is the maximum percent decrease in the resistance of a constantan wire starting at $20.0\ ^\circ\text{C}$?

Exercise:

Problem:

A copper wire has a resistance of $0.500\ \Omega$ at $20.0\ ^\circ\text{C}$, and an iron wire has a resistance of $0.525\ \Omega$ at the same temperature. At what temperature are their resistances equal?

Solution:

$R_{\text{Fe}} = 0.525\ \Omega\,, \quad R_{\text{Cu}} = 0.500\ \Omega\,, \quad \alpha_{\text{Fe}} = 0.0065\ ^\circ\text{C}^{-1} \quad \alpha_{\text{Cu}} = 0.0039\ ^\circ\text{C}^{-1}$
$R_{\text{Fe}} = R_{\text{Cu}}$
$R_{0\,\text{Fe}}\left(1 + \alpha_{\text{Fe}}\left(T - T_0\right)\right) = R_{0\,\text{Cu}}\left(1 + \alpha_{\text{Cu}}\left(T - T_0\right)\right)$
$\frac{R_{0\,\text{Fe}}}{R_{0\,\text{Cu}}}\left(1 + \alpha_{\text{Fe}}\left(T - T_0\right)\right) = 1 + \alpha_{\text{Cu}}\left(T - T_0\right)$
$T = 2.91\ ^\circ\text{C}$

Glossary

electrical conductivity
: measure of a material’s ability to conduct or transmit electricity

ohm
: (Ω) unit of electrical resistance, $1\,\Omega = 1\,\mathrm{V/A}$

resistance
: electric property that impedes current; for ohmic materials, it is the ratio of voltage to current, $R = V/I$

resistivity
: intrinsic property of a material, independent of its shape or size, directly proportional to the resistance, denoted by ρ

Ohm's Law

By the end of this section, you will be able to:

- Describe Ohm’s law
- Recognize when Ohm’s law applies and when it does not

We have been discussing three electrical properties so far in this chapter: current, voltage, and resistance. It turns out that many materials exhibit a simple relationship among the values for these properties, known as Ohm’s law. Many other materials do not show this relationship, so despite being called Ohm’s law, it is not considered a law of nature, like Newton’s laws or the laws of thermodynamics. But it is very useful for calculations involving materials that do obey Ohm’s law.

Description of Ohm’s Law

The current that flows through most substances is directly proportional to the voltage *V* applied to it. The German physicist Georg Simon Ohm (1787–1854) was the first to demonstrate experimentally that the current in a metal wire is *directly proportional to the voltage applied*:

Equation:

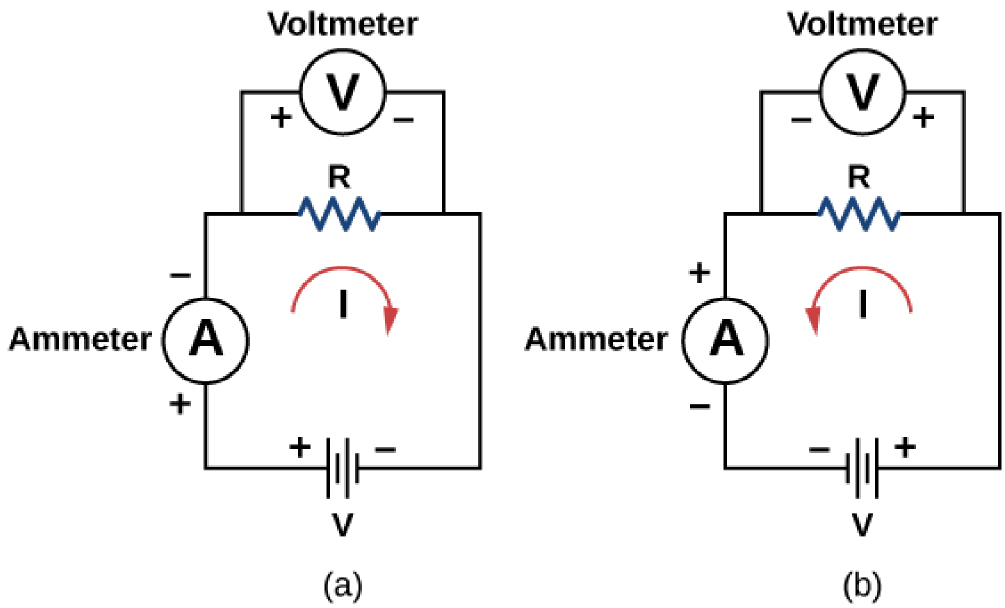
$$I \propto V.$$

This important relationship is the basis for **Ohm’s law**. It can be viewed as a cause-and-effect relationship, with voltage the cause and current the effect. This is an empirical law, which is to say that it is an experimentally observed phenomenon, like friction. Such a linear relationship doesn’t always occur. Any material, component, or device that obeys Ohm’s law, where the current through the device is proportional to the voltage applied, is known as an **ohmic** material or ohmic component. Any material or component that does not obey Ohm’s law is known as a **nonohmic** material or nonohmic component.

Ohm’s Experiment

In a paper published in 1827, Georg Ohm described an experiment in which he measured voltage across and current through various simple electrical circuits containing various lengths of wire. A similar experiment is shown in [link]. This experiment is used to observe the current through a resistor that results from an applied voltage. In this simple circuit, a resistor is connected in series with a battery. The voltage is measured with a voltmeter, which must be placed across the resistor

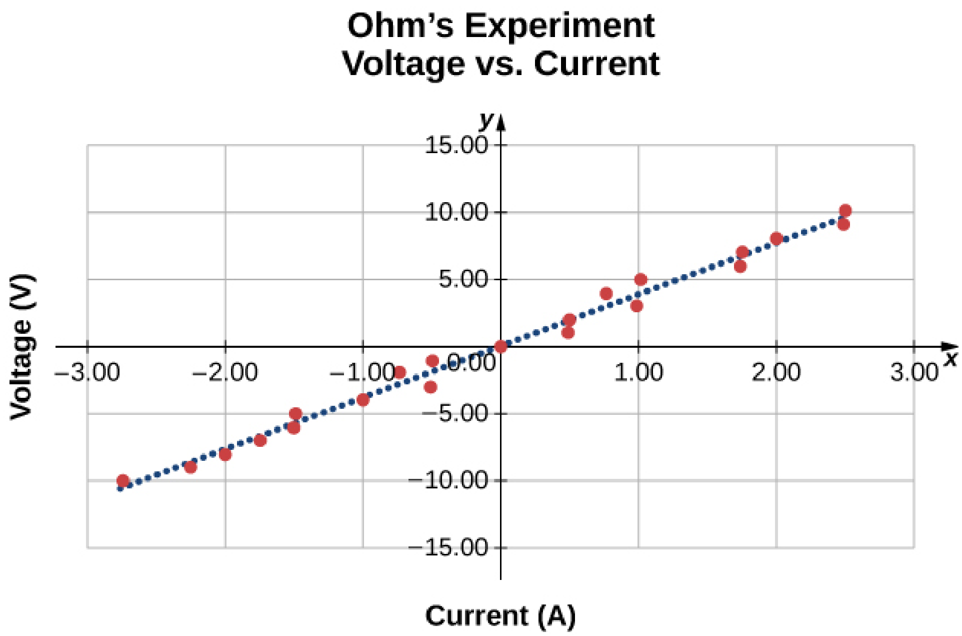
(in parallel with the resistor). The current is measured with an ammeter, which must be in line with the resistor (in series with the resistor).



The experimental set-up used to determine if a resistor is an ohmic or nonohmic device. (a) When the battery is attached, the current flows in the clockwise direction and the voltmeter and ammeter have positive readings. (b) When the leads of the battery are switched, the current flows in the counterclockwise direction and the voltmeter and ammeter have negative readings.

In this updated version of Ohm's original experiment, several measurements of the current were made for several different voltages. When the battery was hooked up as in [link](a), the current flowed in the clockwise direction and the readings of the voltmeter and ammeter were positive. Does the behavior of the current change if the current flowed in the opposite direction? To get the current to flow in the opposite direction, the leads of the battery can be switched. When the leads of the battery were switched, the readings of the voltmeter and ammeter readings were negative because the current flowed in the opposite direction, in this case, counterclockwise. Results of a similar experiment are shown in [link].

I(A)	V(V)
–2.74	–10.00
–2.25	–9.00
–2.00	–8.00
–1.75	–7.00
–1.50	–6.00
–1.49	–5.00
–1.00	–4.00
–0.51	–3.00
–0.74	–2.00
–0.49	–1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00



$V = IR$
$R = 3.84\Omega$

A resistor is placed in a circuit with a battery. The voltage applied varies from −10.00 V to +10.00 V, increased by 1.00-V increments. A plot shows values of the voltage versus the current typical of what a casual experimenter might find.

In this experiment, the voltage applied across the resistor varies from −10.00 to +10.00 V, by increments of 1.00 V. The current through the resistor and the voltage across the resistor are measured. A plot is made of the voltage versus the current, and the result is approximately linear. The slope of the line is the resistance, or the voltage divided by the current. This result is known as Ohm's law:

Note:
Equation:

$$V = IR,$$

where *V* is the voltage measured in volts across the object in question, *I* is the current measured through the object in amps, and *R* is the resistance in units of ohms. As stated previously, any device that shows a linear relationship between the voltage and the current is known as an ohmic device. A resistor is therefore an ohmic device.

Example:
Measuring Resistance
A carbon resistor at room temperature $(20\ ^{\circ}\mathrm{C})$ is attached to a 9.00-V battery and the current measured through the resistor is 3.00 mA. (a) What is the resistance of the resistor measured in ohms? (b) If the temperature of the resistor is increased to $60\ ^{\circ}\mathrm{C}$ by heating the resistor, what is the current through the resistor?
Strategy
(a) The resistance can be found using Ohm's law. Ohm's law states that $V = IR$, so the resistance can be found using $R = V/I$.
(b) First, the resistance is temperature dependent so the new resistance after the resistor has been heated can be found using $R = R_0\,(1 + \alpha\Delta T)$. The current can be found using Ohm's law in the form $I = V/R$.
Solution

a. Using Ohm's law and solving for the resistance yields the resistance at room temperature:
   Equation:

$$R = \frac{V}{I} = \frac{9.00\ \mathrm{V}}{3.00\ \times\ 10^{-3}\ \mathrm{A}} = 3.00\ \times\ 10^{3}\ \Omega\ = 3.00\ \mathrm{k}\ \Omega.$$

b. The resistance at $60\ ^{\circ}\mathrm{C}$ can be found using $R = R_0\,(1 + \alpha\Delta T)$ where the temperature coefficient for carbon is $\alpha = -0.0005$.
   $R = R_0\,(1 + \alpha\Delta T) = 3.00\ \times\ 10^{3}\,(1 - 0.0005\,(60\ ^{\circ}\mathrm{C} - 20\ ^{\circ}\mathrm{C})) = 2.94\ \mathrm{k}\ \Omega$
   .
   The current through the heated resistor is
   Equation:

$$I = \frac{V}{R} = \frac{9.00\ \mathrm{V}}{2.94\ \times\ 10^{3}\ \Omega} = 3.06\ \times\ 10^{-3}\mathrm{A} = 3.06\ \mathrm{mA}.$$

Significance
A change in temperature of $40\ ^{\circ}\mathrm{C}$ resulted in a 2.00% change in current. This may not seem like a very great change, but changing electrical characteristics can have a

strong effect on the circuits. For this reason, many electronic appliances, such as computers, contain fans to remove the heat dissipated by components in the electric circuits.

Note:
Exercise:

Problem:

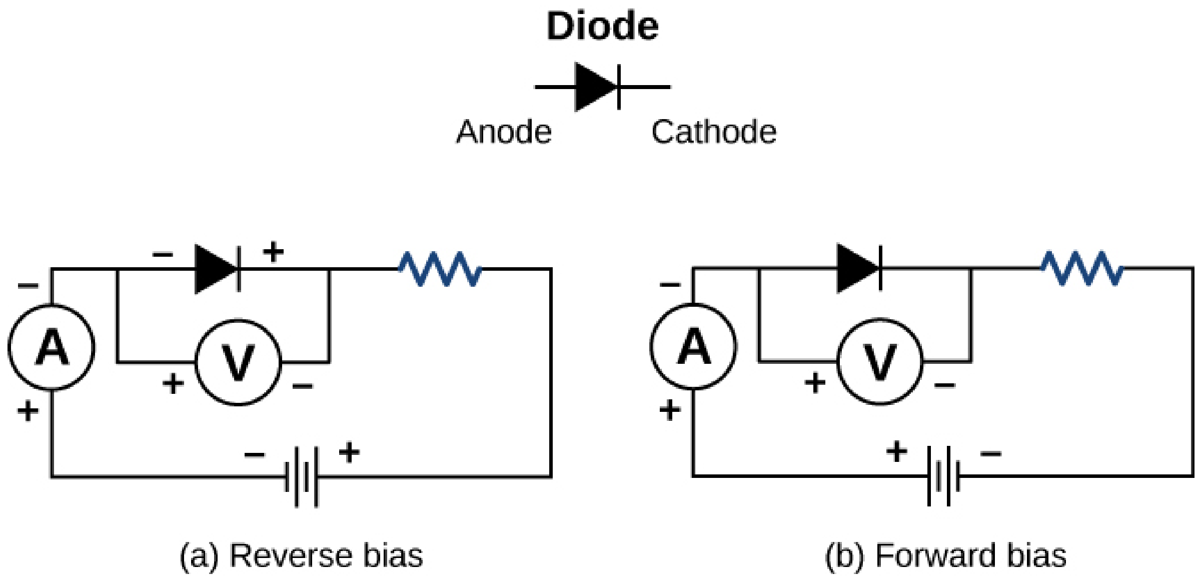
Check Your Understanding The voltage supplied to your house varies as $V\left(t\right)=V_{\text{max}}\,\text{sin}\,\left(2\pi ft\right)$. If a resistor is connected across this voltage, will Ohm's law $V=IR$ still be valid?

Solution:

Yes, Ohm's law is still valid. At every point in time the current is equal to $I(t)=V(t)/R$, so the current is also a function of time, $I\left(t\right)=\frac{V_{\text{max}}}{R}\,\text{sin}\,\left(2\pi ft\right)$.

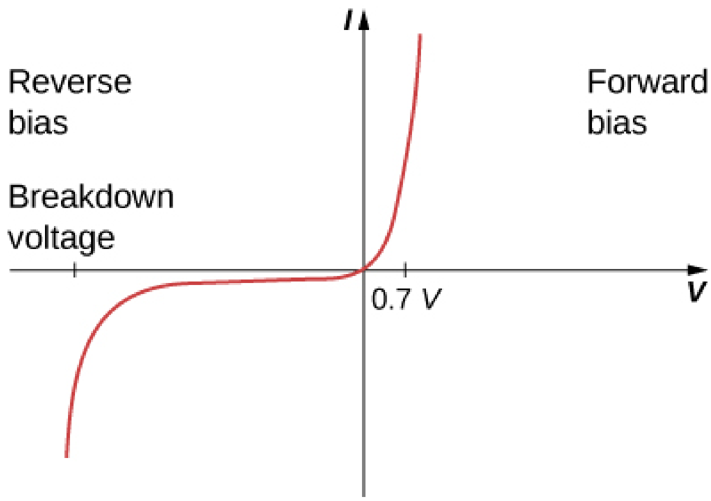
Note:
See how the equation form of Ohm's law relates to a simple circuit. Adjust the voltage and resistance, and see the current change according to Ohm's law. The sizes of the symbols in the equation change to match the circuit diagram.

Nonohmic devices do not exhibit a linear relationship between the voltage and the current. One such device is the semiconducting circuit element known as a diode. A **diode** is a circuit device that allows current flow in only one direction. A diagram of a simple circuit consisting of a battery, a diode, and a resistor is shown in [link]. Although we do not cover the theory of the diode in this section, the diode can be tested to see if it is an ohmic or a nonohmic device.



A diode is a semiconducting device that allows current flow only if the diode is forward biased, which means that the anode is positive and the cathode is negative.

A plot of current versus voltage is shown in [link]. Note that the behavior of the diode is shown as current versus voltage, whereas the resistor operation was shown as voltage versus current. A diode consists of an anode and a cathode. When the anode is at a negative potential and the cathode is at a positive potential, as shown in part (a), the diode is said to have reverse bias. With reverse bias, the diode has an extremely large resistance and there is very little current flow—essentially zero current—through the diode and the resistor. As the voltage applied to the circuit increases, the current remains essentially zero, until the voltage reaches the breakdown voltage and the diode conducts current, as shown in [link]. When the battery and the potential across the diode are reversed, making the anode positive and the cathode negative, the diode conducts and current flows through the diode if the voltage is greater than 0.7 V. The resistance of the diode is close to zero. (This is the reason for the resistor in the circuit; if it were not there, the current would become very large.) You can see from the graph in [link] that the voltage and the current do not have a linear relationship. Thus, the diode is an example of a nonohmic device.



When the voltage across the diode is negative and small, there is very little current flow through the diode. As the voltage reaches the breakdown voltage, the diode conducts. When the voltage across the diode is positive and greater than 0.7 V (the actual voltage value depends on the diode), the diode conducts. As the voltage applied increases, the current through the diode increases, but the voltage across the diode remains approximately 0.7 V.

Ohm's law is commonly stated as $V = IR$, but originally it was stated as a microscopic view, in terms of the current density, the conductivity, and the electrical field. This microscopic view suggests the proportionality $V \propto I$ comes from the drift velocity of the free electrons in the metal that results from an applied electrical field. As stated earlier, the current density is proportional to the applied electrical field. The reformulation of Ohm's law is credited to Gustav Kirchhoff, whose name we will see again in the next chapter.

Summary

- Ohm's law is an empirical relationship for current, voltage, and resistance for some common types of circuit elements, including resistors. It does not apply to other devices, such as diodes.

- One statement of Ohm's law gives the relationship among current I, voltage V, and resistance R in a simple circuit as $V = IR$.
- Another statement of Ohm's law, on a microscopic level, is $J = \sigma E$.

Conceptual Questions

Exercise:

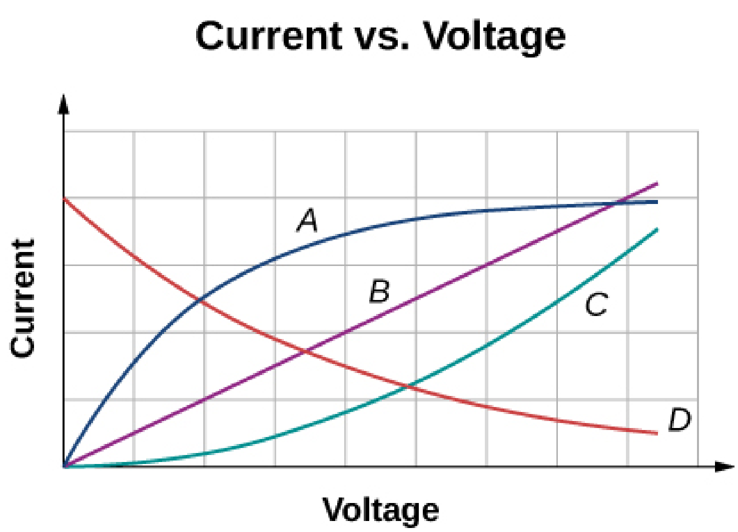
Problem:

In Determining Field from Potential, resistance was defined as $R \equiv \frac{V}{I}$. In this section, we presented Ohm's law, which is commonly expressed as $V = IR$. The equations look exactly alike. What is the difference between Ohm's law and the definition of resistance?

Exercise:

Problem:

Shown below are the results of an experiment where four devices were connected across a variable voltage source. The voltage is increased and the current is measured. Which device, if any, is an ohmic device?



Solution:

Device B shows a linear relationship and the device is ohmic.

Exercise:

Problem:

The current I is measured through a sample of an ohmic material as a voltage V is applied. (a) What is the current when the voltage is doubled to $2V$ (assume the change in temperature of the material is negligible)? (b) What is the voltage applied is the current measured is $0.2I$ (assume the change in temperature of the material is negligible)? What will happen to the current if the material if the voltage remains constant, but the temperature of the material increases significantly?

Problems

Exercise:

Problem:

A $2.2\text{-k}\,\Omega$ resistor is connected across a D cell battery (1.5 V). What is the current through the resistor?

Exercise:

Problem:

A resistor rated at $250\text{ k}\,\Omega$ is connected across two D cell batteries (each 1.50 V) in series, with a total voltage of 3.00 V. The manufacturer advertises that their resistors are within 5% of the rated value. What are the possible minimum current and maximum current through the resistor?

Solution:

$$R_{\text{min}} = 2.375 \times 10^5\,\Omega, \quad I_{\text{min}} = 12.63\,\mu\,\text{A}$$
$$R_{\text{max}} = 2.625 \times 10^5\,\Omega, \quad I_{\text{max}} = 11.43\,\mu\,\text{A}$$

Exercise:

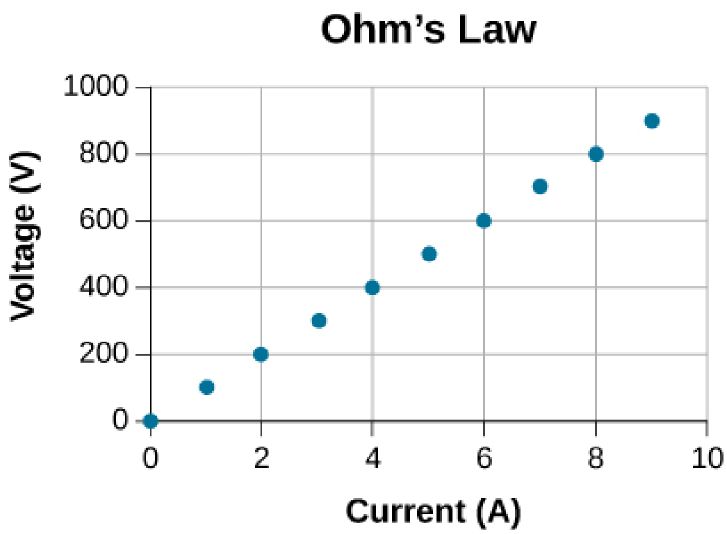
Problem:

A resistor is connected in series with a power supply of 20.00 V. The current measure is 0.50 A. What is the resistance of the resistor?

Exercise:

Problem:

A resistor is placed in a circuit with an adjustable voltage source. The voltage across and the current through the resistor and the measurements are shown below. Estimate the resistance of the resistor.



Solution:

$R = 100\ \Omega$

Exercise:

Problem:

The following table show the measurements of a current through and the voltage across a sample of material. Plot the data, and assuming the object is an ohmic device, estimate the resistance.

***I*(A)**	***V*(V)**
0	3
2	23

I(A)	V(V)
4	39
6	58
8	77
10	100
12	119
14	142
16	162

Glossary

diode
: nonohmic circuit device that allows current flow in only one direction

Ohm's law
: empirical relation stating that the current I is proportional to the potential difference V; it is often written as $V = IR$, where R is the resistance

ohmic
: type of a material for which Ohm's law is valid, that is, the voltage drop across the device is equal to the current times the resistance

nonohmic
: type of a material for which Ohm's law is not valid

Electrical Energy and Power
By the end of this section, you will be able to:

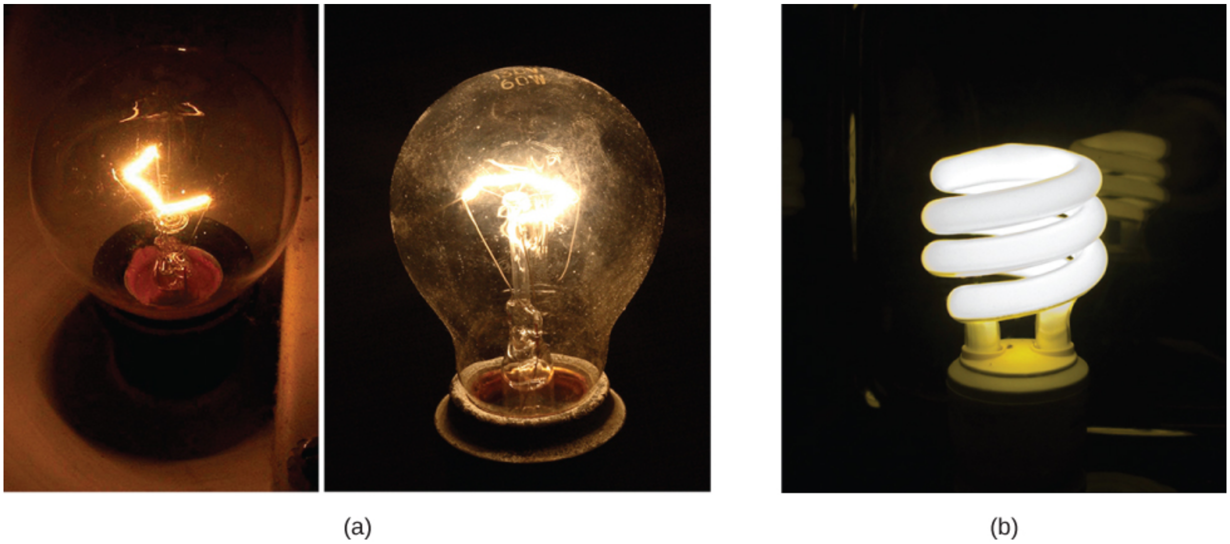
- Express electrical power in terms of the voltage and the current
- Describe the power dissipated by a resistor in an electric circuit
- Calculate the energy efficiency and cost effectiveness of appliances and equipment

In an electric circuit, electrical energy is continuously converted into other forms of energy. For example, when a current flows in a conductor, electrical energy is converted into thermal energy within the conductor. The electrical field, supplied by the voltage source, accelerates the free electrons, increasing their kinetic energy for a short time. This increased kinetic energy is converted into thermal energy through collisions with the ions of the lattice structure of the conductor. In Work and Kinetic Energy, we defined power as the rate at which work is done by a force measured in watts. Power can also be defined as the rate at which energy is transferred. In this section, we discuss the time rate of energy transfer, or power, in an electric circuit.

Power in Electric Circuits

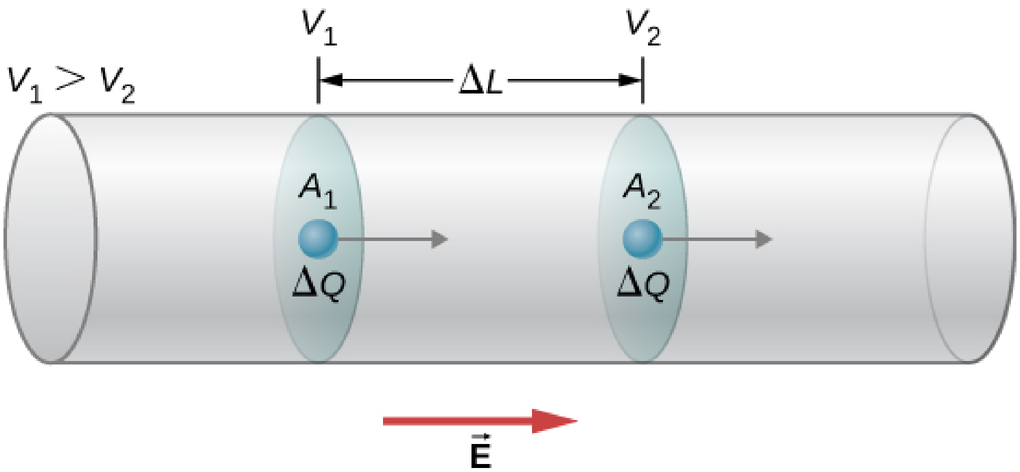
Power is associated by many people with electricity. Power transmission lines might come to mind. We also think of light bulbs in terms of their power ratings in watts. What is the expression for **electric power**?

Let us compare a 25-W bulb with a 60-W bulb ([link](a)). The 60-W bulb glows brighter than the 25-W bulb. Although it is not shown, a 60-W light bulb is also warmer than the 25-W bulb. The heat and light is produced by from the conversion of electrical energy. The kinetic energy lost by the electrons in collisions is converted into the internal energy of the conductor and radiation. How are voltage, current, and resistance related to electric power?



(a) Pictured above are two incandescent bulbs: a 25-W bulb (left) and a 60-W bulb (right). The 60-W bulb provides a higher intensity light than the 25-W bulb. The electrical energy supplied to the light bulbs is converted into heat and light. (b) This compact fluorescent light (CFL) bulb puts out the same intensity of light as the 60-W bulb, but at 1/4 to 1/10 the input power. (credit a: modification of works by “Dickbauch”/Wikimedia Commons and Greg Westfall; credit b: modification of work by “dbgg1979”/Flickr)

To calculate electric power, consider a voltage difference existing across a material ([link]). The electric potential V_1 is higher than the electric potential at V_2, and the voltage difference is negative $V = V_2 - V_1$. As discussed in Electric Potential, an electrical field exists between the two potentials, which points from the higher potential to the lower potential. Recall that the electrical potential is defined as the potential energy per charge, $V = \Delta U/q$, and the charge ΔQ loses potential energy moving through the potential difference.



When there is a potential difference across a conductor, an electrical field is present that points in the direction from the higher potential to the lower potential.

If the charge is positive, the charge experiences a force due to the electrical field $\vec{\mathbf{F}} = m\vec{\mathbf{a}} = \Delta Q\vec{\mathbf{E}}$. This force is necessary to keep the charge moving. This force does not act to accelerate the charge through the entire distance ΔL because of the interactions of the charge with atoms and free electrons in the material. The speed, and therefore the kinetic energy, of the charge do not increase during the entire trip across ΔL, and charge passing through area A_2 has the same drift velocity v_{d} as the charge that passes through area A_1. However, work is done on the charge, by the electrical field, which changes the potential energy. Since the change in the electrical potential difference is negative, the electrical field is found to be

Equation:

$$E = -\frac{(V_2 - V_1)}{\Delta L} = \frac{V}{\Delta L}.$$

The work done on the charge is equal to the electric force times the length at which the force is applied,

Equation:

$$W = F\Delta L = (\Delta QE)\Delta L = \left(\Delta Q\frac{V}{\Delta L}\right)\Delta L = \Delta QV = \Delta U.$$

The charge moves at a drift velocity v_{d} so the work done on the charge results in a loss of potential energy, but the average kinetic energy remains constant. The lost electrical potential energy appears as thermal energy in the material. On a microscopic scale, the energy transfer is due to collisions between the charge and the molecules of the material, which leads to an increase in temperature in the material. The loss of potential energy results in an increase in the temperature of the material, which is dissipated as radiation. In a resistor, it is dissipated as heat, and in a light bulb, it is dissipated as heat and light.

The power dissipated by the material as heat and light is equal to the time rate of change of the work:
Equation:

$$P = \frac{\Delta U}{\Delta t} = -\frac{\Delta QV}{\Delta t} = IV.$$

With a resistor, the voltage drop across the resistor is dissipated as heat. Ohm’s law states that the voltage across the resistor is equal to the current times the resistance, $V = IR$. The power dissipated by the resistor is therefore
Equation:

$$P = IV = I\,(IR) = I^2R \text{ or } P = IV = \left(\frac{V}{R}\right)V = \frac{V^2}{R}.$$

If a resistor is connected to a battery, the power dissipated as radiant energy by the wires and the resistor is equal to $P = IV = I^2R = \frac{V^2}{R}$. The power supplied from the battery is equal to current times the voltage, $P = IV$.

Note:
Electric Power
The electric power gained or lost by any device has the form
Equation:

$$P = IV.$$

The power dissipated by a resistor has the form
Equation:

$$P = I^2R = \frac{V^2}{R}.$$

Different insights can be gained from the three different expressions for electric power. For example, $P = V^2/R$ implies that the lower the resistance connected to a given voltage source, the greater the power delivered. Furthermore, since voltage is squared in $P = V^2/R$, the effect of applying a higher voltage is perhaps greater than expected. Thus, when the voltage is doubled to a 25-W bulb, its power nearly quadruples to about 100 W, burning it out. If the bulb's resistance remained constant, its power would be exactly 100 W, but at the higher temperature, its resistance is higher, too.

Example:

Calculating Power in Electric Devices

A DC winch motor is rated at 20.00 A with a voltage of 115 V. When the motor is running at its maximum power, it can lift an object with a weight of 4900.00 N a distance of 10.00 m, in 30.00 s, at a constant speed. (a) What is the power consumed by the motor? (b) What is the power used in lifting the object? Ignore air resistance. (c) Assuming that the difference in the power consumed by the motor and the power used lifting the object are dissipated as heat by the resistance of the motor, estimate the resistance of the motor?

Strategy

(a) The power consumed by the motor can be found using $P = IV$. (b) The power used in lifting the object at a constant speed can be found using $P = Fv$, where the speed is the distance divided by the time. The upward force supplied by the motor is equal to the weight of the object because the acceleration is zero. (c) The resistance of the motor can be found using $P = I^2R$.

Solution

a. The power consumed by the motor is equal to $P = IV$ and the current is given as 20.00 A and the voltage is 115.00 V:
   Equation:

$$P = IV = (20.00\text{ A})115.00\text{ V} = 2300.00\text{ W}.$$

b. The power used lifting the object is equal to $P = Fv$ where the force is equal to the weight of the object (1960 N) and the magnitude of the velocity is $v = \frac{10.00\text{ m}}{30.00\text{ s}} = 0.33\frac{\text{m}}{\text{s}}$,
Equation:

$$P = Fv = (4900\text{ N})0.33\text{ m/s} = 1633.33\text{ W}.$$

c. The difference in the power equals $2300.00\text{ W} - 1633.33\text{ W} = 666.67\text{ W}$ and the resistance can be found using $P = I^2R$:
Equation:

$$R = \frac{P}{I^2} = \frac{666.67\text{ W}}{(20.00\text{ A})^2} = 1.67\,\Omega\,.$$

Significance
The resistance of the motor is quite small. The resistance of the motor is due to many windings of copper wire. The power dissipated by the motor can be significant since the thermal power dissipated by the motor is proportional to the square of the current $\left(P = I^2R\right)$.

Note:
Exercise:

Problem:

Check Your Understanding Electric motors have a reasonably high efficiency. A 100-hp motor can have an efficiency of 90% and a 1-hp motor can have an efficiency of 80%. Why is it important to use high-performance motors?

Solution:

Even though electric motors are highly efficient 10–20% of the power consumed is wasted, not being used for doing useful work. Most of the 10–20% of the power lost is transferred into heat dissipated by the copper wires used to make the coils of the motor. This heat adds to the heat of the environment and adds to the demand on power plants providing the power.

The demand on the power plant can lead to increased greenhouse gases, particularly if the power plant uses coal or gas as fuel.

A fuse ([link]) is a device that protects a circuit from currents that are too high. A fuse is basically a short piece of wire between two contacts. As we have seen, when a current is running through a conductor, the kinetic energy of the charge carriers is converted into thermal energy in the conductor. The piece of wire in the fuse is under tension and has a low melting point. The wire is designed to heat up and break at the rated current. The fuse is destroyed and must be replaced, but it protects the rest of the circuit. Fuses act quickly, but there is a small time delay while the wire heats up and breaks.

A fuse consists of a piece of wire between two contacts. When a current passes through the wire that is greater than the rated current, the wire melts, breaking the connection. Pictured is a “blown” fuse where the wire broke protecting a circuit (credit: modification of work by “Shardayyy”/Flickr).

Circuit breakers are also rated for a maximum current, and open to protect the circuit, but can be reset. Circuit breakers react much faster. The operation of circuit breakers is not within the scope of this chapter and will be discussed in later chapters. Another method of protecting equipment and people is the ground

fault circuit interrupter (GFCI), which is common in bathrooms and kitchens. The GFCI outlets respond very quickly to changes in current. These outlets open when there is a change in magnetic field produced by current-carrying conductors, which is also beyond the scope of this chapter and is covered in a later chapter.

The Cost of Electricity

The more electric appliances you use and the longer they are left on, the higher your electric bill. This familiar fact is based on the relationship between energy and power. You pay for the energy used. Since $P = \frac{dE}{dt}$, we see that

Equation:

$$E = \int P dt$$

is the energy used by a device using power *P* for a time interval *t*. If power is delivered at a constant rate, then then the energy can be found by $E = Pt$. For example, the more light bulbs burning, the greater *P* used; the longer they are on, the greater *t* is.

The energy unit on electric bills is the kilowatt-hour $(\mathrm{kW} \cdot \mathrm{h})$, consistent with the relationship $E = Pt$. It is easy to estimate the cost of operating electrical appliances if you have some idea of their power consumption rate in watts or kilowatts, the time they are on in hours, and the cost per kilowatt-hour for your electric utility. Kilowatt-hours, like all other specialized energy units such as food calories, can be converted into joules. You can prove to yourself that $1 \, \mathrm{kW} \cdot \mathrm{h} = 3.6 \, \times \, 10^6 \, \mathrm{J}$.

The electrical energy (*E*) used can be reduced either by reducing the time of use or by reducing the power consumption of that appliance or fixture. This not only reduces the cost but also results in a reduced impact on the environment. Improvements to lighting are some of the fastest ways to reduce the electrical energy used in a home or business. About 20% of a home’s use of energy goes to lighting, and the number for commercial establishments is closer to 40%. Fluorescent lights are about four times more efficient than incandescent lights—this is true for both the long tubes and the compact fluorescent lights (CFLs). (See [link](b).) Thus, a 60-W incandescent bulb can be replaced by a 15-W CFL, which has the same brightness and color. CFLs have a bent tube inside a globe or a spiral-shaped tube, all connected to a standard screw-in base that fits standard

incandescent light sockets. (Original problems with color, flicker, shape, and high initial investment for CFLs have been addressed in recent years.)

The heat transfer from these CFLs is less, and they last up to 10 times longer than incandescent bulbs. The significance of an investment in such bulbs is addressed in the next example. New white LED lights (which are clusters of small LED bulbs) are even more efficient (twice that of CFLs) and last five times longer than CFLs.

Example:
Calculating the Cost Effectiveness of LED Bulb
The typical replacement for a 100-W incandescent bulb is a 20-W LED bulb. The 20-W LED bulb can provide the same amount of light output as the 100-W incandescent light bulb. What is the cost savings for using the LED bulb in place of the incandescent bulb for one year, assuming $0.10 per kilowatt-hour is the average energy rate charged by the power company? Assume that the bulb is turned on for three hours a day.

Strategy
(a) Calculate the energy used during the year for each bulb, using $E = Pt$.
(b) Multiply the energy by the cost.

Solution

a. Calculate the power for each bulb.
   Equation:

$$\begin{aligned} E_{\text{Incandescent}} &= Pt = 100\ \text{W}\left(\frac{1\ \text{kW}}{1000\ \text{W}}\right)\left(\frac{3\ \text{h}}{\text{day}}\right)(365\ \text{days}) = 109.5\ \text{kW}\cdot\text{h} \\ E_{\text{LED}} &= Pt = 20\ \text{W}\left(\frac{1\ \text{kW}}{1000\ \text{W}}\right)\left(\frac{3\ \text{h}}{\text{day}}\right)(365\ \text{days}) = 21.90\ \text{kW}\cdot\text{h} \end{aligned}.$$

b. Calculate the cost for each.
   Equation:

$$\begin{aligned} \text{cost}_{\text{Incandescent}} &= 109.5\ \text{kW-h}\left(\frac{\$0.10}{\text{kW}\cdot\text{h}}\right) = \$10.95 \\ \text{cost}_{\text{LED}} &= 21.90\ \text{kW-h}\left(\frac{\$0.10}{\text{kW}\cdot\text{h}}\right) = \$2.19 \end{aligned}.$$

Significance

A LED bulb uses 80% less energy than the incandescent bulb, saving $8.76 over the incandescent bulb for one year. The LED bulb can cost $20.00 and the 100-W incandescent bulb can cost $0.75, which should be calculated into the computation. A typical lifespan of an incandescent bulb is 1200 hours and is 50,000 hours for the LED bulb. The incandescent bulb would last 1.08 years at 3 hours a day and the LED bulb would last 45.66 years. The initial cost of the LED bulb is high, but the cost to the home owner will be $0.69 for the incandescent bulbs versus $0.44 for the LED bulbs per year. (Note that the LED bulbs are coming down in price.) The cost savings per year is approximately $8.50, and that is just for one bulb.

Note:
Exercise:

Problem:

Check Your Understanding Is the efficiency of the various light bulbs the only consideration when comparing the various light bulbs?

Solution:

No, the efficiency is a very important consideration of the light bulbs, but there are many other considerations. As mentioned above, the cost of the bulbs and the life span of the bulbs are important considerations. For example, CFL bulbs contain mercury, a neurotoxin, and must be disposed of as hazardous waste. When replacing incandescent bulbs that are being controlled by a dimmer switch with LED, the dimmer switch may need to be replaced. The dimmer switches for LED lights are comparably priced to the incandescent light switches, but this is an initial cost which should be considered. The spectrum of light should also be considered, but there is a broad range of color temperatures available, so you should be able to find one that fits your needs. None of these considerations mentioned are meant to discourage the use of LED or CFL light bulbs, but they are considerations.

Changing light bulbs from incandescent bulbs to CFL or LED bulbs is a simple way to reduce energy consumption in homes and commercial sites. CFL bulbs operate with a much different mechanism than do incandescent lights. The

mechanism is complex and beyond the scope of this chapter, but here is a very general description of the mechanism. CFL bulbs contain argon and mercury vapor housed within a spiral-shaped tube. The CFL bulbs use a “ballast” that increases the voltage used by the CFL bulb. The ballast produce an electrical current, which passes through the gas mixture and excites the gas molecules. The excited gas molecules produce ultraviolet (UV) light, which in turn stimulates the fluorescent coating on the inside of the tube. This coating fluoresces in the visible spectrum, emitting visible light. Traditional fluorescent tubes and CFL bulbs had a short time delay of up to a few seconds while the mixture was being “warmed up” and the molecules reached an excited state. It should be noted that these bulbs do contain mercury, which is poisonous, but if the bulb is broken, the mercury is never released. Even if the bulb is broken, the mercury tends to remain in the fluorescent coating. The amount is also quite small and the advantage of the energy saving may outweigh the disadvantage of using mercury.

The CFL light bulbs are being replaced with LED light bulbs, where LED stands for “light-emitting diode.” The diode was briefly discussed as a nonohmic device, made of semiconducting material, which essentially permits current flow in one direction. LEDs are a special type of diode made of semiconducting materials infused with impurities in combinations and concentrations that enable the extra energy from the movement of the electrons during electrical excitation to be converted into visible light. Semiconducting devices will be explained in greater detail in Condensed Matter Physics.

Commercial LEDs are quickly becoming the standard for commercial and residential lighting, replacing incandescent and CFL bulbs. They are designed for the visible spectrum and are constructed from gallium doped with arsenic and phosphorous atoms. The color emitted from an LED depends on the materials used in the semiconductor and the current. In the early years of LED development, small LEDs found on circuit boards were red, green, and yellow, but LED light bulbs can now be programmed to produce millions of colors of light as well as many different hues of white light.

Comparison of Incandescent, CFL, and LED Light Bulbs

The energy savings can be significant when replacing an incandescent light bulb or a CFL light bulb with an LED light. Light bulbs are rated by the amount of power that the bulb consumes, and the amount of light output is measured in lumens. The lumen (lm) is the SI -derived unit of luminous flux and is a measure of the total quantity of visible light emitted by a source. A 60-W incandescent

light bulb can be replaced with a 13- to 15-W CFL bulb or a 6- to 8-W LED bulb, all three of which have a light output of approximately 800 lm. A table of light output for some commonly used light bulbs appears in [link].

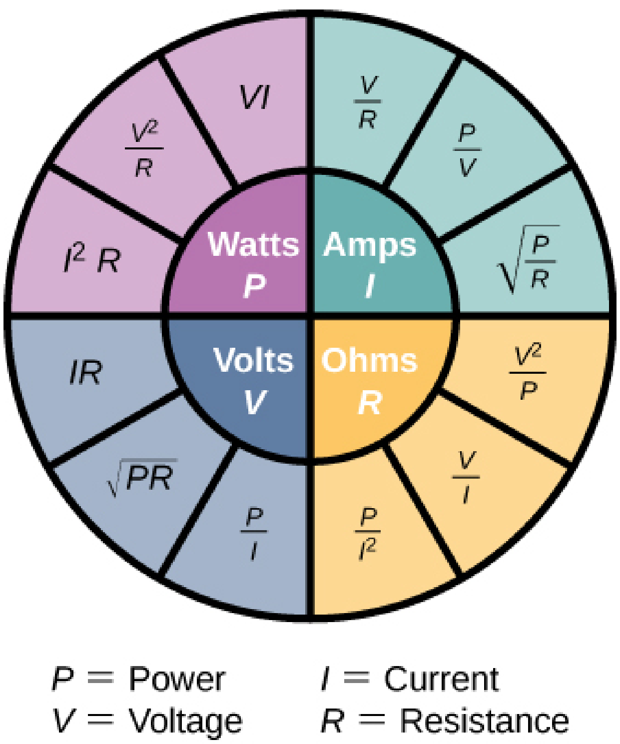
The life spans of the three types of bulbs are significantly different. An LED bulb has a life span of 50,000 hours, whereas the CFL has a lifespan of 8000 hours and the incandescent lasts a mere 1200 hours. The LED bulb is the most durable, easily withstanding rough treatment such as jarring and bumping. The incandescent light bulb has little tolerance to the same treatment since the filament and glass can easily break. The CFL bulb is also less durable than the LED bulb because of its glass construction. The amount of heat emitted is 3.4 btu/h for the 8-W LED bulb, 85 btu/h for the 60-W incandescent bulb, and 30 btu/h for the CFL bulb. As mentioned earlier, a major drawback of the CFL bulb is that it contains mercury, a neurotoxin, and must be disposed of as hazardous waste. From these data, it is easy to understand why the LED light bulb is quickly becoming the standard in lighting.

Light Output (lumens)	**LED Light Bulbs (watts)**	**Incandescent Light Bulbs (watts)**	**CFL Light Bulbs (watts)**
450	4−5	40	9−13
800	6−8	60	13−15
1100	9−13	75	18−25
1600	16−20	100	23−30
2600	25−28	150	30−55

Light Output of LED, Incandescent, and CFL Light Bulbs

Summary of Relationships

In this chapter, we have discussed relationships between voltages, current, resistance, and power. [link] shows a summary of the relationships between these measurable quantities for ohmic devices. (Recall that ohmic devices follow Ohm's law $V = IR$.) For example, if you need to calculate the power, use the pink section, which shows that $P = VI$, $P = \frac{V^2}{R}$, and $P = I^2R$.



This circle shows a summary of the equations for the relationships between power, current, voltage, and resistance.

Which equation you use depends on what values you are given, or you measure. For example if you are given the current and the resistance, use $P = I^2R$. Although all the possible combinations may seem overwhelming, don't forget that they all are combinations of just two equations, Ohm's law $(V = IR)$ and power $(P = IV)$.

Summary

- Electric power is the rate at which electric energy is supplied to a circuit or consumed by a load.
- Power dissipated by a resistor depends on the square of the current through the resistor and is equal to $P = I^2R = \frac{V^2}{R}$.
- The SI unit for electric power is the watt and the SI unit for electric energy is the joule. Another common unit for electric energy, used by power companies, is the kilowatt-hour ($\text{kW} \cdot \text{h}$).
- The total energy used over a time interval can be found by $E = \int P dt$.

Conceptual Questions

Exercise:

Problem:

Common household appliances are rated at 110 V, but power companies deliver voltage in the kilovolt range and then step the voltage down using transformers to 110 V to be used in homes. You will learn in later chapters that transformers consist of many turns of wire, which warm up as current flows through them, wasting some of the energy that is given off as heat. This sounds inefficient. Why do the power companies transport electric power using this method?

Solution:

Although the conductors have a low resistance, the lines from the power company can be kilometers long. Using a high voltage reduces the current that is required to supply the power demand and that reduces line losses.

Exercise:

Problem:

Your electric bill gives your consumption in units of kilowatt-hour ($\text{kW} \cdot \text{h}$). Does this unit represent the amount of charge, current, voltage, power, or energy you buy?

Exercise:

Problem:

Resistors are commonly rated at $\frac{1}{8}\,\mathrm{W}$, $\frac{1}{4}\,\mathrm{W}$, $\frac{1}{2}\,\mathrm{W}$, 1 W and 2 W for use in electrical circuits. If a current of $I = 2.00\ \mathrm{A}$ is accidentally passed through a $R = 1.00\ \Omega$ resistor rated at 1 W, what would be the most probable outcome? Is there anything that can be done to prevent such an accident?

Solution:

The resistor would overheat, possibly to the point of causing the resistor to burn. Fuses are commonly added to circuits to prevent such accidents.

Exercise:

Problem:

An immersion heater is a small appliance used to heat a cup of water for tea by passing current through a resistor. If the voltage applied to the appliance is doubled, will the time required to heat the water change? By how much? Is this a good idea?

Problems

Exercise:

Problem:

A 20.00-V battery is used to supply current to a $10\text{-k}\ \Omega$ resistor. Assume the voltage drop across any wires used for connections is negligible. (a) What is the current through the resistor? (b) What is the power dissipated by the resistor? (c) What is the power input from the battery, assuming all the electrical power is dissipated by the resistor? (d) What happens to the energy dissipated by the resistor?

Solution:

a. $I = 2\ \mathrm{mA}$; b. $P = 0.04\ \mathrm{W}$; c. $P = 0.04\ \mathrm{W}$; d. It is converted into heat.

Exercise:

Problem:

What is the maximum voltage that can be applied to a $20\text{-k}\,\Omega$ resistor rated at $\frac{1}{4}\,\text{W}$?

Exercise:

Problem:

A heater is being designed that uses a coil of 14-gauge nichrome wire to generate 300 W using a voltage of $V = 110\,\text{V}$. How long should the engineer make the wire?

Solution:

$$\begin{array}{l} P = \frac{V^2}{R} \\ R = 40\,\Omega \end{array}, \begin{array}{l} A = 2.08\text{mm}^2 \\ \rho = 100\ \times\ 10^{-8}\,\Omega\ \cdot \text{m} \\ R = \rho\frac{L}{A} \\ L = 83\,\text{m} \end{array}$$

Exercise:

Problem:

An alternative to CFL bulbs and incandescent bulbs are light-emitting diode (LED) bulbs. A 100-W incandescent bulb can be replaced by a 16-W LED bulb. Both produce 1600 lumens of light. Assuming the cost of electricity is $0.10 per kilowatt-hour, how much does it cost to run the bulb for one year if it runs for four hours a day?

Exercise:

Problem:

The power dissipated by a resistor with a resistance of $R = 100\,\Omega$ is $P = 2.0\,\text{W}$. What are the current through and the voltage drop across the resistor?

Solution:

$$I = 0.14\,\text{A}, \quad V = 14\,\text{V}$$

Exercise:

Problem:

Running late to catch a plane, a driver accidentally leaves the headlights on after parking the car in the airport parking lot. During takeoff, the driver realizes the mistake. Having just replaced the battery, the driver knows that the battery is a 12-V automobile battery, rated at 100 $\text{A} \cdot \text{h}$. The driver, knowing there is nothing that can be done, estimates how long the lights will shine, assuming there are two 12-V headlights, each rated at 40 W. What did the driver conclude?

Exercise:

Problem:

A physics student has a single-occupancy dorm room. The student has a small refrigerator that runs with a current of 3.00 A and a voltage of 110 V, a lamp that contains a 100-W bulb, an overhead light with a 60-W bulb, and various other small devices adding up to 3.00 W. (a) Assuming the power plant that supplies 110 V electricity to the dorm is 10 km away and the two aluminum transmission cables use 0-gauge wire with a diameter of 8.252 mm, estimate the percentage of the total power supplied by the power company that is lost in the transmission. (b) What would be the result is the power company delivered the electric power at 110 kV?

Solution:

a.
$$I \approx 3.00 \text{ A} + \frac{100 \text{ W}}{110 \text{ V}} + \frac{60 \text{ W}}{110 \text{ V}} + \frac{3.00 \text{ W}}{110 \text{ V}} = 4.48 \text{ A}$$
$$P = 493 \text{ W}$$
$$R = 9.91 \, \Omega \, ,$$
$$P_{\text{loss}} = 200. \text{ W}$$
$$\% \text{loss} = 40\%$$

b.
$$P = 493 \text{ W}$$
$$I = 0.0045 \text{ A}$$
$$R = 9.91 \, \Omega$$
$$P_{\text{loss}} = 201 \mu \text{ W}$$
$$\% \text{loss} = 0.00004\%$$

Exercise:

Problem:

A 0.50-W, 220- Ω resistor carries the maximum current possible without damaging the resistor. If the current were reduced to half the value, what would be the power consumed?

Glossary

electrical power
: time rate of change of energy in an electric circuit

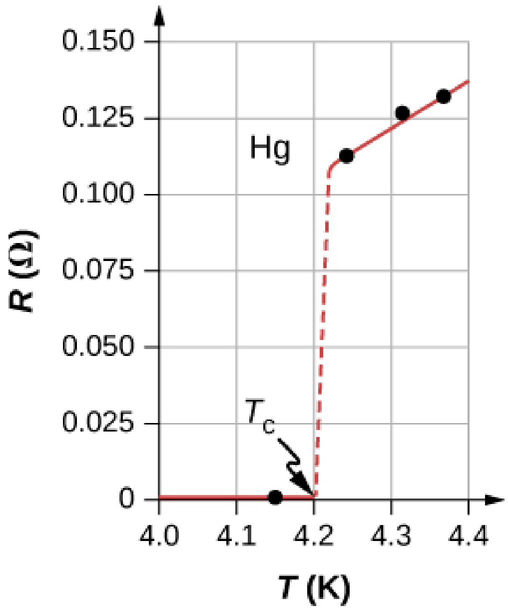
Superconductors
By the end of this section, you will be able to:

- Describe the phenomenon of superconductivity
- List applications of superconductivity

Touch the power supply of your laptop computer or some other device. It probably feels slightly warm. That heat is an unwanted byproduct of the process of converting household electric power into a current that can be used by your device. Although electric power is reasonably efficient, other losses are associated with it. As discussed in the section on power and energy, transmission of electric power produces I^2R line losses. These line losses exist whether the power is generated from conventional power plants (using coal, oil, or gas), nuclear plants, solar plants, hydroelectric plants, or wind farms. These losses can be reduced, but not eliminated, by transmitting using a higher voltage. It would be wonderful if these line losses could be eliminated, but that would require transmission lines that have zero resistance. In a world that has a global interest in not wasting energy, the reduction or elimination of this unwanted thermal energy would be a significant achievement. Is this possible?

The Resistance of Mercury

In 1911, Heike Kamerlingh Onnes of Leiden University, a Dutch physicist, was looking at the temperature dependence of the resistance of the element mercury. He cooled the sample of mercury and noticed the familiar behavior of a linear dependence of resistance on temperature; as the temperature decreased, the resistance decreased. Kamerlingh Onnes continued to cool the sample of mercury, using liquid helium. As the temperature approached $4.2\ \text{K}\ (-269.2\ ^\circ\text{C})$, the resistance abruptly went to zero ([link]). This temperature is known as the **critical temperature** T_c for mercury. The sample of mercury entered into a phase where the resistance was absolutely zero. This phenomenon is known as **superconductivity**. (*Note:* If you connect the leads of a three-digit ohmmeter across a conductor, the reading commonly shows up as $0.00\ \Omega$. The resistance of the conductor is not actually zero, it is less than $0.01\ \Omega$.) There are various methods to measure very small resistances, such as the four-point method, but an ohmmeter is not an acceptable method to use for testing resistance in superconductivity.



The resistance of a sample of mercury is zero at very low temperatures—it is a superconductor up to the temperature of about 4.2 K. Above that critical temperature, its resistance makes a sudden jump and then increases nearly linearly with temperature.

Other Superconducting Materials

As research continued, several other materials were found to enter a superconducting phase, when the temperature reached near absolute zero. In 1941, an alloy of niobium-nitride was found that could become superconducting at $T_{\text{c}} = 16\,\text{K}\,(-257\,^{\circ}\text{C})$ and in 1953, vanadium-silicon was found to become superconductive at $T_{\text{c}} = 17.5\,\text{K}(-255.7\,^{\circ}\text{C})$. The temperatures for the transition into superconductivity were slowly creeping higher. Strangely, many materials that make good conductors, such as copper, silver, and gold, do not exhibit superconductivity. Imagine the energy savings if transmission lines for electric power-generating stations could be made to be superconducting at temperatures near room temperature! A resistance of zero ohms means no I^2R losses and a great boost to reducing energy consumption. The problem

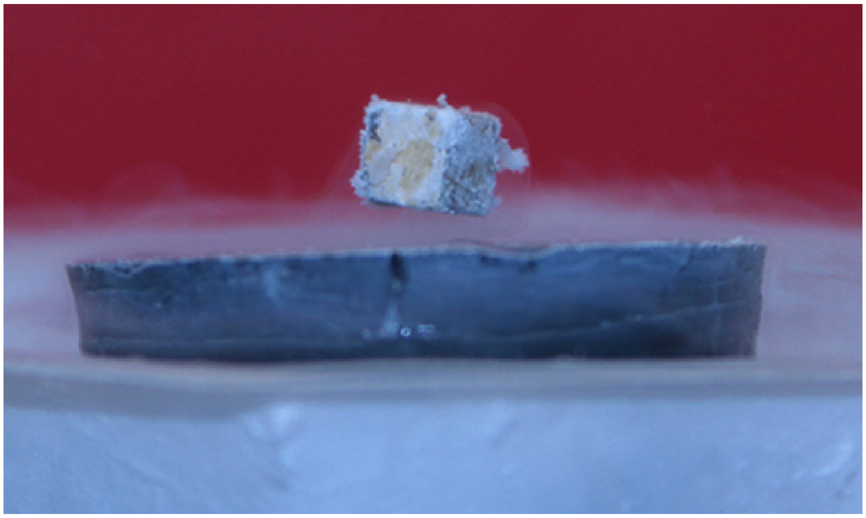
is that $T_c = 17.5\,\mathrm{K}$ is still very cold and in the range of liquid helium temperatures. At this temperature, it is not cost effective to transmit electrical energy because of the cooling requirements.

A large jump was seen in 1986, when a team of researchers, headed by Dr. Ching Wu Chu of Houston University, fabricated a brittle, ceramic compound with a transition temperature of $T_c = 92\,\mathrm{K}\,(-181\,^\circ\mathrm{C})$. The ceramic material, composed of yttrium barium copper oxide (YBCO), was an insulator at room temperature. Although this temperature still seems quite cold, it is near the boiling point of liquid nitrogen, a liquid commonly used in refrigeration. You may have noticed refrigerated trucks traveling down the highway labeled as “Liquid Nitrogen Cooled.”

YBCO ceramic is a material that could be useful for transmitting electrical energy because the cost saving of reducing the I^2R losses are larger than the cost of cooling the superconducting cable, making it financially feasible. There were and are many engineering problems to overcome. For example, unlike traditional electrical cables, which are flexible and have a decent tensile strength, ceramics are brittle and would break rather than stretch under pressure. Processes that are rather simple with traditional cables, such as making connections, become difficult when working with ceramics. The problems are difficult and complex, and material scientists and engineers are coming up with innovative solutions.

An interesting consequence of the resistance going to zero is that once a current is established in a superconductor, it persists without an applied voltage source. Current loops in a superconductor have been set up and the current loops have been observed to persist for years without decaying.

Zero resistance is not the only interesting phenomenon that occurs as the materials reach their transition temperatures. A second effect is the exclusion of magnetic fields. This is known as the **Meissner effect** ([link]). A light, permanent magnet placed over a superconducting sample will levitate in a stable position above the superconductor. High-speed trains have been developed that levitate on strong superconducting magnets, eliminating the friction normally experienced between the train and the tracks. In Japan, the Yamanashi Maglev test line opened on April 3, 1997. In April 2015, the MLX01 test vehicle attained a speed of 374 mph (603 km/h).



A small, strong magnet levitates over a superconductor cooled to liquid nitrogen temperature. The magnet levitates because the superconductor excludes magnetic fields. (credit: Joseph J. Trout)

[link] shows a select list of elements, compounds, and high-temperature superconductors, along with the critical temperatures for which they become superconducting. Each section is sorted from the highest critical temperature to the lowest. Also listed is the critical magnetic field for some of the materials. This is the strength of the magnetic field that destroys superconductivity. Finally, the type of the superconductor is listed.

There are two types of superconductors. There are 30 pure metals that exhibit zero resistivity below their critical temperature and exhibit the Meissner effect, the property of excluding magnetic fields from the interior of the superconductor while the superconductor is at a temperature below the critical temperature. These metals are called Type I superconductors. The superconductivity exists only below their critical temperatures and below a critical magnetic field strength. Type I superconductors are well described by the BCS theory (described next). Type I superconductors have limited practical applications because the strength of the critical magnetic field needed to destroy the superconductivity is quite low.

Type II superconductors are found to have much higher critical magnetic fields and therefore can carry much higher current densities while remaining in the superconducting state. A collection of various ceramics containing barium-copper-oxide have much higher critical temperatures for the transition into a superconducting

state. Superconducting materials that belong to this subcategory of the Type II superconductors are often categorized as high-temperature superconductors.

Introduction to BCS Theory

Type I superconductors, along with some Type II superconductors can be modeled using the BCS theory, proposed by John Bardeen, Leon Cooper, and Robert Schrieffer. Although the theory is beyond the scope of this chapter, a short summary of the theory is provided here. (More detail is provided in Condensed Matter Physics.) The theory considers pairs of electrons and how they are coupled together through lattice-vibration interactions. Through the interactions with the crystalline lattice, electrons near the Fermi energy level feel a small attractive force and form pairs (Cooper pairs), and the coupling is known as a phonon interaction. Single electrons are fermions, which are particles that obey the Pauli exclusion principle. The Pauli exclusion principle in quantum mechanics states that two identical fermions (particles with half-integer spin) cannot occupy the same quantum state simultaneously. Each electron has four quantum numbers (n, l, m_l, m_s). The principal quantum number (n) describes the energy of the electron, the orbital angular momentum quantum number (l) indicates the most probable distance from the nucleus, the magnetic quantum number (m_l) describes the energy levels in the subshell, and the electron spin quantum number (m_s) describes the orientation of the spin of the electron, either up or down. As the material enters a superconducting state, pairs of electrons act more like bosons, which can condense into the same energy level and need not obey the Pauli exclusion principle. The electron pairs have a slightly lower energy and leave an energy gap above them on the order of 0.001 eV. This energy gap inhibits collision interactions that lead to ordinary resistivity. When the material is below the critical temperature, the thermal energy is less than the band gap and the material exhibits zero resistivity.

Material	**Symbol or Formula**	**Critical Temperature T_c (K)**	**Critical Magnetic Field H_c (T)**	**Type**
Elements				

Material	Symbol or Formula	Critical Temperature T_c (K)	Critical Magnetic Field H_c (T)	Type
Lead	Pb	7.19	0.08	I
Lanthanum	La	(α) 4.90 − (β) 6.30		I
Tantalum	Ta	4.48	0.09	I
Mercury	Hg	(α) 4.15 − (β) 3.95	0.04	I
Tin	Sn	3.72	0.03	I
Indium	In	3.40	0.03	I
Thallium	Tl	2.39	0.03	I
Rhenium	Re	2.40	0.03	I
Thorium	Th	1.37	0.013	I
Protactinium	Pa	1.40		I
Aluminum	Al	1.20	0.01	I
Gallium	Ga	1.10	0.005	I
Zinc	Zn	0.86	0.014	I
Titanium	Ti	0.39	0.01	I
Uranium	U	(α) 0.68 − (β) 1.80		I
Cadmium	Cd	11.4	4.00	I
Compounds				

Material	Symbol or Formula	Critical Temperature T_c (K)	Critical Magnetic Field H_c (T)	Type
Niobium-germanium	Nb_3Ge	23.20	37.00	II
Niobium-tin	Nb_3Sn	18.30	30.00	II
Niobium-nitrite	NbN	16.00		II
Niobium-titanium	NbTi	10.00	15.00	II
High-Temperature Oxides				
	$HgBa_2CaCu_2O_8$	134.00		II
	$Tl_2Ba_2Ca_2Cu_3O_{10}$	125.00		II
	$YBa_2Cu_3O_7$	92.00	120.00	II

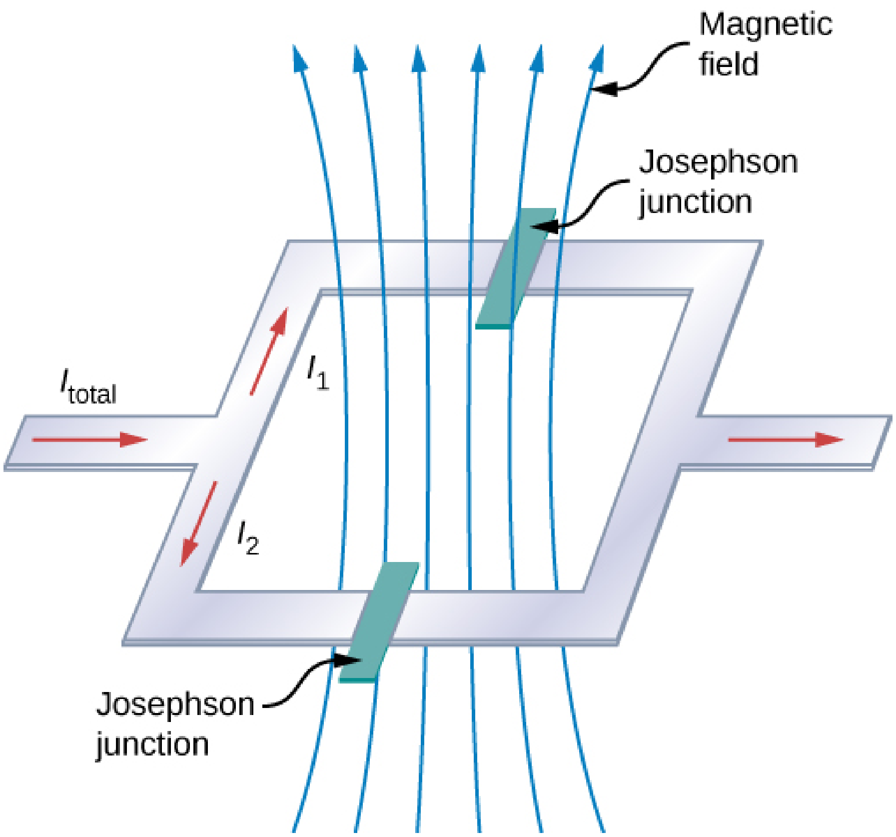
Superconductor Critical Temperatures

Applications of Superconductors

Superconductors can be used to make superconducting magnets. These magnets are 10 times stronger than the strongest electromagnets. These magnets are currently in use in magnetic resonance imaging (MRI), which produces high-quality images of the body interior without dangerous radiation.

Another interesting application of superconductivity is the **SQUID** (superconducting quantum interference device). A SQUID is a very sensitive magnetometer used to measure extremely subtle magnetic fields. The operation of the SQUID is based on superconducting loops containing Josephson junctions. A **Josephson junction** is the result of a theoretical prediction made by B. D. Josephson in an article published in 1962. In the article, Josephson described how a supercurrent can flow between two pieces of superconductor separated by a thin layer of insulator. This phenomenon is now called the Josephson effect. The SQUID consists of a superconducting current

loop containing two Josephson junctions, as shown in [link]. When the loop is placed in even a very weak magnetic field, there is an interference effect that depends on the strength of the magnetic field.



The SQUID (superconducting quantum interference device) uses a superconducting current loop and two Josephson junctions to detect magnetic fields as low as $10^{-14}\ \text{T}$ (Earth's magnet field is on the order of $0.3\ \times\ 10^{-5}\ \text{T}$).

Superconductivity is a fascinating and useful phenomenon. At critical temperatures near the boiling point of liquid nitrogen, superconductivity has special applications in MRIs, particle accelerators, and high-speed trains. Will we reach a state where we can have materials enter the superconducting phase at near room temperatures? It seems a long way off, but if scientists in 1911 were asked if we would reach liquid-nitrogen temperatures with a ceramic, they might have thought it implausible.

Summary

- Superconductivity is a phenomenon that occurs in some materials when cooled to very low critical temperatures, resulting in a resistance of exactly zero and the expulsion of all magnetic fields.
- Materials that are normally good conductors (such as copper, gold, and silver) do not experience superconductivity.
- Superconductivity was first observed in mercury by Heike Kamerlingh Onnes in 1911. In 1986, Dr. Ching Wu Chu of Houston University fabricated a brittle, ceramic compound with a critical temperature close to the temperature of liquid nitrogen.
- Superconductivity can be used in the manufacture of superconducting magnets for use in MRIs and high-speed, levitated trains.

Key Equations

Average electrical current	$I_{\text{ave}} = \frac{\Delta Q}{\Delta t}$
Definition of an ampere	$1\text{ A} = 1\text{ C/s}$
Electrical current	$I = \frac{dQ}{dt}$
Drift velocity	$v_d = \frac{I}{nqA}$
Current density	$I = \iint\limits_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$
Resistivity	$\rho = \frac{E}{J}$
Common expression of Ohm's law	$V = IR$
Resistivity as a function of temperature	$\rho = \rho_0 \left[1 + \alpha \left(T - T_0\right)\right]$
Definition of resistance	$R \equiv \frac{V}{I}$

Resistance of a cylinder of material	$R=\rho\frac{L}{A}$
Temperature dependence of resistance	$R=R_0\left(1+\alpha\Delta T\right)$
Electric power	$P=IV$
Power dissipated by a resistor	$P=I^2R=\frac{V^2}{R}$

Conceptual Questions

Exercise:

Problem:

What requirement for superconductivity makes current superconducting devices expensive to operate?

Solution:

Very low temperatures necessitate refrigeration. Some materials require liquid nitrogen to cool them below their critical temperatures. Other materials may need liquid helium, which is even more costly.

Exercise:

Problem:

Name two applications for superconductivity listed in this section and explain how superconductivity is used in the application. Can you think of a use for superconductivity that is not listed?

Problems

Exercise:

Problem:

Consider a power plant is located 60 km away from a residential area uses 0-gauge $\left(A=42.40\,\text{mm}^2\right)$ wire of copper to transmit power at a current of $I=100.00\text{ A}$. How much more power is dissipated in the copper wires than it would be in superconducting wires?

Solution:

$$R_{\text{copper}} = 23.77\,\Omega$$
$$P = 2.377\ \times\ 10^5\text{W}$$

Exercise:

Problem:

A wire is drawn through a die, stretching it to four times its original length. By what factor does its resistance increase?

Exercise:

Problem:

Digital medical thermometers determine temperature by measuring the resistance of a semiconductor device called a thermistor (which has $\alpha = -0.06/^\circ\text{C}$) when it is at the same temperature as the patient. What is a patient's temperature if the thermistor's resistance at that temperature is 82.0% of its value at $37\ ^\circ\text{C}$ (normal body temperature)?

Solution:

$$R = R_0\left(1 + \alpha\left(T - T_0\right)\right)$$
$$0.82R_0 = R_0\left(1 + \alpha\left(T - T_0\right)\right),\quad 0.82 = 1 - 0.06\left(T - 37\ ^\circ\text{C}\right),\quad T = 40\ ^\circ\text{C}$$

Exercise:

Problem:

Electrical power generators are sometimes "load tested" by passing current through a large vat of water. A similar method can be used to test the heat output of a resistor. A $R = 30\ \Omega$ resistor is connected to a 9.0-V battery and the resistor leads are waterproofed and the resistor is placed in 1.0 kg of room temperature water $\left(T = 20\ ^\circ\text{C}\right)$. Current runs through the resistor for 20 minutes. Assuming all the electrical energy dissipated by the resistor is converted to heat, what is the final temperature of the water?

Exercise:

Problem:

A 12-gauge gold wire has a length of 1 meter. (a) What would be the length of a silver 12-gauge wire with the same resistance? (b) What are their respective resistances at the temperature of boiling water?

Solution:

a. $R_{\text{Au}} = R_{\text{Ag}}, \quad \rho_{\text{Au}} \frac{L_{\text{Au}}}{A_{\text{Au}}} = \rho_{\text{Ag}} \frac{L_{\text{Ag}}}{A_{\text{Ag}}}, \quad L_{\text{Ag}} = 1.53\,\text{m};$
b. $R_{\text{Au},20\,^{\circ}\text{C}} = 0.0074\,\Omega\,, \quad R_{\text{Au},100\,^{\circ}\text{C}} = 0.0094\,\Omega\,, \quad R_{\text{Ag},100\,^{\circ}\text{C}} = 0.0096\,\Omega$

Exercise:

Problem:

What is the change in temperature required to decrease the resistance for a carbon resistor by 10%?

Additional Problems

Exercise:

Problem:

A coaxial cable consists of an inner conductor with radius $r_{\text{i}} = 0.25\,\text{cm}$ and an outer radius of $r_{\text{o}} = 0.5\,\text{cm}$ and has a length of 10 meters. Plastic, with a resistivity of $\rho = 2.00 \times 10^{13}\,\Omega \cdot \text{m}$, separates the two conductors. What is the resistance of the cable?

Solution:

$$\begin{aligned} dR &= \frac{\rho}{2\pi rL}dr \\ R &= \frac{\rho}{2\pi L}\ln\frac{r_{\text{o}}}{r_{\text{i}}} \\ R &= 2.21 \times 10^{11}\,\Omega \end{aligned}$$

Exercise:

Problem:

A 10.00-meter long wire cable that is made of copper has a resistance of 0.051 ohms. (a) What is the weight if the wire was made of copper? (b) What is the weight of a 10.00-meter-long wire of the same gauge made of aluminum? (c)What is the resistance of the aluminum wire? The density of copper is $8960\,\text{kg/m}^3$ and the density of aluminum is $2760\,\text{kg/m}^3$.

Exercise:

Problem:

A nichrome rod that is 3.00 mm long with a cross-sectional area of $1.00\ \text{mm}^2$ is used for a digital thermometer. (a) What is the resistance at room temperature? (b) What is the resistance at body temperature?

Solution:

a.
$R_0 = 0.003\ \Omega$; b.

$$\begin{aligned} T_\text{c} &= 37.0\ ^\circ\text{C} \\ R &= 0.00302\ \Omega \end{aligned}$$

Exercise:

Problem:

The temperature in Philadelphia, PA can vary between $68.00\ ^\circ\text{F}$ and $100.00\ ^\circ\text{F}$ in one summer day. By what percentage will an aluminum wire's resistance change during the day?

Exercise:

Problem:

When 100.0 V is applied across a 5-gauge (diameter 4.621 mm) wire that is 10 m long, the magnitude of the current density is $2.0\ \times\ 10^8 \text{A/m}^2$. What is the resistivity of the wire?

Solution:

$\rho = 5.00\ \times\ 10^{-8}\ \Omega\ \cdot \text{m}$

Exercise:

Problem:

A wire with a resistance of $5.0\ \Omega$ is drawn out through a die so that its new length is twice times its original length. Find the resistance of the longer wire. You may assume that the resistivity and density of the material are unchanged.

Exercise:

Problem:

What is the resistivity of a wire of 5-gauge wire $(A = 16.8\ \times\ 10^{-6} \text{m}^2)$, 5.00 m length, and $5.10\ \text{m}\,\Omega$ resistance?

Solution:

$\rho = 1.71 \times 10^{-8}\ \Omega \cdot \mathrm{m}$

Exercise:

Problem:

Coils are often used in electrical and electronic circuits. Consider a coil which is formed by winding 1000 turns of insulated 20-gauge copper wire (area $0.52\ \mathrm{mm}^2$) in a single layer on a cylindrical non-conducting core of radius 2.0 mm. What is the resistance of the coil? Neglect the thickness of the insulation.

Exercise:

Problem:

Currents of approximately 0.06 A can be potentially fatal. Currents in that range can make the heart fibrillate (beat in an uncontrolled manner). The resistance of a dry human body can be approximately $100\ \mathrm{k}\ \Omega$. (a) What voltage can cause 0.06 A through a dry human body? (b) When a human body is wet, the resistance can fall to $100\ \Omega$. What voltage can cause harm to a wet body?

Solution:

a. $V = 6000$ V; b. $V = 6$ V

Exercise:

Problem:

A 20.00-ohm, 5.00-watt resistor is placed in series with a power supply. (a) What is the maximum voltage that can be applied to the resistor without harming the resistor? (b) What would be the current through the resistor?

Exercise:

Problem:

A battery with an emf of 24.00 V delivers a constant current of 2.00 mA to an appliance. How much work does the battery do in three minutes?

Solution:

$P = \frac{W}{t}, \quad W = 8.64\ \mathrm{J}$

Exercise:

Problem:

A 12.00-V battery has an internal resistance of a tenth of an ohm. (a) What is the current if the battery terminals are momentarily shorted together? (b) What is the terminal voltage if the battery delivers 0.25 amps to a circuit?

Challenge Problems

Exercise:

Problem:

A 10-gauge copper wire has a cross-sectional area $A = 5.26\ \text{mm}^2$ and carries a current of $I = 5.00\ \text{A}$. The density of copper is $\rho = 8.95\ \text{g/cm}^3$. One mole of copper atoms $\left(6.02\ \times\ 10^{23} \text{atoms}\right)$ has a mass of approximately 63.50 g. What is the magnitude of the drift velocity of the electrons, assuming that each copper atom contributes one free electron to the current?

Solution:

$$V = 7.09\ \text{cm}^3$$
$$n = 8.49\ \times\ 10^{28}\ \frac{\text{electrons}}{\text{m}^3}$$
$$v_\text{d} = 7.00\ \times\ 10^{-5}\ \frac{\text{m}}{\text{s}}$$

Exercise:

Problem:

The current through a 12-gauge wire is given as $I\left(t\right) = \left(5.00\ \text{A}\right) \sin\ \left(2\pi 60\ \text{Hz}\ t\right)$. What is the current density at time 15.00 ms?

Exercise:

Problem:

A particle accelerator produces a beam with a radius of 1.25 mm with a current of 2.00 mA. Each proton has a kinetic energy of 10.00 MeV. (a) What is the velocity of the protons? (b) What is the number (n) of protons per unit volume? (b) How many electrons pass a cross sectional area each second?

Solution:

a. 4.38 x 10^7 m/s b. $v = 5.81 \times 10^{13} \, \frac{\text{protons}}{\text{m}^3}$ c. $1.25 \, \frac{\text{electrons}}{\text{m}^3}$

Exercise:

Problem:

In this chapter, most examples and problems involved direct current (DC). DC circuits have the current flowing in one direction, from positive to negative. When the current was changing, it was changed linearly from $I = -I_{\text{max}}$ to $I = +I_{\text{max}}$ and the voltage changed linearly from $V = -V_{\text{max}}$ to $V = +V_{\text{max}}$, where $V_{\text{max}} = I_{\text{max}}R$. Suppose a voltage source is placed in series with a resistor of $R = 10\,\Omega$ that supplied a current that alternated as a sine wave, for example, $I\,(t) = (3.00\text{ A})\,\sin\,\left(\frac{2\pi}{4.00\text{ s}}t\right)$. (a) What would a graph of the voltage drop across the resistor *V*(*t*) versus time look like? (b) What would a plot of *V*(*t*) versus *I*(*t*) for one period look like? (*Hint*: If you are not sure, try plotting *V*(*t*) versus *I*(*t*) using a spreadsheet.)

Exercise:

Problem:

A current of $I = 25A$ is drawn from a 100-V battery for 30 seconds. By how much is the chemical energy reduced?

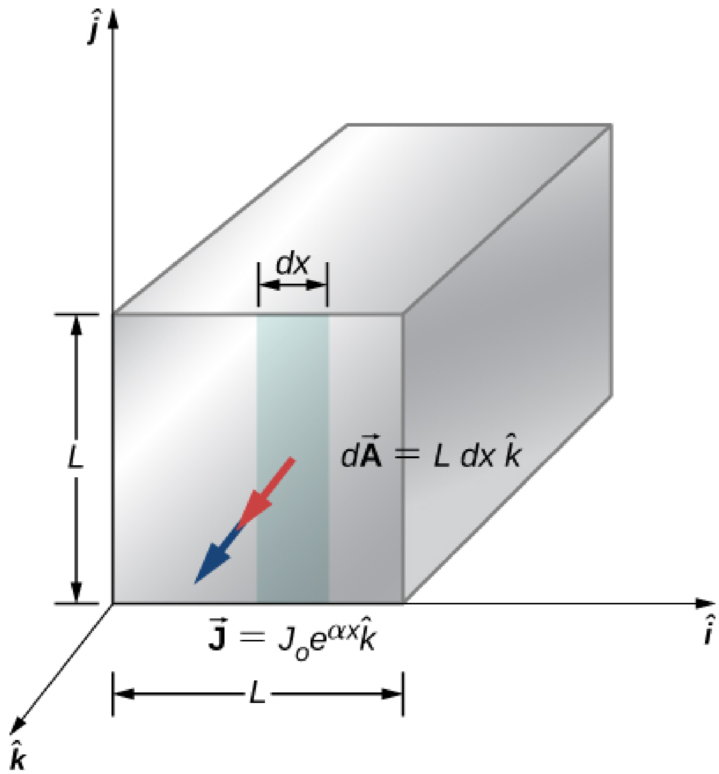
Solution:

$E = 75\text{ kJ}$

Exercise:

Problem:

Consider a square rod of material with sides of length $L = 3.00\text{ cm}$ with a current density of $\vec{\mathbf{J}} = J_0 e^{\alpha x}\hat{k} = \left(0.35\,\frac{\text{A}}{\text{m}^2}\right)e^{\left(2.1\,\times\,10^{-3}\text{m}^{-1}\right)x}\hat{\mathbf{k}}$ as shown below. Find the current that passes through the face of the rod.



Exercise:

Problem:

A resistor of an unknown resistance is placed in an insulated container filled with 0.75 kg of water. A voltage source is connected in series with the resistor and a current of 1.2 amps flows through the resistor for 10 minutes. During this time, the temperature of the water is measured and the temperature change during this time is $\Delta T = 10.00\ ^{\circ}\mathrm{C}$. (a) What is the resistance of the resistor? (b) What is the voltage supplied by the power supply?

Solution:

a. $P = 52\ \mathrm{W}$; b. $V = 43.54\ \mathrm{V}$
$R = 36\ \Omega$

Exercise:

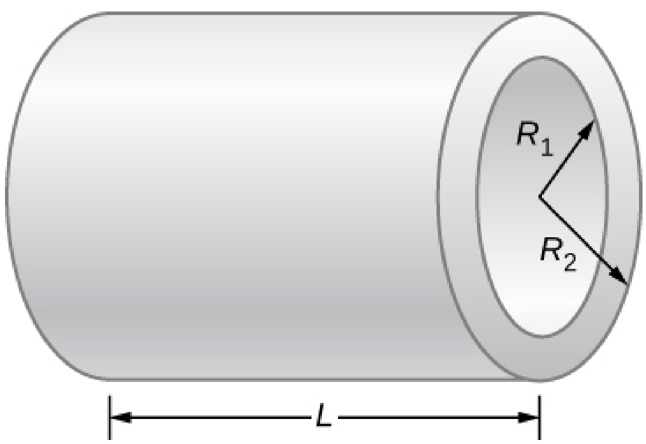
Problem:

The charge that flows through a point in a wire as a function of time is modeled as $q\left(t\right)={q}_{0}{e}^{-t/T}=10.0\,\text{C}{e}^{-t/5\,\text{s}}$. (a) What is the initial current through the wire at time $t=0.00\,\text{s}$? (b) Find the current at time $t=\frac{1}{2}T$. (c) At what time t will the current be reduced by one-half $I=\frac{1}{2}{I}_{0}$?

Exercise:

Problem:

Consider a resistor made from a hollow cylinder of carbon as shown below. The inner radius of the cylinder is ${R}_{i}=0.20\,\text{mm}$ and the outer radius is ${R}_{0}=0.30\,\text{mm}$. The length of the resistor is $L=0.90\,\text{mm}$. The resistivity of the carbon is $\rho =3.5\,\times\,{10}^{-5}\,\Omega\,\cdot\text{m}$. (a) Prove that the resistance perpendicular from the axis is $R=\frac{\rho }{2\pi L}\text{ln}\left(\frac{{R}_{0}}{{R}_{i}}\right)$. (b) What is the resistance?



Solution:

a. $R=\frac{\rho }{2\pi L}\text{ln}\left(\frac{{R}_{0}}{{R}_{i}}\right)$; b. $R=2.5\,\text{m}\,\Omega$

Exercise:

Problem:

What is the current through a cylindrical wire of radius $R=0.1\,\text{mm}$ if the current density is $J=\frac{{J}_{0}}{R}r$, where ${J}_{0}=32000\frac{\text{A}}{{\text{m}}^{2}}$?

Exercise:

Problem:

A student uses a 100.00-W, 115.00-V radiant heater to heat the student's dorm room, during the hours between sunset and sunrise, 6:00 p.m. to 7:00 a.m. (a) What current does the heater operate at? (b) How many electrons move through the heater? (c) What is the resistance of the heater? (d) How much heat was added to the dorm room?

Solution:

(a) 0.870 A
(b) #electrons = 2.54×10^{23} electrons
(c) 132 ohms
(d) $q = 4.68 \times 10^{6}$ J

Exercise:

Problem:

A 12-V car battery is used to power a 20.00-W, 12.00-V lamp during the physics club camping trip/star party. The cable to the lamp is 2.00 meters long, 14-gauge copper wire with a charge density of $n = 9.50\ \times\ 10^{28}\text{m}^{-3}$. (a) What is the current draw by the lamp? (b) How long would it take an electron to get from the battery to the lamp?

Exercise:

Problem:

A physics student uses a 115.00-V immersion heater to heat 400.00 grams (almost two cups) of water for herbal tea. During the two minutes it takes the water to heat, the physics student becomes bored and decides to figure out the resistance of the heater. The student starts with the assumption that the water is initially at the temperature of the room $T_i = 25.00\ ^\circ\text{C}$ and reaches $T_f = 100.00\ ^\circ\text{C}$. The specific heat of the water is $c = 4180\frac{\text{J}}{\text{kg}\bullet\text{K}}$. What is the resistance of the heater?

Solution:

$$P = 1045\text{ W}, \quad P = \frac{V^2}{R}, \quad R = 12.27\,\Omega$$

Glossary

critical temperature
: temperature at which a material reaches superconductivity

Josephson junction
: junction of two pieces of superconducting material separated by a thin layer of insulating material, which can carry a supercurrent

Meissner effect
: phenomenon that occurs in a superconducting material where all magnetic fields are expelled

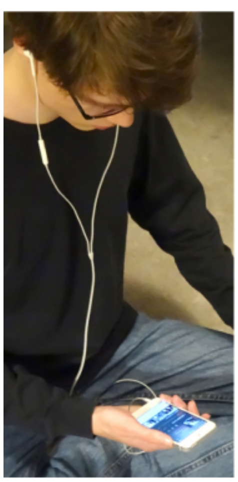
SQUID
: (Superconducting Quantum Interference Device) device that is a very sensitive magnetometer, used to measure extremely subtle magnetic fields

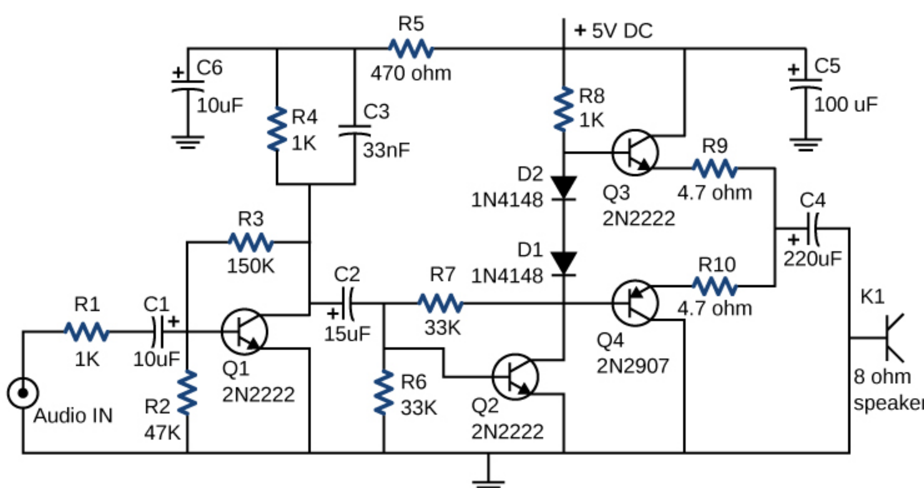
superconductivity
: phenomenon that occurs in some materials where the resistance goes to exactly zero and all magnetic fields are expelled, which occurs dramatically at some low critical temperature (T_C)

Introduction class="introduction"

This circuit shown is used to amplify small signals and power the earbud speakers attached to a cellular phone. This circuit’s components include resistors, capacitors, and diodes, all of which have been covered in previous chapters, as well as transistors, which are semi-conducting devices covered in Condensed Matter Physics. Circuits

using similar components are found in all types of equipment and appliances you encounter in everyday life, such as alarm clocks, televisions, computers, and refrigerators. (credit left: modification of work by Jane Whitney)





In the preceding few chapters, we discussed electric components, including capacitors, resistors, and diodes. In this chapter, we use these electric components in circuits. A circuit is a collection of electrical components connected to accomplish a specific task. [link] shows an amplifier circuit, which takes a small-amplitude signal and amplifies it to power the speakers in earbuds. Although the circuit looks complex, it actually consists of a set of series, parallel, and series-parallel circuits. The second section of this chapter covers the analysis of series and parallel circuits that consist of resistors. Later in this chapter, we introduce the basic equations and techniques to analyze any circuit, including those that are not reducible through simplifying parallel and series elements. But first, we need to understand how to power a circuit.

Electromotive Force

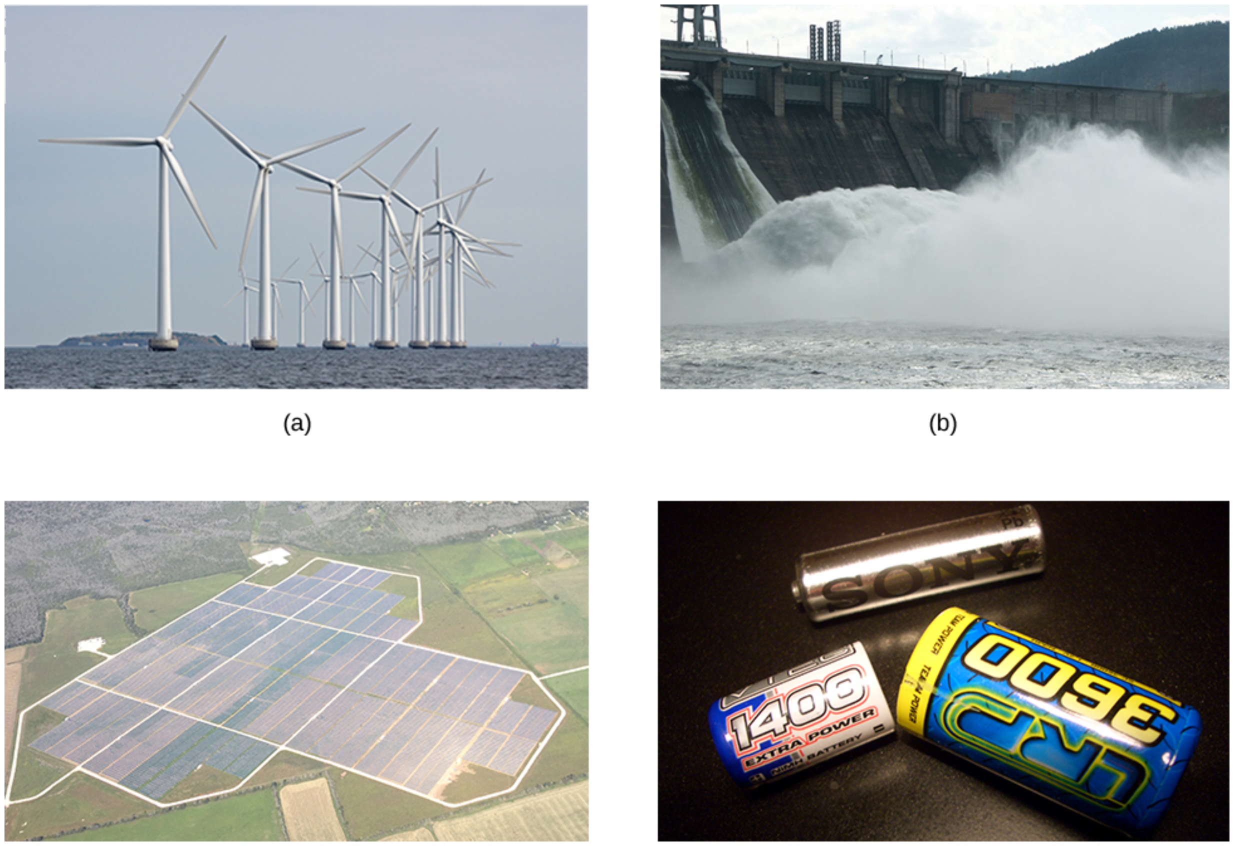
By the end of the section, you will be able to:

- Describe the electromotive force (emf) and the internal resistance of a battery
- Explain the basic operation of a battery

If you forget to turn off your car lights, they slowly dim as the battery runs down. Why don’t they suddenly blink off when the battery’s energy is gone? Their gradual dimming implies that the battery output voltage decreases as the battery is depleted. The reason for the decrease in output voltage for depleted batteries is that all voltage sources have two fundamental parts—a source of electrical energy and an internal resistance. In this section, we examine the energy source and the internal resistance.

Introduction to Electromotive Force

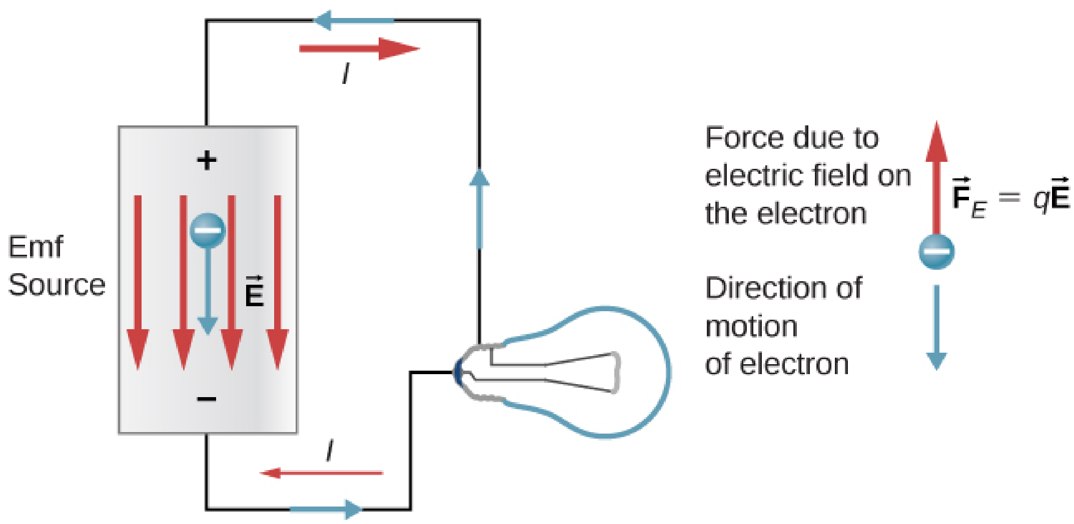
Voltage has many sources, a few of which are shown in [link]. All such devices create a **potential difference** and can supply current if connected to a circuit. A special type of potential difference is known as **electromotive force (emf)**. The emf is not a force at all, but the term ‘electromotive force’ is used for historical reasons. It was coined by Alessandro Volta in the 1800s, when he invented the first battery, also known as the voltaic pile. Because the electromotive force is not a force, it is common to refer to these sources simply as sources of emf (pronounced as the letters “ee-em-eff”), instead of sources of electromotive force.



(a) (b) (c) (d)

A variety of voltage sources. (a) The Brazos Wind Farm in Fluvanna, Texas; (b) the Krasnoyarsk Dam in Russia; (c) a solar farm; (d) a group of nickel metal hydride batteries. The voltage output of each device depends on its construction and load. The voltage output equals emf only if there is no load. (credit a: modification of work by Stig Nygaard; credit b: modification of work by "vadimpl"/Wikimedia Commons; credit c: modification of work by "The tdog"/Wikimedia Commons; credit d: modification of work by "Itrados"/Wikimedia Commons)

If the electromotive force is not a force at all, then what is the emf and what is a source of emf? To answer these questions, consider a simple circuit of a 12-V lamp attached to a 12-V battery, as shown in [link]. The battery can be modeled as a two-terminal device that keeps one terminal at a higher electric potential than the second terminal. The higher electric potential is sometimes called the positive terminal and is labeled with a plus sign. The lower-potential terminal is sometimes called the negative terminal and labeled with a minus sign. This is the source of the emf.



A source of emf maintains one terminal at a higher electric potential than the other terminal, acting as a source of current in a circuit.

When the emf source is not connected to the lamp, there is no net flow of charge within the emf source. Once the battery is connected to the lamp, charges flow from one terminal of the battery, through the lamp (causing the lamp to light), and back to the other terminal of the battery. If we consider positive (conventional) current flow, positive charges leave the positive terminal, travel through the lamp, and enter the negative terminal.

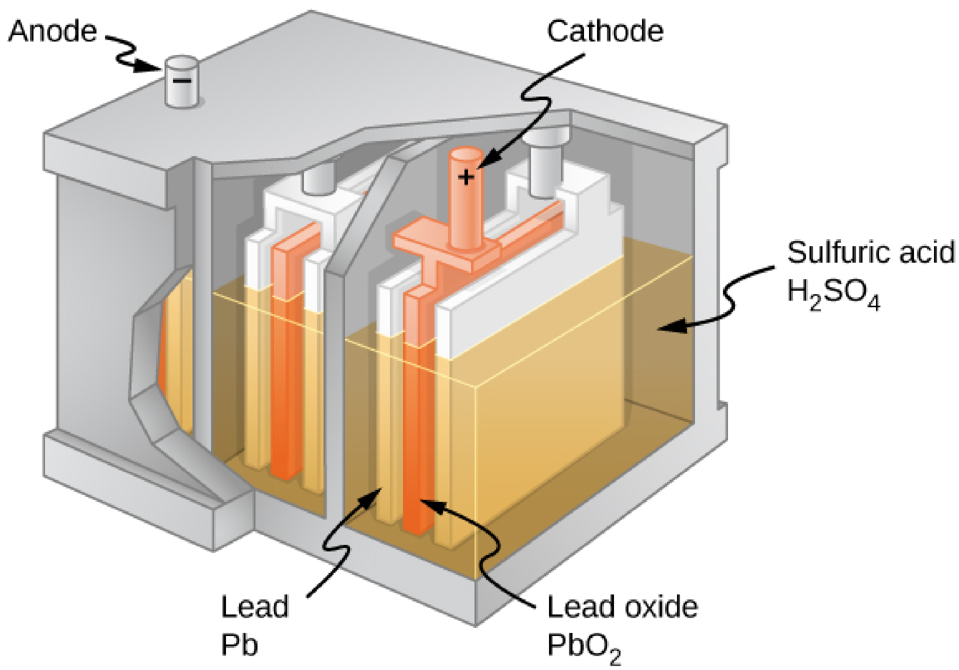
Positive current flow is useful for most of the circuit analysis in this chapter, but in metallic wires and resistors, electrons contribute the most to current, flowing in the opposite direction of positive current flow. Therefore, it is more realistic to consider the movement of electrons for the analysis of the circuit in [link]. The electrons leave the negative terminal, travel through the lamp, and return to the positive terminal. In order for the emf source to maintain the potential difference between the two terminals, negative charges (electrons) must be moved from the positive terminal to the negative terminal. The emf source acts as a charge pump, moving negative charges from the positive terminal to the negative terminal to maintain the potential difference. This increases the potential energy of the charges and, therefore, the electric potential of the charges.

The force on the negative charge from the electric field is in the opposite direction of the electric field, as shown in [link]. In order for the negative charges to be moved to the negative terminal, work must be done on the negative charges. This requires energy, which comes from chemical reactions in the battery. The potential is kept high on the positive terminal and low on the negative terminal to maintain the potential difference between the two terminals. The emf is equal to the work done on the charge per unit charge $\left(\varepsilon = \frac{dW}{dq}\right)$ when there is no current flowing. Since the unit for work is the joule and the unit for charge is the coulomb, the unit for emf is the volt $(1\ \text{V} = 1\ \text{J/C})$.

The **terminal voltage** V_{terminal} of a battery is voltage measured across the terminals of the battery when there is no load connected to the terminal. An ideal battery is an emf source that maintains a constant terminal voltage, independent of the current between the two terminals. An ideal battery has no internal resistance, and the terminal voltage is equal to the emf of the battery. In the next section, we will show that a real battery does have internal resistance and the terminal voltage is always less than the emf of the battery.

The Origin of Battery Potential

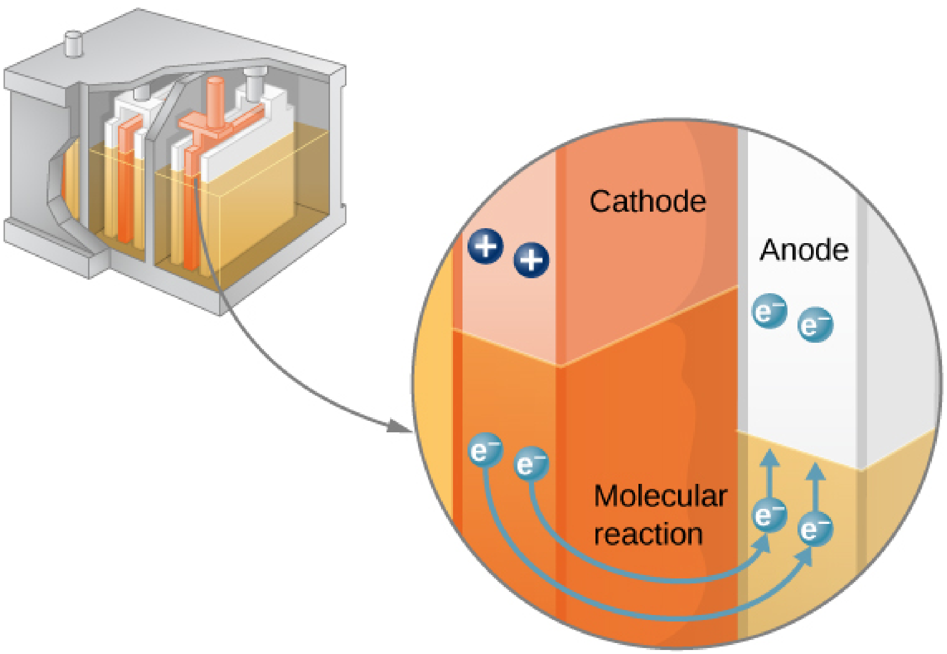
The combination of chemicals and the makeup of the terminals in a battery determine its emf. The lead acid battery used in cars and other vehicles is one of the most common combinations of chemicals. [link] shows a single cell (one of six) of this battery. The cathode (positive) terminal of the cell is connected to a lead oxide plate, whereas the anode (negative) terminal is connected to a lead plate. Both plates are immersed in sulfuric acid, the electrolyte for the system.



Chemical reactions in a lead-acid cell separate charge, sending negative charge to the anode, which is connected to the lead plates. The lead oxide plates are connected to the positive or cathode terminal of the cell. Sulfuric acid conducts the charge, as well as participates in the chemical reaction.

Knowing a little about how the chemicals in a lead-acid battery interact helps in understanding the potential created by the battery. [link] shows the result of a single chemical reaction. Two electrons are placed on the anode, making it negative, provided that the cathode supplies two electrons. This leaves the cathode positively charged, because it has lost two electrons. In short, a separation of charge has been driven by a chemical reaction.

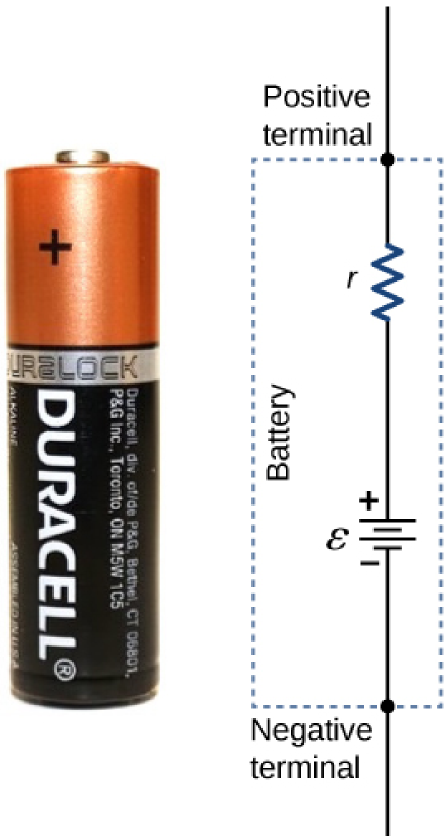
Note that the reaction does not take place unless there is a complete circuit to allow two electrons to be supplied to the cathode. Under many circumstances, these electrons come from the anode, flow through a resistance, and return to the cathode. Note also that since the chemical reactions involve substances with resistance, it is not possible to create the emf without an internal resistance.



In a lead-acid battery, two electrons are forced onto the anode of a cell, and two electrons are removed from the cathode of the cell. The chemical reaction in a lead-acid battery places two electrons on the anode and removes two from the cathode. It requires a closed circuit to proceed, since the two electrons must be supplied to the cathode.

Internal Resistance and Terminal Voltage

The amount of resistance to the flow of current within the voltage source is called the **internal resistance**. The internal resistance r of a battery can behave in complex ways. It generally increases as a battery is depleted, due to the oxidation of the plates or the reduction of the acidity of the electrolyte. However, internal resistance may also depend on the magnitude and direction of the current through a voltage source, its temperature, and even its history. The internal resistance of rechargeable nickel-cadmium cells, for example, depends on how many times and how deeply they have been depleted. A simple model for a battery consists of an idealized emf source ε and an internal resistance r ([link]).



A battery can be modeled as an idealized emf (ε) with an internal resistance (r). The terminal voltage of the battery is $V_{\text{terminal}} = \varepsilon - Ir$.

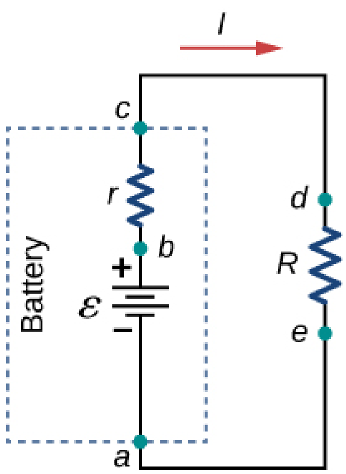
Suppose an external resistor, known as the load resistance R, is connected to a voltage source such as a battery, as in [link]. The figure shows a model of a battery with an emf ε,

an internal resistance r, and a load resistor R connected across its terminals. Using conventional current flow, positive charges leave the positive terminal of the battery, travel through the resistor, and return to the negative terminal of the battery. The terminal voltage of the battery depends on the emf, the internal resistance, and the current, and is equal to

Note:
Equation:

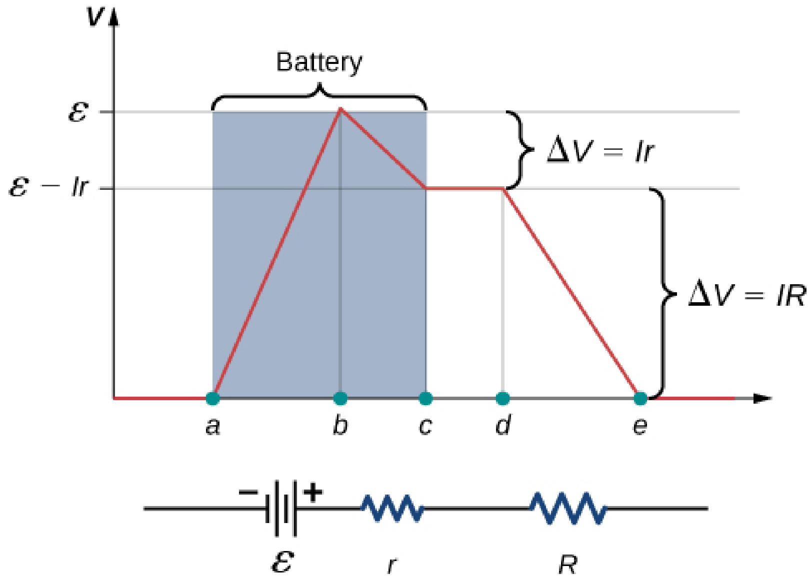
$$V_{\text{terminal}} = \varepsilon - Ir.$$

For a given emf and internal resistance, the terminal voltage decreases as the current increases due to the potential drop Ir of the internal resistance.



Schematic of a voltage source and its load resistor R. Since the internal resistance r is in series with the load, it can significantly affect the terminal voltage and the current delivered to the load.

A graph of the potential difference across each element the circuit is shown in [link]. A current I runs through the circuit, and the potential drop across the internal resistor is equal to Ir. The terminal voltage is equal to $\varepsilon - Ir$, which is equal to the **potential drop** across the load resistor $IR = \varepsilon - Ir$. As with potential energy, it is the change in voltage that is important. When the term "voltage" is used, we assume that it is actually the change in the potential, or ΔV. However, Δ is often omitted for convenience.



A graph of the voltage through the circuit of a battery and a load resistance. The electric potential increases the emf of the battery due to the chemical reactions doing work on the charges. There is a decrease in the electric potential in the battery due to the internal resistance. The potential decreases due to the internal resistance $(-Ir)$, making the terminal voltage of the battery equal to $(\varepsilon - Ir)$. The voltage then decreases by (IR). The current is equal to $I = \frac{\varepsilon}{r+R}$.

The current through the load resistor is $I = \frac{\varepsilon}{r+R}$. We see from this expression that the smaller the internal resistance r, the greater the current the voltage source supplies to its load R. As batteries are depleted, r increases. If r becomes a significant fraction of the load resistance, then the current is significantly reduced, as the following example illustrates.

Example:

Analyzing a Circuit with a Battery and a Load

A given battery has a 12.00-V emf and an internal resistance of $0.100\ \Omega$. (a) Calculate its terminal voltage when connected to a $10.00\text{-}\Omega$ load. (b) What is the terminal voltage when connected to a $0.500\text{-}\Omega$ load? (c) What power does the $0.500\text{-}\Omega$ load dissipate? (d) If the internal resistance grows to $0.500\ \Omega$, find the current, terminal voltage, and power dissipated by a $0.500\text{-}\Omega$ load.

Strategy

The analysis above gave an expression for current when internal resistance is taken into account. Once the current is found, the terminal voltage can be calculated by using the equation $V_{\text{terminal}} = \varepsilon - Ir$. Once current is found, we can also find the power dissipated by the resistor.

Solution

a. Entering the given values for the emf, load resistance, and internal resistance into the expression above yields
   Equation:

$$I = \frac{\varepsilon}{R + r} = \frac{12.00\ \text{V}}{10.10\ \Omega} = 1.188\ \text{A}.$$

   Enter the known values into the equation $V_{\text{terminal}} = \varepsilon - Ir$ to get the terminal voltage:
   Equation:

$$V_{\text{terminal}} = \varepsilon - Ir = 12.00\ \text{V} - (1.188\ \text{A})(0.100\ \Omega) = 11.90\ \text{V}.$$

   The terminal voltage here is only slightly lower than the emf, implying that the current drawn by this light load is not significant.

b. Similarly, with $R_{\text{load}} = 0.500\ \Omega$, the current is
   Equation:

$$I = \frac{\varepsilon}{R + r} = \frac{12.00\ \text{V}}{0.600\ \Omega} = 20.00\ \text{A}.$$

   The terminal voltage is now
   Equation:

$$V_{\text{terminal}} = \varepsilon - Ir = 12.00\ \text{V} - (20.00\ \text{A})\,(0.100\ \Omega) = 10.00\ \text{V}.$$

   The terminal voltage exhibits a more significant reduction compared with emf, implying $0.500\ \Omega$ is a heavy load for this battery. A “heavy load” signifies a larger draw of current from the source but not a larger resistance.

c. The power dissipated by the $0.500\text{-}\Omega$ load can be found using the formula $P = I^2R$. Entering the known values gives
Equation:

$$P = I^2R = (20.0\text{ A})^2(0.500\ \Omega) = 2.00\ \times\ 10^2\text{ W}.$$

Note that this power can also be obtained using the expression $\frac{V^2}{R}$ or IV, where V is the terminal voltage (10.0 V in this case).

d. Here, the internal resistance has increased, perhaps due to the depletion of the battery, to the point where it is as great as the load resistance. As before, we first find the current by entering the known values into the expression, yielding
Equation:

$$I = \frac{\varepsilon}{R + r} = \frac{12.00\text{ V}}{1.00\ \Omega} = 12.00\text{ A}.$$

Now the terminal voltage is
Equation:

$$V_{\text{terminal}} = \varepsilon - Ir = 12.00\text{ V} - (12.00\text{ A})(0.500\ \Omega) = 6.00\text{ V},$$

and the power dissipated by the load is
Equation:

$$P = I^2R = (12.00\text{ A})^2(0.500\ \Omega) = 72.00\text{ W}.$$

We see that the increased internal resistance has significantly decreased the terminal voltage, current, and power delivered to a load.

Significance
The internal resistance of a battery can increase for many reasons. For example, the internal resistance of a rechargeable battery increases as the number of times the battery is recharged increases. The increased internal resistance may have two effects on the battery. First, the terminal voltage will decrease. Second, the battery may overheat due to the increased power dissipated by the internal resistance.

Note:
Exercise:

Problem:

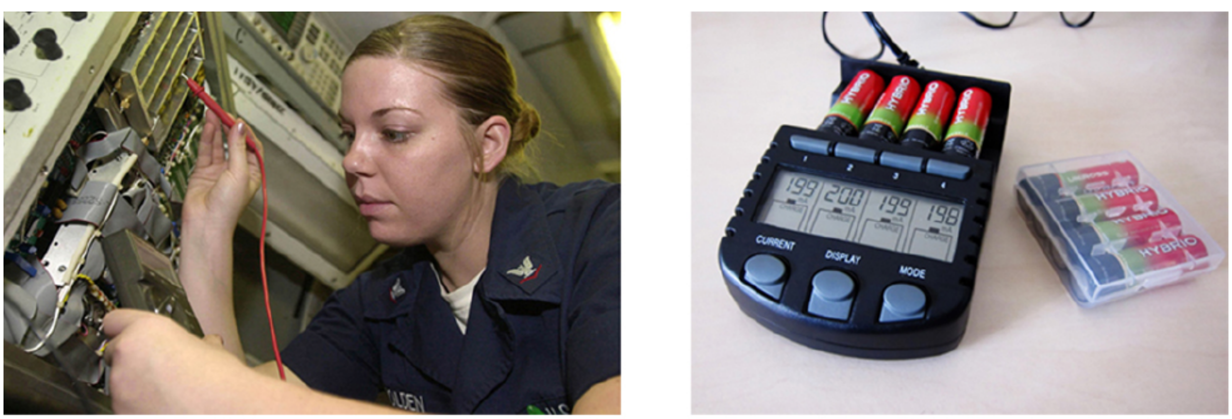
Check Your Understanding If you place a wire directly across the two terminal of a battery, effectively shorting out the terminals, the battery will begin to get hot. Why do you suppose this happens?

Solution:

If a wire is connected across the terminals, the load resistance is close to zero, or at least considerably less than the internal resistance of the battery. Since the internal resistance is small, the current through the circuit will be large, $I = \frac{\varepsilon}{R+r} = \frac{\varepsilon}{0+r} = \frac{\varepsilon}{r}$. The large current causes a high power to be dissipated by the internal resistance $\left(P = I^2 r\right)$. The power is dissipated as heat.

Battery Testers

Battery testers, such as those in [link], use small load resistors to intentionally draw current to determine whether the terminal potential drops below an acceptable level. Although it is difficult to measure the internal resistance of a battery, battery testers can provide a measurement of the internal resistance of the battery. If internal resistance is high, the battery is weak, as evidenced by its low terminal voltage.

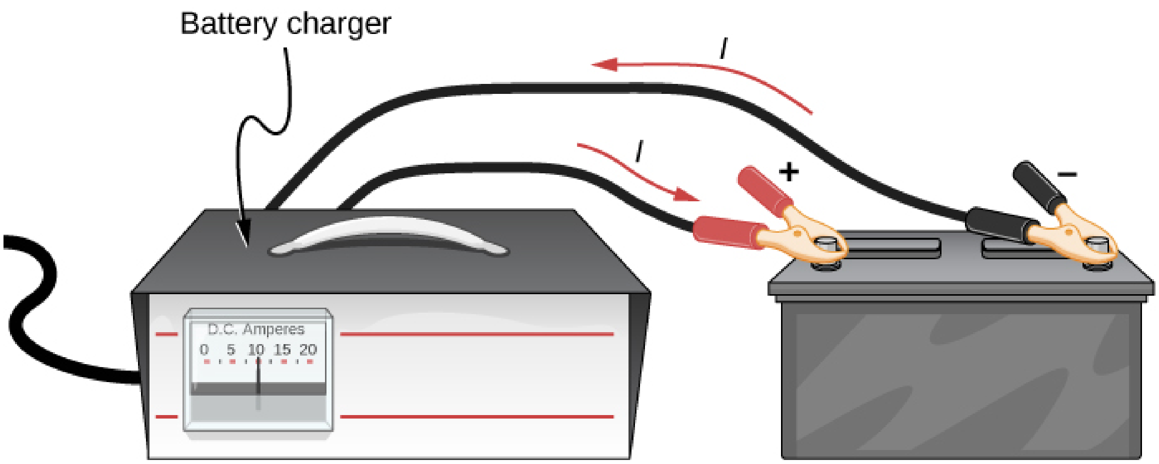


(a) (b)

Battery testers measure terminal voltage under a load to determine the condition of a battery. (a) A US Navy electronics technician uses a battery tester to test large batteries aboard the aircraft carrier USS *Nimitz*. The battery tester she uses has a small resistance that can dissipate large amounts of power. (b) The small device shown is used on small batteries and has a digital display to indicate the acceptability of the

terminal voltage. (credit a: modification of work by Jason A. Johnston; credit b: modification of work by Keith Williamson)

Some batteries can be recharged by passing a current through them in the direction opposite to the current they supply to an appliance. This is done routinely in cars and in batteries for small electrical appliances and electronic devices ([link]). The voltage output of the battery charger must be greater than the emf of the battery to reverse the current through it. This causes the terminal voltage of the battery to be greater than the emf, since $V = \varepsilon - Ir$ and I is now negative.



A car battery charger reverses the normal direction of current through a battery, reversing its chemical reaction and replenishing its chemical potential.

It is important to understand the consequences of the internal resistance of emf sources, such as batteries and solar cells, but often, the analysis of circuits is done with the terminal voltage of the battery, as we have done in the previous sections. The terminal voltage is referred to as simply as V, dropping the subscript "terminal." This is because the internal resistance of the battery is difficult to measure directly and can change over time.

Summary

- All voltage sources have two fundamental parts: a source of electrical energy that has a characteristic electromotive force (emf), and an internal resistance r. The emf is the work done per charge to keep the potential difference of a source constant. The emf is equal to the potential difference across the terminals when no current is flowing. The

internal resistance *r* of a voltage source affects the output voltage when a current flows.
- The voltage output of a device is called its terminal voltage V_{terminal} and is given by $V_{\text{terminal}} = \varepsilon - Ir$, where *I* is the electric current and is positive when flowing away from the positive terminal of the voltage source and *r* is the internal resistance.

Conceptual Questions

Exercise:

Problem:

What effect will the internal resistance of a rechargeable battery have on the energy being used to recharge the battery?

Solution:

Some of the energy being used to recharge the battery will be dissipated as heat by the internal resistance.

Exercise:

Problem:

A battery with an internal resistance of *r* and an emf of 10.00 V is connected to a load resistor $R = r$. As the battery ages, the internal resistance triples. How much is the current through the load resistor reduced?

Exercise:

Problem:

Show that the power dissipated by the load resistor is maximum when the resistance of the load resistor is equal to the internal resistance of the battery.

Solution:

$$P = I^2R = \left(\frac{\varepsilon}{r+R}\right)^2 R = \varepsilon^2 R(r+R)^{-2}, \quad \frac{dP}{dR} = \varepsilon^2\left[(r+R)^{-2} - 2R(r+R)^{-3}\right] = 0,$$
$$\left[\frac{(r+R)-2R}{(r+R)^3}\right] = 0, \quad r = R$$

Problems

Exercise:

Problem:

A car battery with a 12-V emf and an internal resistance of $0.050\ \Omega$ is being charged with a current of 60 A. Note that in this process, the battery is being charged. (a) What is the potential difference across its terminals? (b) At what rate is thermal energy being dissipated in the battery? (c) At what rate is electric energy being converted into chemical energy?

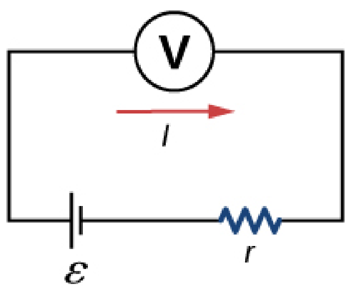
Exercise:

Problem:

The label on a battery-powered radio recommends the use of a rechargeable nickel-cadmium cell (nicads), although it has a 1.25-V emf, whereas an alkaline cell has a 1.58-V emf. The radio has a $3.20\ \Omega$ resistance. (a) Draw a circuit diagram of the radio and its battery. Now, calculate the power delivered to the radio (b) when using a nicad cells, each having an internal resistance of $0.0400\ \Omega$, and (c) when using an alkaline cell, having an internal resistance of $0.200\ \Omega$. (d) Does this difference seem significant, considering that the radio's effective resistance is lowered when its volume is turned up?

Solution:

a.



b. 0.476W; c. 0.691 W; d. As R_L is lowered, the power difference decreases; therefore, at higher volumes, there is no significant difference.

Exercise:

Problem:

An automobile starter motor has an equivalent resistance of $0.0500\ \Omega$ and is supplied by a 12.0-V battery with a $0.0100\text{-}\Omega$ internal resistance. (a) What is the current to the motor? (b) What voltage is applied to it? (c) What power is supplied to the motor? (d) Repeat these calculations for when the battery connections are corroded and add $0.0900\ \Omega$ to the circuit. (Significant problems are caused by even small amounts of unwanted resistance in low-voltage, high-current applications.)

Exercise:

Problem:

(a) What is the internal resistance of a voltage source if its terminal potential drops by 2.00 V when the current supplied increases by 5.00 A? (b) Can the emf of the voltage source be found with the information supplied?

Solution:

a. $0.400\ \Omega$; b. No, there is only one independent equation, so only *r* can be found.

Exercise:

Problem:

A person with body resistance between his hands of $10.0\ \text{k}\Omega$ accidentally grasps the terminals of a 20.0-kV power supply. (Do NOT do this!) (a) Draw a circuit diagram to represent the situation. (b) If the internal resistance of the power supply is $2000\ \Omega$, what is the current through his body? (c) What is the power dissipated in his body? (d) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in this situation to be 1.00 mA or less? (e) Will this modification compromise the effectiveness of the power supply for driving low-resistance devices? Explain your reasoning.

Exercise:

Problem:

A 12.0-V emf automobile battery has a terminal voltage of 16.0 V when being charged by a current of 10.0 A. (a) What is the battery's internal resistance? (b) What power is dissipated inside the battery? (c) At what rate (in $^\circ\text{C/min}$) will its temperature increase if its mass is 20.0 kg and it has a specific heat of $0.300\ \text{kcal/kg}\ \cdot\ ^\circ\text{C}$, assuming no heat escapes?

Solution:

a. $0.400\ \Omega$; b. 40.0 W; c. $0.0956\ ^\circ\text{C/min}$

Glossary

electromotive force (emf)
: energy produced per unit charge, drawn from a source that produces an electrical current

internal resistance
: amount of resistance to the flow of current within the voltage source

potential difference
: difference in electric potential between two points in an electric circuit, measured in volts

potential drop
: loss of electric potential energy as a current travels across a resistor, wire, or other component

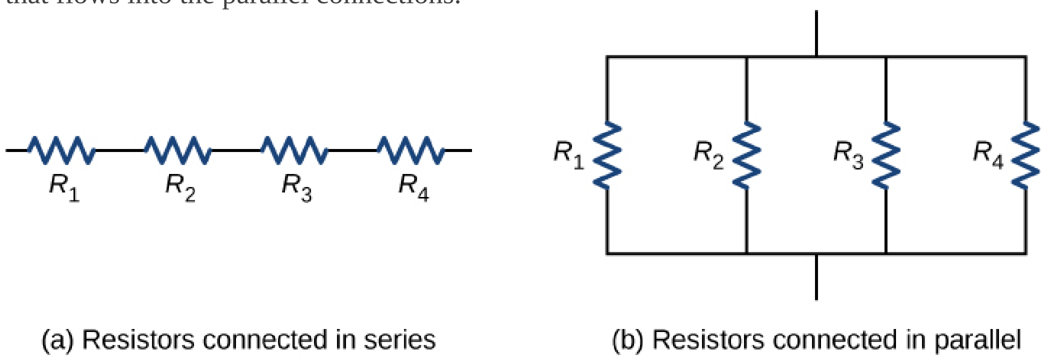
terminal voltage
: potential difference measured across the terminals of a source when there is no load attached

Resistors in Series and Parallel
By the end of the section, you will be able to:

- Define the term equivalent resistance
- Calculate the equivalent resistance of resistors connected in series
- Calculate the equivalent resistance of resistors connected in parallel

In Current and Resistance, we described the term 'resistance' and explained the basic design of a resistor. Basically, a resistor limits the flow of charge in a circuit and is an ohmic device where $V = IR$. Most circuits have more than one resistor. If several resistors are connected together and connected to a battery, the current supplied by the battery depends on the **equivalent resistance** of the circuit.

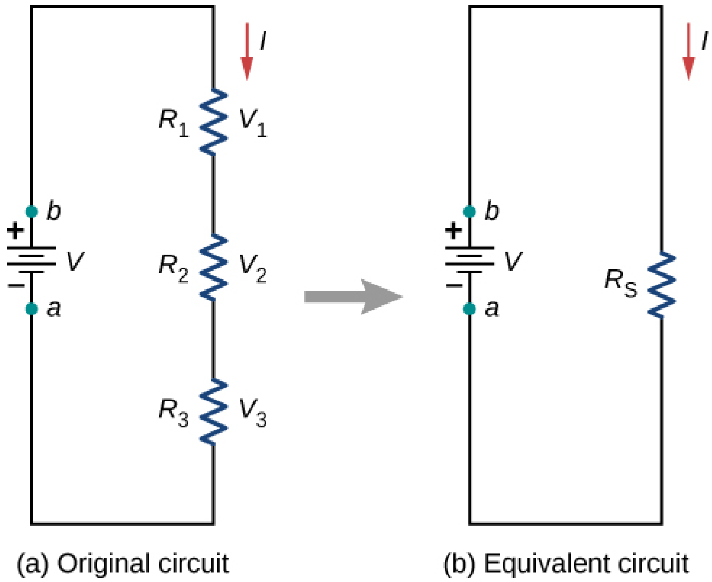
The equivalent resistance of a combination of resistors depends on both their individual values and how they are connected. The simplest combinations of resistors are series and parallel connections ([link]). In a series circuit, the output current of the first resistor flows into the input of the second resistor; therefore, the current is the same in each resistor. In a parallel circuit, all of the resistor leads on one side of the resistors are connected together and all the leads on the other side are connected together. In the case of a parallel configuration, each resistor has the same potential drop across it, and the currents through each resistor may be different, depending on the resistor. The sum of the individual currents equals the current that flows into the parallel connections.



(a) For a series connection of resistors, the current is the same in each resistor. (b) For a parallel connection of resistors, the voltage is the same across each resistor.

Resistors in Series

Resistors are said to be in series whenever the current flows through the resistors sequentially. Consider [link], which shows three resistors in series with an applied voltage equal to V_{ab}. Since there is only one path for the charges to flow through, the current is the same through each resistor. The equivalent resistance of a set of resistors in a series connection is equal to the algebraic sum of the individual resistances.



(a) Three resistors connected in series to a voltage source. (b) The original circuit is reduced to an equivalent resistance and a voltage source.

In [link], the current coming from the voltage source flows through each resistor, so the current through each resistor is the same. The current through the circuit depends on the voltage supplied by the voltage source and the resistance of the resistors. For each resistor, a potential drop occurs that is equal to the loss of electric potential energy as a current travels through each resistor. According to Ohm's law, the potential drop V across a resistor when a current flows through it is calculated using the equation $V = IR$, where I is the current in amps (A) and R is the resistance in ohms (Ω). Since energy is conserved, and the voltage is equal to the potential energy per charge, the sum of the voltage applied to the circuit by the source and the potential drops across the individual resistors around a loop should be equal to zero:

Equation:

$$\sum_{i=1}^{N} V_i = 0.$$

This equation is often referred to as Kirchhoff's loop law, which we will look at in more detail later in this chapter. For [link], the sum of the potential drop of each resistor and the voltage supplied by the voltage source should equal zero:

Equation:

$$\begin{aligned} V - V_1 - V_2 - V_3 &= 0, \\ V &= V_1 + V_2 + V_3, \\ &= IR_1 + IR_2 + IR_3, \\ I &= \frac{V}{R_1+R_2+R_3} = \frac{V}{R_S}. \end{aligned}$$

Since the current through each component is the same, the equality can be simplified to an equivalent resistance, which is just the sum of the resistances of the individual resistors.

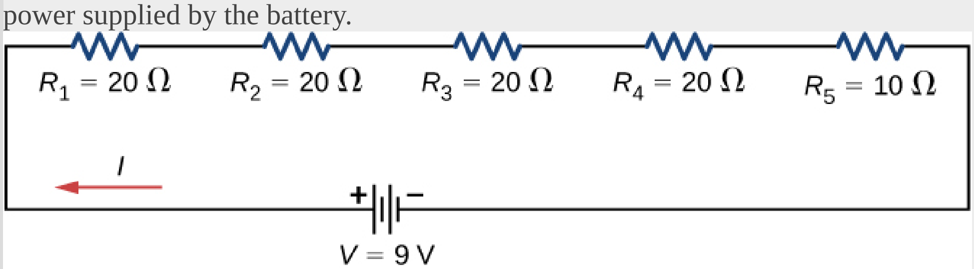
Any number of resistors can be connected in series. If *N* resistors are connected in series, the equivalent resistance is

Note:
Equation:

$$R_{\text{S}} = R_1 + R_2 + R_3 + \cdots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i.$$

One result of components connected in a series circuit is that if something happens to one component, it affects all the other components. For example, if several lamps are connected in series and one bulb burns out, all the other lamps go dark.

Example:
Equivalent Resistance, Current, and Power in a Series Circuit
A battery with a terminal voltage of 9 V is connected to a circuit consisting of four $20\text{-}\Omega$ and one $10\text{-}\Omega$ resistors all in series ([link]). Assume the battery has negligible internal resistance. (a) Calculate the equivalent resistance of the circuit. (b) Calculate the current through each resistor. (c) Calculate the potential drop across each resistor. (d) Determine the total power dissipated by the resistors and the power supplied by the battery.



A simple series circuit with five resistors.

Strategy
In a series circuit, the equivalent resistance is the algebraic sum of the resistances. The current through the circuit can be found from Ohm's law and is equal to the voltage divided by the equivalent resistance. The potential drop across each resistor can be found using Ohm's law. The power dissipated by each resistor can be found using $P = I^2R$, and the total power dissipated by the resistors is equal to the sum of the power dissipated by each resistor. The power supplied by the battery can be found using $P = I\varepsilon$.
Solution

a. The equivalent resistance is the algebraic sum of the resistances:
   Equation:

$$R_S = R_1 + R_2 + R_3 + R_4 + R_5 = 20\,\Omega + 20\,\Omega + 20\,\Omega + 20\,\Omega + 10\,\Omega = 90\,\Omega.$$

b. The current through the circuit is the same for each resistor in a series circuit and is equal to the applied voltage divided by the equivalent resistance:
Equation:

$$I = \frac{V}{R_S} = \frac{9\text{ V}}{90\,\Omega} = 0.1\text{ A}.$$

c. The potential drop across each resistor can be found using Ohm's law:
Equation:

$$\begin{aligned} &V_1 = V_2 = V_3 = V_4 = (0.1\text{ A})20\,\Omega = 2\text{ V}, \\ &V_5 = (0.1\text{ A})10\,\Omega = 1\text{ V}, \\ &V_1 + V_2 + V_3 + V_4 + V_5 = 9\text{ V}. \end{aligned}$$

Note that the sum of the potential drops across each resistor is equal to the voltage supplied by the battery.

d. The power dissipated by a resistor is equal to $P = I^2R$, and the power supplied by the battery is equal to $P = I\varepsilon$:
Equation:

$$\begin{aligned} &P_1 = P_2 = P_3 = P_4 = (0.1\text{ A})^2\,(20\,\Omega) = 0.2\text{ W}, \\ &P_5 = (0.1\text{ A})^2\,(10\,\Omega) = 0.1\text{ W}, \\ &P_{\text{dissipated}} = 0.2\text{ W} + 0.2\text{ W} + 0.2\text{ W} + 0.2\text{ W} + 0.1\text{ W} = 0.9\text{ W}, \\ &P_{\text{source}} = I\varepsilon = (0.1\text{ A})\,(9\text{ V}) = 0.9\text{ W}. \end{aligned}$$

Significance
There are several reasons why we would use multiple resistors instead of just one resistor with a resistance equal to the equivalent resistance of the circuit. Perhaps a resistor of the required size is not available, or we need to dissipate the heat generated, or we want to minimize the cost of resistors. Each resistor may cost a few cents to a few dollars, but when multiplied by thousands of units, the cost saving may be appreciable.

Note:
Exercise:

Problem:

Check Your Understanding Some strings of miniature holiday lights are made to short out when a bulb burns out. The device that causes the short is called a shunt, which allows current to flow around the open circuit. A "short" is like putting a piece of wire across the component. The bulbs are usually grouped in series of nine bulbs. If too many bulbs burn out, the shunts eventually open. What causes this?

Solution:

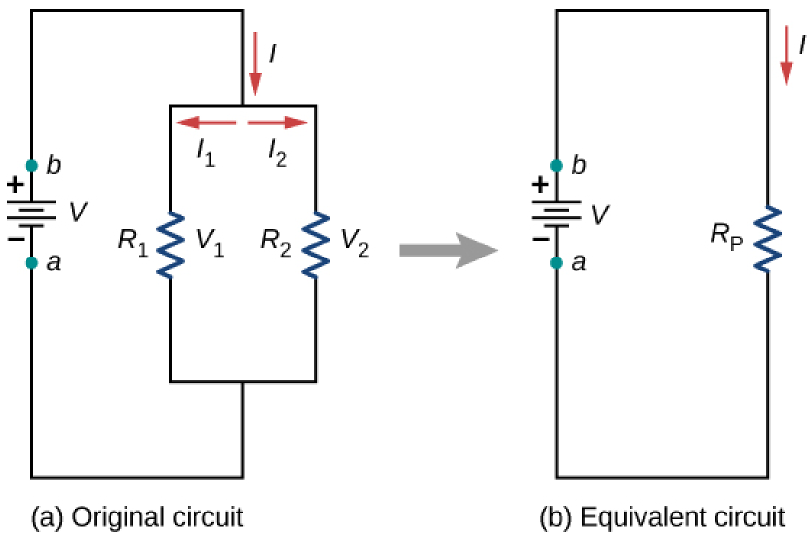
The equivalent resistance of nine bulbs connected in series is $9R$. The current is $I = V/9\,R$. If one bulb burns out, the equivalent resistance is $8R$, and the voltage does not change, but the current increases $(I = V/8\,R)$. As more bulbs burn out, the current becomes even higher. Eventually, the current becomes too high, burning out the shunt.

Let's briefly summarize the major features of resistors in series:

1. Series resistances add together to get the equivalent resistance:
   Equation:
   $$R_{\mathrm{S}} = R_1 + R_2 + R_3 + \cdots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i.$$
2. The same current flows through each resistor in series.
3. Individual resistors in series do not get the total source voltage, but divide it. The total potential drop across a series configuration of resistors is equal to the sum of the potential drops across each resistor.

Resistors in Parallel

[link] shows resistors in parallel, wired to a voltage source. Resistors are in parallel when one end of all the resistors are connected by a continuous wire of negligible resistance and the other end of all the resistors are also connected to one another through a continuous wire of negligible resistance. The potential drop across each resistor is the same. Current through each resistor can be found using Ohm's law $I = V/R$, where the voltage is constant across each resistor. For example, an automobile's headlights, radio, and other systems are wired in parallel, so that each subsystem utilizes the full voltage of the source and can operate completely independently. The same is true of the wiring in your house or any building.



(a) Two resistors connected in parallel to a voltage source. (b)

The original circuit is reduced to an equivalent resistance and a voltage source.

The current flowing from the voltage source in [link] depends on the voltage supplied by the voltage source and the equivalent resistance of the circuit. In this case, the current flows from the voltage source and enters a junction, or node, where the circuit splits flowing through resistors R_1 and R_2. As the charges flow from the battery, some go through resistor R_1 and some flow through resistor R_2. The sum of the currents flowing into a junction must be equal to the sum of the currents flowing out of the junction:

Equation:

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

This equation is referred to as Kirchhoff's junction rule and will be discussed in detail in the next section. In [link], the junction rule gives $I = I_1 + I_2$. There are two loops in this circuit, which leads to the equations $V = I_1 R_1$ and $I_1 R_1 = I_2 R_2$. Note the voltage across the resistors in parallel are the same $(V = V_1 = V_2)$ and the current is additive:

Equation:

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{V_1}{R_1} + \frac{V_2}{R_2} \\ &= \frac{V}{R_1} + \frac{V}{R_2} \\ &= V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R_\text{P}} \\ R_\text{P} &= \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}. \end{aligned}$$

Generalizing to any number of *N* resistors, the equivalent resistance R_P of a parallel connection is related to the individual resistances by

Note:

Equation:

$$R_\text{P} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}.$$

This relationship results in an equivalent resistance R_P that is less than the smallest of the individual resistances. When resistors are connected in parallel, more current flows from the source than would flow for any of them individually, so the total resistance is lower.

Example:

Analysis of a Parallel Circuit

Three resistors $R_1 = 1.00\,\Omega$, $R_2 = 2.00\,\Omega$, and $R_3 = 2.00\,\Omega$, are connected in parallel. The parallel connection is attached to a $V = 3.00\text{ V}$ voltage source. (a) What is the equivalent resistance? (b) Find the current supplied by the source to the parallel circuit. (c) Calculate the currents in each resistor and show that these add together to equal the current output of the source. (d) Calculate the power dissipated by each resistor. (e) Find the power output of the source and show that it equals the total power dissipated by the resistors.

Strategy

(a) The total resistance for a parallel combination of resistors is found using $R_{\text{P}} = \left(\sum_i \frac{1}{R_i}\right)^{-1}$.

(Note that in these calculations, each intermediate answer is shown with an extra digit.)

(b) The current supplied by the source can be found from Ohm's law, substituting R_{P} for the total resistance $I = \frac{V}{R_{\text{P}}}$.

(c) The individual currents are easily calculated from Ohm's law $\left(I_i = \frac{V_i}{R_i}\right)$, since each resistor gets the full voltage. The total current is the sum of the individual currents: $I = \sum_i I_i$.

(d) The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P_i = V^2/R_i$, since each resistor gets full voltage.

(e) The total power can also be calculated in several ways, use $P = IV$.

Solution

a. The total resistance for a parallel combination of resistors is found using [link]. Entering known values gives
   Equation:

$$R_{\text{P}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{1.00\,\Omega} + \frac{1}{2.00\,\Omega} + \frac{1}{2.00\,\Omega}\right)^{-1} = 0.50\,\Omega.$$

   The total resistance with the correct number of significant digits is $R_{\text{P}} = 0.50\,\Omega$. As predicted, R_{P} is less than the smallest individual resistance.

b. The total current can be found from Ohm's law, substituting R_{P} for the total resistance. This gives
   Equation:

$$I = \frac{V}{R_{\text{P}}} = \frac{3.00\text{ V}}{0.50\,\Omega} = 6.00\text{ A}.$$

   Current *I* for each device is much larger than for the same devices connected in series (see the previous example). A circuit with parallel connections has a smaller total resistance than the resistors connected in series.

c. The individual currents are easily calculated from Ohm's law, since each resistor gets the full voltage. Thus,
   Equation:

$$I_1 = \frac{V}{R_1} = \frac{3.00\text{ V}}{1.00\,\Omega} = 3.00\text{ A}.$$

Similarly,
Equation:

$$I_2 = \frac{V}{R_2} = \frac{3.00\ \text{V}}{2.00\ \Omega} = 1.50\ \text{A}$$

and
Equation:

$$I_3 = \frac{V}{R_3} = \frac{6.00\ \text{V}}{2.00\ \Omega} = 1.50\ \text{A}.$$

The total current is the sum of the individual currents:
Equation:

$$I_1 + I_2 + I_3 = 6.00\ \text{A}.$$

d. The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P = V^2/R$, since each resistor gets full voltage. Thus,
Equation:

$$P_1 = \frac{V^2}{R_1} = \frac{(3.00\ \text{V})^2}{1.00\ \Omega} = 9.00\ \text{W}.$$

Similarly,
Equation:

$$P_2 = \frac{V^2}{R_2} = \frac{(3.00\ \text{V})^2}{2.00\ \Omega} = 4.50\ \text{W}$$

and
Equation:

$$P_3 = \frac{V^2}{R_3} = \frac{(3.00\ \text{V})^2}{2.00\ \Omega} = 4.50\ \text{W}.$$

e. The total power can also be calculated in several ways. Choosing $P = IV$ and entering the total current yields
Equation:

$$P = IV = (6.00\ \text{A})(3.00\ \text{V}) = 18.00\ \text{W}.$$

Significance
Total power dissipated by the resistors is also 18.00 W:
Equation:

$$P_1 + P_2 + P_3 = 9.00\ \text{W} + 4.50\ \text{W} + 4.50\ \text{W} = 18.00\ \text{W}.$$

Notice that the total power dissipated by the resistors equals the power supplied by the source.

Note:
Exercise:

Problem:

Check Your Understanding Consider the same potential difference $(V = 3.00\ \text{V})$ applied to the same three resistors connected in series. Would the equivalent resistance of the series circuit be higher, lower, or equal to the three resistor in parallel? Would the current through the series circuit be higher, lower, or equal to the current provided by the same voltage applied to the parallel circuit? How would the power dissipated by the resistor in series compare to the power dissipated by the resistors in parallel?

Solution:

The equivalent of the series circuit would be $R_{\text{eq}} = 1.00\ \Omega + 2.00\ \Omega + 2.00\ \Omega = 5.00\ \Omega$, which is higher than the equivalent resistance of the parallel circuit $R_{\text{eq}} = 0.50\ \Omega$. The equivalent resistor of any number of resistors is always higher than the equivalent resistance of the same resistors connected in parallel. The current through for the series circuit would be $I = \frac{3.00\ \text{V}}{5.00\ \Omega} = 0.60\ \text{A}$, which is lower than the sum of the currents through each resistor in the parallel circuit, $I = 6.00\ \text{A}$. This is not surprising since the equivalent resistance of the series circuit is higher. The current through a series connection of any number of resistors will always be lower than the current into a parallel connection of the same resistors, since the equivalent resistance of the series circuit will be higher than the parallel circuit. The power dissipated by the resistors in series would be $P = 1.80\ \text{W}$, which is lower than the power dissipated in the parallel circuit $P = 18.00\ \text{W}$.

Note:
Exercise:

Problem:

Check Your Understanding How would you use a river and two waterfalls to model a parallel configuration of two resistors? How does this analogy break down?

Solution:

A river, flowing horizontally at a constant rate, splits in two and flows over two waterfalls. The water molecules are analogous to the electrons in the parallel circuits. The number of water molecules that flow in the river and falls must be equal to the number of molecules that flow over each waterfall, just like sum of the current through each resistor must be equal to the current flowing into the parallel circuit. The water molecules in the river have energy due to their motion and height. The potential energy of the water molecules in the river is constant due to their equal heights. This is analogous to the constant change in voltage across a parallel circuit. Voltage is the potential energy across each resistor.
The analogy quickly breaks down when considering the energy. In the waterfall, the potential energy is converted into kinetic energy of the water molecules. In the case of electrons flowing

through a resistor, the potential drop is converted into heat and light, not into the kinetic energy of the electrons.

Let us summarize the major features of resistors in parallel:

1. Equivalent resistance is found from
   Equation:

$$R_{\mathrm{P}}=\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\cdots+\frac{1}{R_{N-1}}+\frac{1}{R_N}\right)^{-1}=\left(\sum_{i=1}^{N}\frac{1}{R_i}\right)^{-1},$$

   and is smaller than any individual resistance in the combination.
2. The potential drop across each resistor in parallel is the same.
3. Parallel resistors do not each get the total current; they divide it. The current entering a parallel combination of resistors is equal to the sum of the current through each resistor in parallel.

In this chapter, we introduced the equivalent resistance of resistors connect in series and resistors connected in parallel. You may recall that in Capacitance, we introduced the equivalent capacitance of capacitors connected in series and parallel. Circuits often contain both capacitors and resistors. [link] summarizes the equations used for the equivalent resistance and equivalent capacitance for series and parallel connections.

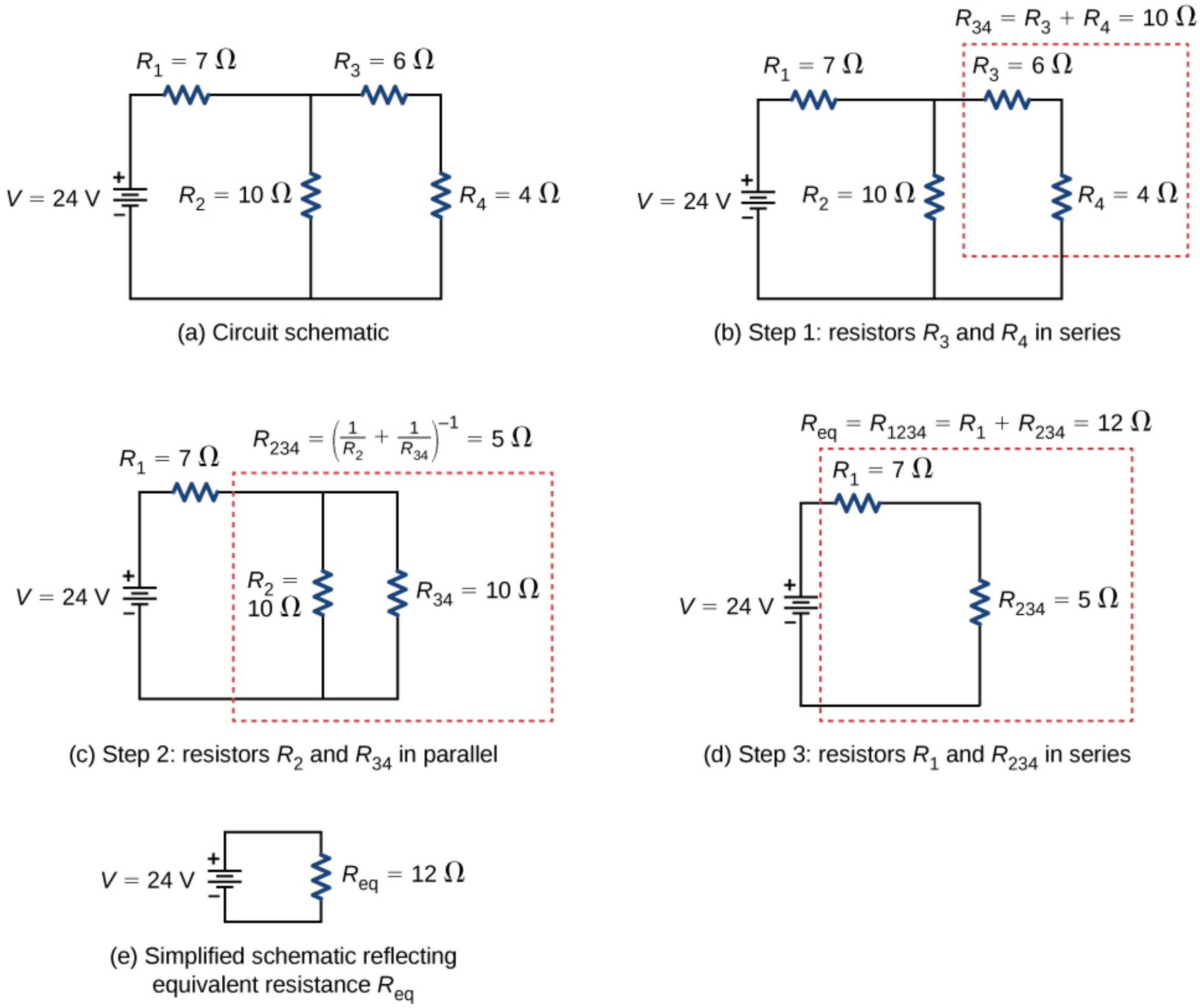
	Series combination	Parallel combination
Equivalent capacitance	$\frac{1}{C_{\mathrm{S}}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}+\cdots$	$C_{\mathrm{P}}=C_1+C_2+C_3+\cdots$
Equivalent resistance	$R_{\mathrm{S}}=R_1+R_2+R_3+\cdots=\sum_{i=1}^{N}R_i$	$\frac{1}{R_{\mathrm{P}}}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\cdots$

Summary for Equivalent Resistance and Capacitance in Series and Parallel Combinations

Combinations of Series and Parallel

More complex connections of resistors are often just combinations of series and parallel connections. Such combinations are common, especially when wire resistance is considered. In that case, wire resistance is in series with other resistances that are in parallel.

Combinations of series and parallel can be reduced to a single equivalent resistance using the technique illustrated in [link]. Various parts can be identified as either series or parallel connections, reduced to their equivalent resistances, and then further reduced until a single equivalent resistance is left. The process is more time consuming than difficult. Here, we note the equivalent resistance as R_{eq}.



(a) The original circuit of four resistors. (b) Step 1: The resistors R_3 and R_4 are in series and the equivalent resistance is $R_{34} = 10\ \Omega$. (c) Step 2: The reduced circuit shows resistors R_2 and R_{34} are in parallel, with an equivalent resistance of $R_{234} = 5\ \Omega$. (d) Step 3: The reduced circuit shows that R_1 and R_{234} are in series with an equivalent resistance of $R_{1234} = 12\ \Omega$, which is the equivalent resistance R_{eq}. (e) The reduced circuit with a voltage source of $V = 24\ \text{V}$ with an equivalent resistance of $R_{\text{eq}} = 12\ \Omega$. This results in a current of $I = 2\ \text{A}$ from the voltage source.

Notice that resistors R_3 and R_4 are in series. They can be combined into a single equivalent resistance. One method of keeping track of the process is to include the resistors as subscripts. Here the equivalent resistance of R_3 and R_4 is

Equation:

$$R_{34} = R_3 + R_4 = 6\ \Omega + 4\ \Omega = 10\ \Omega.$$

The circuit now reduces to three resistors, shown in [link](c). Redrawing, we now see that resistors R_2 and R_{34} constitute a parallel circuit. Those two resistors can be reduced to an equivalent resistance:

Equation:

$$R_{234} = \left(\frac{1}{R_2} + \frac{1}{R_{34}}\right)^{-1} = \left(\frac{1}{10\,\Omega} + \frac{1}{10\,\Omega}\right)^{-1} = 5\,\Omega.$$

This step of the process reduces the circuit to two resistors, shown in in [link](d). Here, the circuit reduces to two resistors, which in this case are in series. These two resistors can be reduced to an equivalent resistance, which is the equivalent resistance of the circuit:

Equation:

$$R_{\text{eq}} = R_{1234} = R_1 + R_{234} = 7\,\Omega + 5\,\Omega = 12\,\Omega.$$

The main goal of this circuit analysis is reached, and the circuit is now reduced to a single resistor and single voltage source.

Now we can analyze the circuit. The current provided by the voltage source is $I = \frac{V}{R_{\text{eq}}} = \frac{24\,\text{V}}{12\,\Omega} = 2\,\text{A}$. This current runs through resistor R_1 and is designated as I_1. The potential drop across R_1 can be found using Ohm's law:

Equation:

$$V_1 = I_1 R_1 = (2\,\text{A})\,(7\,\Omega) = 14\,\text{V}.$$

Looking at [link](c), this leaves $24\,\text{V} - 14\,\text{V} = 10\,\text{V}$ to be dropped across the parallel combination of R_2 and R_{34}. The current through R_2 can be found using Ohm's law:

Equation:

$$I_2 = \frac{V_2}{R_2} = \frac{10\,\text{V}}{10\,\Omega} = 1\,\text{A}.$$

The resistors R_3 and R_4 are in series so the currents I_3 and I_4 are equal to

Equation:

$$I_3 = I_4 = I - I_2 = 2\,\text{A} - 1\,\text{A} = 1\,\text{A}.$$

Using Ohm's law, we can find the potential drop across the last two resistors. The potential drops are $V_3 = I_3 R_3 = 6\,\text{V}$ and $V_4 = I_4 R_4 = 4\,\text{V}$. The final analysis is to look at the power supplied by the voltage source and the power dissipated by the resistors. The power dissipated by the resistors is

Equation:

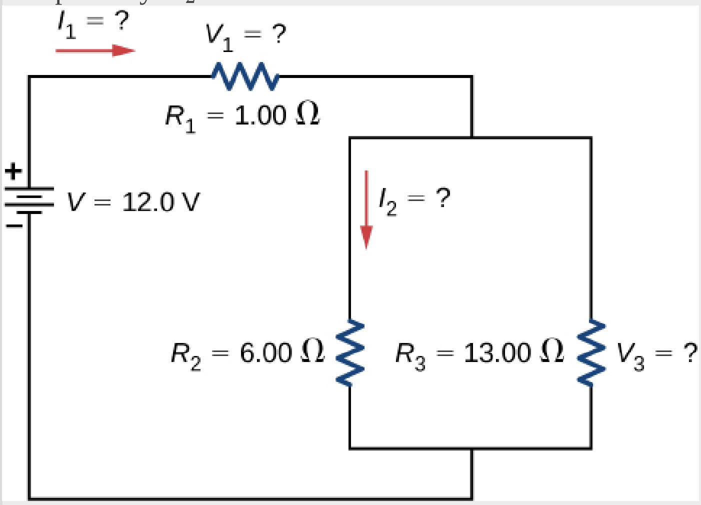
$$\begin{aligned} P_1 &= I_1^2 R_1 = (2\,\text{A})^2\,(7\,\Omega) = 28\,\text{W}, \\ P_2 &= I_2^2 R_2 = (1\,\text{A})^2\,(10\,\Omega) = 10\,\text{W}, \\ P_3 &= I_3^2 R_3 = (1\,\text{A})^2\,(6\,\Omega) = 6\,\text{W}, \\ P_4 &= I_4^2 R_4 = (1\,\text{A})^2\,(4\,\Omega) = 4\,\text{W}, \\ P_{\text{dissipated}} &= P_1 + P_2 + P_3 + P_4 = 48\,\text{W}. \end{aligned}$$

The total energy is constant in any process. Therefore, the power supplied by the voltage source is $P_s = IV = (2\text{ A})(24\text{ V}) = 48\text{ W}$. Analyzing the power supplied to the circuit and the power dissipated by the resistors is a good check for the validity of the analysis; they should be equal.

Example:

Combining Series and Parallel Circuits

[link] shows resistors wired in a combination of series and parallel. We can consider R_1 to be the resistance of wires leading to R_2 and R_3. (a) Find the equivalent resistance of the circuit. (b) What is the potential drop V_1 across resistor R_1? (c) Find the current I_2 through resistor R_2. (d) What power is dissipated by R_2?



These three resistors are connected to a voltage source so that R_2 and R_3 are in parallel with one another and that combination is in series with R_1.

Strategy

(a) To find the equivalent resistance, first find the equivalent resistance of the parallel connection of R_2 and R_3. Then use this result to find the equivalent resistance of the series connection with R_1.

(b) The current through R_1 can be found using Ohm's law and the voltage applied. The current through R_1 is equal to the current from the battery. The potential drop V_1 across the resistor R_1 (which represents the resistance in the connecting wires) can be found using Ohm's law.

(c) The current through R_2 can be found using Ohm's law $I_2 = \frac{V_2}{R_2}$. The voltage across R_2 can be found using $V_2 = V - V_1$.

(d) Using Ohm's law $(V_2 = I_2 R_2)$, the power dissipated by the resistor can also be found using $P_2 = I_2^2 R_2 = \frac{V_2^2}{R_2}$.

Solution

a. To find the equivalent resistance of the circuit, notice that the parallel connection of R_2 and R_3 is in series with R_1, so the equivalent resistance is
   Equation:

$$R_{\mathrm{eq}} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = 1.00\,\Omega + \left(\frac{1}{6.00\,\Omega} + \frac{1}{13.00\,\Omega}\right)^{-1} = 5.10\,\Omega.$$

The total resistance of this combination is intermediate between the pure series and pure parallel values ($20.0\,\Omega$ and $0.804\,\Omega$, respectively).

b. The current through R_1 is equal to the current supplied by the battery:
Equation:

$$I_1 = I = \frac{V}{R_{\mathrm{eq}}} = \frac{12.0\text{ V}}{5.10\,\Omega} = 2.35\text{ A}.$$

The voltage across R_1 is
Equation:

$$V_1 = I_1 R_1 = (2.35\text{ A})(1\,\Omega) = 2.35\text{ V}.$$

The voltage applied to R_2 and R_3 is less than the voltage supplied by the battery by an amount V_1. When wire resistance is large, it can significantly affect the operation of the devices represented by R_2 and R_3.

c. To find the current through R_2, we must first find the voltage applied to it. The voltage across the two resistors in parallel is the same:
Equation:

$$V_2 = V_3 = V - V_1 = 12.0\text{ V} - 2.35\text{ V} = 9.65\text{ V}.$$

Now we can find the current I_2 through resistance R_2 using Ohm's law:
Equation:

$$I_2 = \frac{V_2}{R_2} = \frac{9.65\text{ V}}{6.00\,\Omega} = 1.61\text{ A}.$$

The current is less than the 2.00 A that flowed through R_2 when it was connected in parallel to the battery in the previous parallel circuit example.

d. The power dissipated by R_2 is given by
Equation:

$$P_2 = I_2^2 R_2 = (1.61\text{ A})^2(6.00\,\Omega) = 15.5\text{ W}.$$

Significance
The analysis of complex circuits can often be simplified by reducing the circuit to a voltage source and an equivalent resistance. Even if the entire circuit cannot be reduced to a single voltage source and a single equivalent resistance, portions of the circuit may be reduced, greatly simplifying the analysis.

Note:
Exercise:

Problem:

Check Your Understanding Consider the electrical circuits in your home. Give at least two examples of circuits that must use a combination of series and parallel circuits to operate efficiently.

Solution:

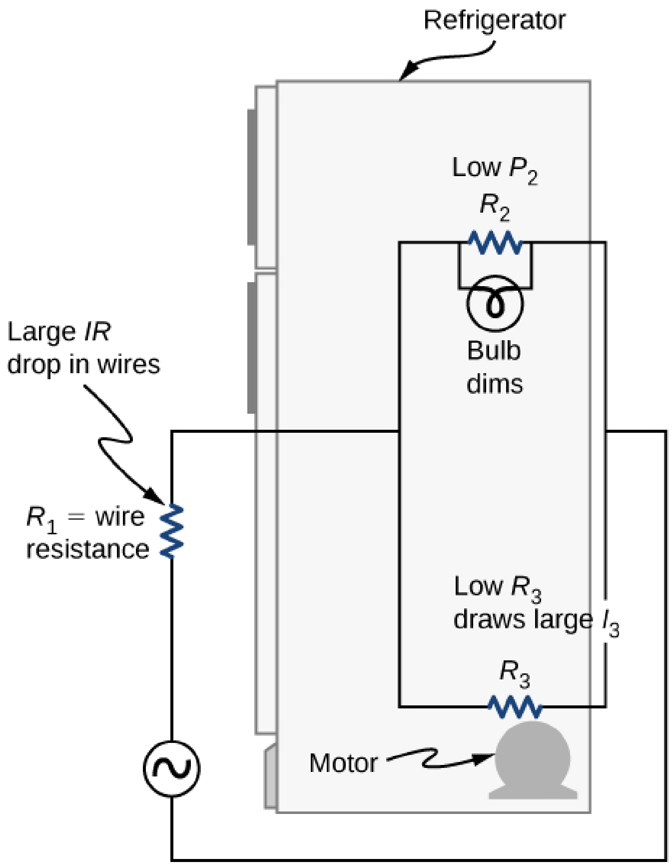
1. All the overhead lighting circuits are in parallel and connected to the main supply line, so when one bulb burns out, all the overhead lighting does not go dark. Each overhead light will have at least one switch in series with the light, so you can turn it on and off. 2. A refrigerator has a compressor and a light that goes on when the door opens. There is usually only one cord for the refrigerator to plug into the wall. The circuit containing the compressor and the circuit containing the lighting circuit are in parallel, but there is a switch in series with the light. A thermostat controls a switch that is in series with the compressor to control the temperature of the refrigerator.

Practical Implications

One implication of this last example is that resistance in wires reduces the current and power delivered to a resistor. If wire resistance is relatively large, as in a worn (or a very long) extension cord, then this loss can be significant. If a large current is drawn, the *IR* drop in the wires can also be significant and may become apparent from the heat generated in the cord.

For example, when you are rummaging in the refrigerator and the motor comes on, the refrigerator light dims momentarily. Similarly, you can see the passenger compartment light dim when you start the engine of your car (although this may be due to resistance inside the battery itself).

What is happening in these high-current situations is illustrated in [link]. The device represented by R_3 has a very low resistance, so when it is switched on, a large current flows. This increased current causes a larger *IR* drop in the wires represented by R_1, reducing the voltage across the light bulb (which is R_2), which then dims noticeably.



Why do lights dim when a large appliance is switched on? The answer is that the large current the appliance motor draws causes a significant IR drop in the wires and reduces the voltage across the light.

Note:
Problem-Solving Strategy: Series and Parallel Resistors

1. Draw a clear circuit diagram, labeling all resistors and voltage sources. This step includes a list of the known values for the problem, since they are labeled in your circuit diagram.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
3. Determine whether resistors are in series, parallel, or a combination of both series and parallel. Examine the circuit diagram to make this assessment. Resistors are in series if the same current must pass sequentially through them.
4. Use the appropriate list of major features for series or parallel connections to solve for the unknowns. There is one list for series and another for parallel.
5. Check to see whether the answers are reasonable and consistent.

Example:

Combining Series and Parallel Circuits

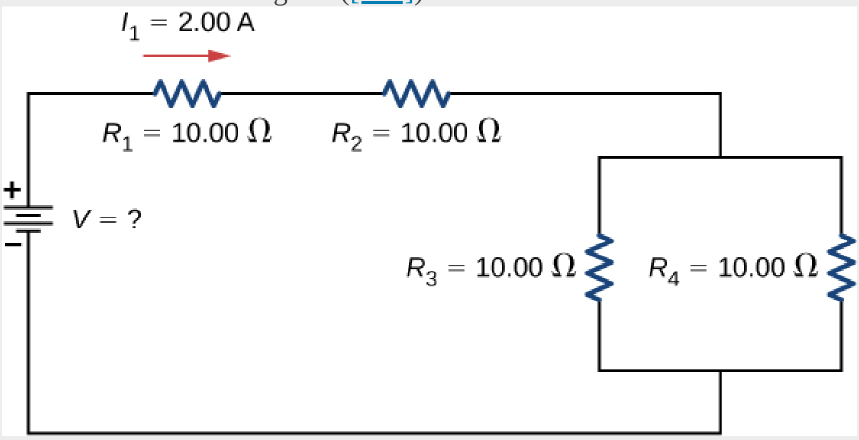
Two resistors connected in series (R_1, R_2) are connected to two resistors that are connected in parallel (R_3, R_4). The series-parallel combination is connected to a battery. Each resistor has a resistance of 10.00 Ohms. The wires connecting the resistors and battery have negligible resistance. A current of 2.00 Amps runs through resistor R_1. What is the voltage supplied by the voltage source?

Strategy

Use the steps in the preceding problem-solving strategy to find the solution for this example.

Solution

1. Draw a clear circuit diagram ([link]).



To find the unknown voltage, we must first find the equivalent resistance of the circuit.

2. The unknown is the voltage of the battery. In order to find the voltage supplied by the battery, the equivalent resistance must be found.
3. In this circuit, we already know that the resistors R_1 and R_2 are in series and the resistors R_3 and R_4 are in parallel. The equivalent resistance of the parallel configuration of the resistors R_3 and R_4 is in series with the series configuration of resistors R_1 and R_2.
4. The voltage supplied by the battery can be found by multiplying the current from the battery and the equivalent resistance of the circuit. The current from the battery is equal to the current through R_1 and is equal to 2.00 A. We need to find the equivalent resistance by reducing the circuit. To reduce the circuit, first consider the two resistors in parallel. The equivalent resistance is $R_{34} = \left(\frac{1}{10.00\ \Omega} + \frac{1}{10.00\ \Omega}\right)^{-1} = 5.00\ \Omega$. This parallel combination is in series with the other two resistors, so the equivalent resistance of the circuit is $R_{\text{eq}} = R_1 + R_2 + R_{34} = 25.00\ \Omega$. The voltage supplied by the battery is therefore $V = IR_{\text{eq}} = 2.00\ \text{A}\ (25.00\ \Omega) = 50.00\ \text{V}$.
5. One way to check the consistency of your results is to calculate the power supplied by the battery and the power dissipated by the resistors. The power supplied by the battery is
   $P_{\text{batt}} = IV = 100.00\ \text{W}$.
   Since they are in series, the current through R_2 equals the current through R_1. Since $R_3 = R_4$, the current through each will be 1.00 Amps. The power dissipated by the resistors is equal to the sum of the power dissipated by each resistor:
   Equation:

   $$P = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_4^2 R_4 = 40.00\ \text{W} + 40.00\ \text{W} + 10.00\ \text{W} + 10.00\ \text{W} = 100.00\ \text{W}.$$

Since the power dissipated by the resistors equals the power supplied by the battery, our solution seems consistent.

Significance
If a problem has a combination of series and parallel, as in this example, it can be reduced in steps by using the preceding problem-solving strategy and by considering individual groups of series or parallel connections. When finding R_{eq} for a parallel connection, the reciprocal must be taken with care. In addition, units and numerical results must be reasonable. Equivalent series resistance should be greater, whereas equivalent parallel resistance should be smaller, for example. Power should be greater for the same devices in parallel compared with series, and so on.

Summary

- The equivalent resistance of an electrical circuit with resistors wired in a series is the sum of the individual resistances: $R_{\text{s}} = R_1 + R_2 + R_3 + \cdots = \sum_{i=1}^{N} R_i$.
- Each resistor in a series circuit has the same amount of current flowing through it.
- The potential drop, or power dissipation, across each individual resistor in a series is different, and their combined total is the power source input.
- The equivalent resistance of an electrical circuit with resistors wired in parallel is less than the lowest resistance of any of the components and can be determined using the formula
  Equation:

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}.$$

- Each resistor in a parallel circuit has the same full voltage of the source applied to it.
- The current flowing through each resistor in a parallel circuit is different, depending on the resistance.
- If a more complex connection of resistors is a combination of series and parallel, it can be reduced to a single equivalent resistance by identifying its various parts as series or parallel, reducing each to its equivalent, and continuing until a single resistance is eventually reached.

Conceptual Questions

Exercise:

Problem: A voltage occurs across an open switch. What is the power dissipated by the open switch?

Exercise:

Problem:

The severity of a shock depends on the magnitude of the current through your body. Would you prefer to be in series or in parallel with a resistance, such as the heating element of a toaster, if you were shocked by it? Explain.

Solution:

It would probably be better to be in series because the current will be less than if it were in parallel.

Exercise:

Problem:

Suppose you are doing a physics lab that asks you to put a resistor into a circuit, but all the resistors supplied have a larger resistance than the requested value. How would you connect the available resistances to attempt to get the smaller value asked for?

Exercise:

Problem:

Some light bulbs have three power settings (not including zero), obtained from multiple filaments that are individually switched and wired in parallel. What is the minimum number of filaments needed for three power settings?

Solution:

two filaments, a low resistance and a high resistance, connected in parallel

Problems

Exercise:

Problem:

(a) What is the resistance of a $1.00 \times 10^{2}\text{-}\Omega$, a $2.50\text{-k}\Omega$, and a $4.00\text{-k}\Omega$ resistor connected in series? (b) In parallel?

Exercise:

Problem:

What are the largest and smallest resistances you can obtain by connecting a $36.0\text{-}\Omega$, a $50.0\text{-}\Omega$, and a $700\text{-}\Omega$ resistor together?

Solution:

largest, $786\ \Omega$, smallest, $20.32\ \Omega$

Exercise:

Problem:

An 1800-W toaster, a 1400-W speaker, and a 75-W lamp are plugged into the same outlet in a 15-A fuse and 120-V circuit. (The three devices are in parallel when plugged into the same socket.) (a) What current is drawn by each device? (b) Will this combination blow the 15-A fuse?

Exercise:

Problem:

Your car’s 30.0-W headlight and 2.40-kW starter are ordinarily connected in parallel in a 12.0-V system. What power would one headlight and the starter consume if connected in series to a 12.0-V battery? (Neglect any other resistance in the circuit and any change in resistance in the two devices.)

Solution:

29.6 W

Exercise:

Problem:

(a) Given a 48.0-V battery and $24.0\text{-}\Omega$ and $96.0\text{-}\Omega$ resistors, find the current and power for each when connected in series. (b) Repeat when the resistances are in parallel.

Exercise:

Problem:

Referring to the example combining series and parallel circuits and [link], calculate I_3 in the following two different ways: (a) from the known values of I and I_2; (b) using Ohm's law for R_3. In both parts, explicitly show how you follow the steps in the [link].

Solution:

a. 0.74 A; b. 0.742 A

Exercise:

Problem:

Referring to [link], (a) Calculate P_3 and note how it compares with P_3 found in the first two example problems in this module. (b) Find the total power supplied by the source and compare it with the sum of the powers dissipated by the resistors.

Exercise:

Problem:

Refer to [link] and the discussion of lights dimming when a heavy appliance comes on. (a) Given the voltage source is 120 V, the wire resistance is $0.800\ \Omega$, and the bulb is nominally 75.0 W, what power will the bulb dissipate if a total of 15.0 A passes through the wires when the motor comes on? Assume negligible change in bulb resistance. (b) What power is consumed by the motor?

Solution:

a. 60.8 W; b. 1.56 kW

Exercise:

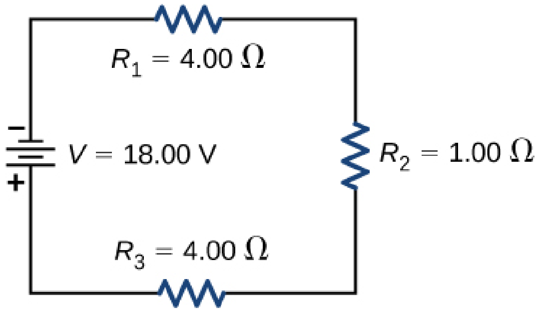
Problem:

Show that if two resistors R_1 and R_2 are combined and one is much greater than the other $(R_1 \gg R_2)$, (a) their series resistance is very nearly equal to the greater resistance R_1 and (b) their parallel resistance is very nearly equal to the smaller resistance R_2.

Exercise:

Problem:

Consider the circuit shown below. The terminal voltage of the battery is $V = 18.00\ \text{V}$. (a) Find the equivalent resistance of the circuit. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the power supplied by the battery.



Solution:

a. $R_s = 9.00\ \Omega$; b.$I_1 = I_2 = I_3 = 2.00\ \text{A}$;
c. $V_1 = 8.00\ \text{V}, V_2 = 2.00\ \text{V}, V_3 = 8.00\ \text{V}$; d. $P_1 = 16.00\ \text{W}, P_2 = 4.00\ \text{W}, P_3 = 16.00\ \text{W}$; e. $P = 36.00\ \text{W}$

Glossary

equivalent resistance
: resistance of a combination of resistors; it can be thought of as the resistance of a single resistor that can replace a combination of resistors in a series and/or parallel circuit

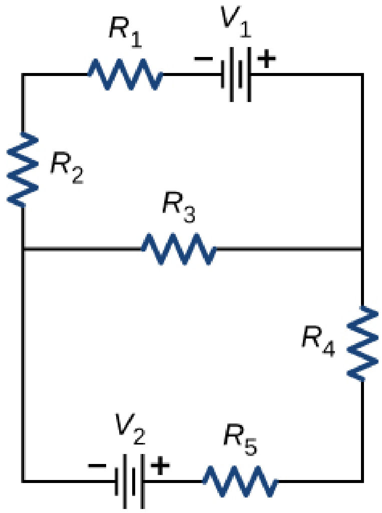
Kirchhoff's Rules

By the end of the section, you will be able to:

- State Kirchhoff’s junction rule
- State Kirchhoff’s loop rule
- Analyze complex circuits using Kirchhoff’s rules

We have just seen that some circuits may be analyzed by reducing a circuit to a single voltage source and an equivalent resistance. Many complex circuits cannot be analyzed with the series-parallel techniques developed in the preceding sections. In this section, we elaborate on the use of Kirchhoff’s rules to analyze more complex circuits. For example, the circuit in [link] is known as a multi-loop circuit, which consists of junctions. A junction, also known as a node, is a connection of three or more wires. In this circuit, the previous methods cannot be used, because not all the resistors are in clear series or parallel configurations that can be reduced. Give it a try. The resistors R_1 and R_2 are in series and can be reduced to an equivalent resistance. The same is true of resistors R_4 and R_5. But what do you do then?

Even though this circuit cannot be analyzed using the methods already learned, two circuit analysis rules can be used to analyze any circuit, simple or complex. The rules are known as **Kirchhoff’s rules**, after their inventor Gustav Kirchhoff (1824–1887).



This circuit cannot be reduced to a combination of series and parallel connections. However, we can use Kirchhoff’s rules to analyze it.

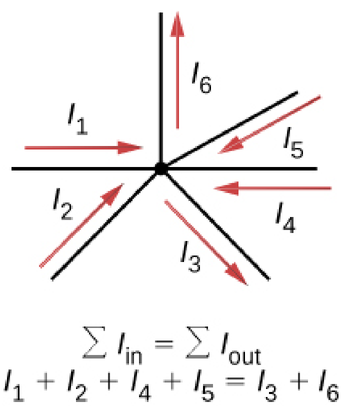
Note:
Kirchhoff's Rules

- Kirchhoff's first rule—the junction rule. The sum of all currents entering a junction must equal the sum of all currents leaving the junction:
  Equation:
  $$\sum I_{\text{in}} = \sum I_{\text{out}}.$$
- Kirchhoff's second rule—the loop rule. The algebraic sum of changes in potential around any closed circuit path (loop) must be zero:
  Equation:
  $$\sum V = 0.$$

We now provide explanations of these two rules, followed by problem-solving hints for applying them and a worked example that uses them.

Kirchhoff's First Rule

Kirchhoff's first rule (the **junction rule**) applies to the charge entering and leaving a junction ([link]). As stated earlier, a junction, or node, is a connection of three or more wires. Current is the flow of charge, and charge is conserved; thus, whatever charge flows into the junction must flow out.



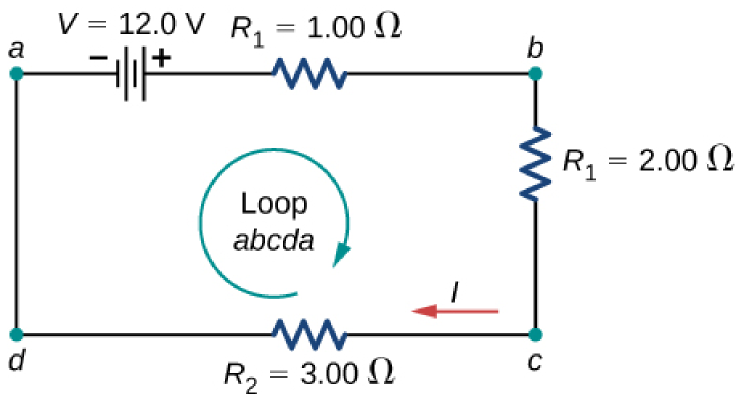
Charge must be conserved, so the sum of currents into a junction must be

equal to the sum of currents out of the junction.

Although it is an over-simplification, an analogy can be made with water pipes connected in a plumbing junction. If the wires in [link] were replaced by water pipes, and the water was assumed to be incompressible, the volume of water flowing into the junction must equal the volume of water flowing out of the junction.

Kirchhoff's Second Rule

Kirchhoff's second rule (the **loop rule**) applies to potential differences. The loop rule is stated in terms of potential *V* rather than potential energy, but the two are related since $U = qV$. In a closed loop, whatever energy is supplied by a voltage source, the energy must be transferred into other forms by the devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. Kirchhoff's loop rule states that the algebraic sum of potential differences, including voltage supplied by the voltage sources and resistive elements, in any loop must be equal to zero. For example, consider a simple loop with no junctions, as in [link].

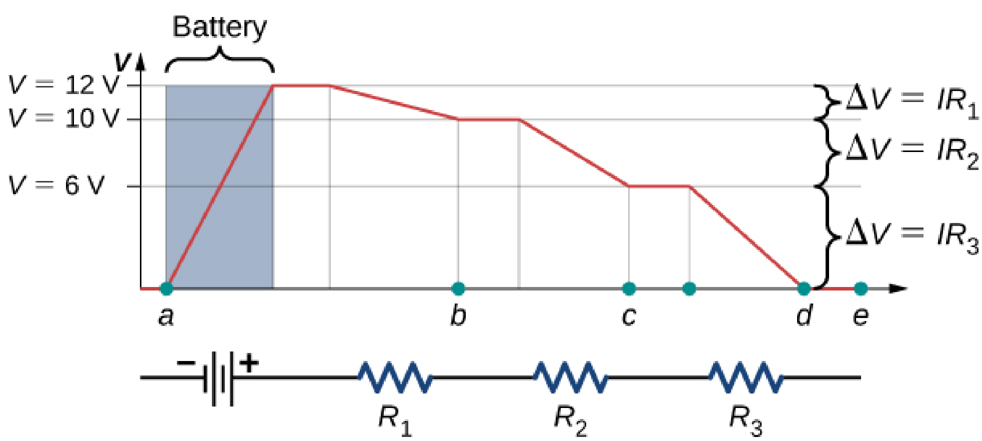


A simple loop with no junctions. Kirchhoff's loop rule states that the algebraic sum of the voltage differences is equal to zero.

The circuit consists of a voltage source and three external load resistors. The labels *a*, *b*, *c*, and *d* serve as references, and have no other significance. The usefulness of these labels will become apparent soon. The loop is designated as Loop *abcda*, and the labels help keep track of the voltage differences as we travel around the circuit. Start at point *a* and travel to point *b*.

The voltage of the voltage source is added to the equation and the potential drop of the resistor R_1 is subtracted. From point *b* to *c*, the potential drop across R_2 is subtracted. From *c* to *d*, the potential drop across R_3 is subtracted. From points *d* to *a*, nothing is done because there are no components.

[link] shows a graph of the voltage as we travel around the loop. Voltage increases as we cross the battery, whereas voltage decreases as we travel across a resistor. The potential drop, or change in the electric potential, is equal to the current through the resistor times the resistance of the resistor. Since the wires have negligible resistance, the voltage remains constant as we cross the wires connecting the components.



A voltage graph as we travel around the circuit. The voltage increases as we cross the battery and decreases as we cross each resistor. Since the resistance of the wire is quite small, we assume that the voltage remains constant as we cross the wires connecting the components.

Then Kirchhoff's loop rule states
Equation:

$$V - IR_1 - IR_2 - IR_3 = 0.$$

The loop equation can be used to find the current through the loop:
Equation:

$$I = \frac{V}{R_1 + R_2 + R_2} = \frac{12.00\ \text{V}}{1.00\ \Omega + 2.00\ \Omega + 3.00\ \Omega} = 2.00\ \text{A}.$$

This loop could have been analyzed using the previous methods, but we will demonstrate the power of Kirchhoff's method in the next section.

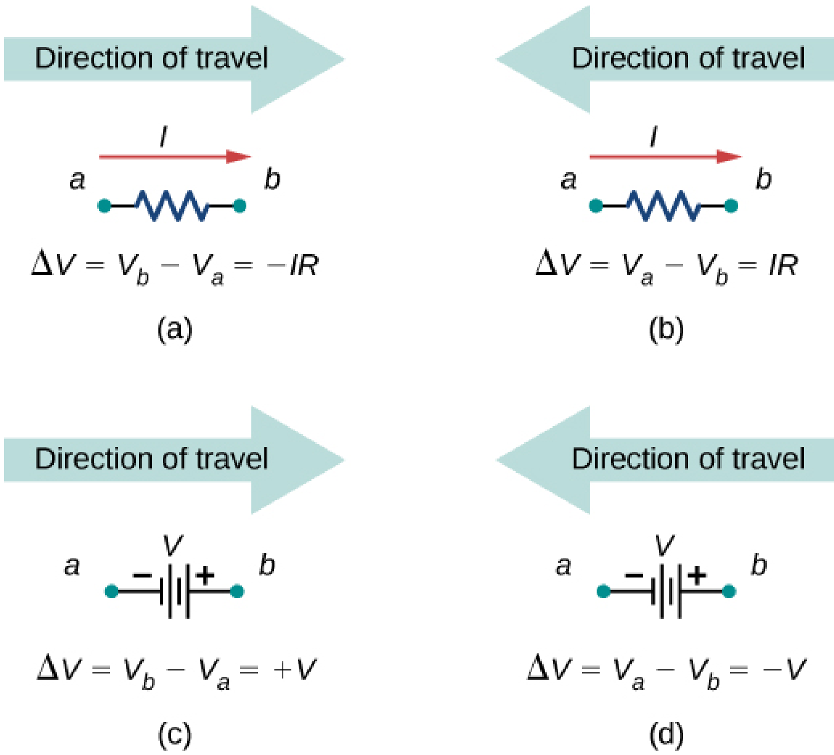
Applying Kirchhoff’s Rules

By applying Kirchhoff’s rules, we generate a set of linear equations that allow us to find the unknown values in circuits. These may be currents, voltages, or resistances. Each time a rule is applied, it produces an equation. If there are as many independent equations as unknowns, then the problem can be solved.

Using Kirchhoff’s method of analysis requires several steps, as listed in the following procedure.

Note:
Problem-Solving Strategy: Kirchhoff’s Rules

1. Label points in the circuit diagram using lowercase letters *a*, *b*, *c*, …. These labels simply help with orientation.
2. Locate the junctions in the circuit. The junctions are points where three or more wires connect. Label each junction with the currents and directions into and out of it. Make sure at least one current points into the junction and at least one current points out of the junction.
3. Choose the loops in the circuit. Every component must be contained in at least one loop, but a component may be contained in more than one loop.
4. Apply the junction rule. Again, some junctions should not be included in the analysis. You need only use enough nodes to include every current.
5. Apply the loop rule. Use the map in [link].



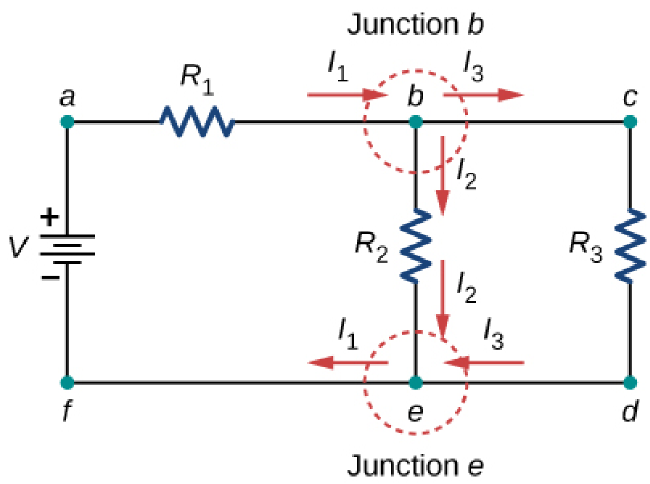
Each of these resistors and voltage sources is traversed from *a* to *b*. (a) When moving across a resistor in the same direction as the current flow, subtract the potential drop. (b) When moving across a resistor in the opposite direction as the current flow, add the potential drop. (c) When moving across a voltage source from the negative terminal to the positive terminal, add the potential drop. (d) When moving across a voltage source from the positive terminal to the negative terminal, subtract the potential drop.

Let's examine some steps in this procedure more closely. When locating the junctions in the circuit, do not be concerned about the direction of the currents. If the direction of current flow is not obvious, choosing any direction is sufficient as long as at least one current points into the junction and at least one current points out of the junction. If the arrow is in the opposite direction of the conventional current flow, the result for the current in question will be negative but the answer will still be correct.

The number of nodes depends on the circuit. Each current should be included in a node and thus included in at least one junction equation. Do not include nodes that are not linearly independent, meaning nodes that contain the same information.

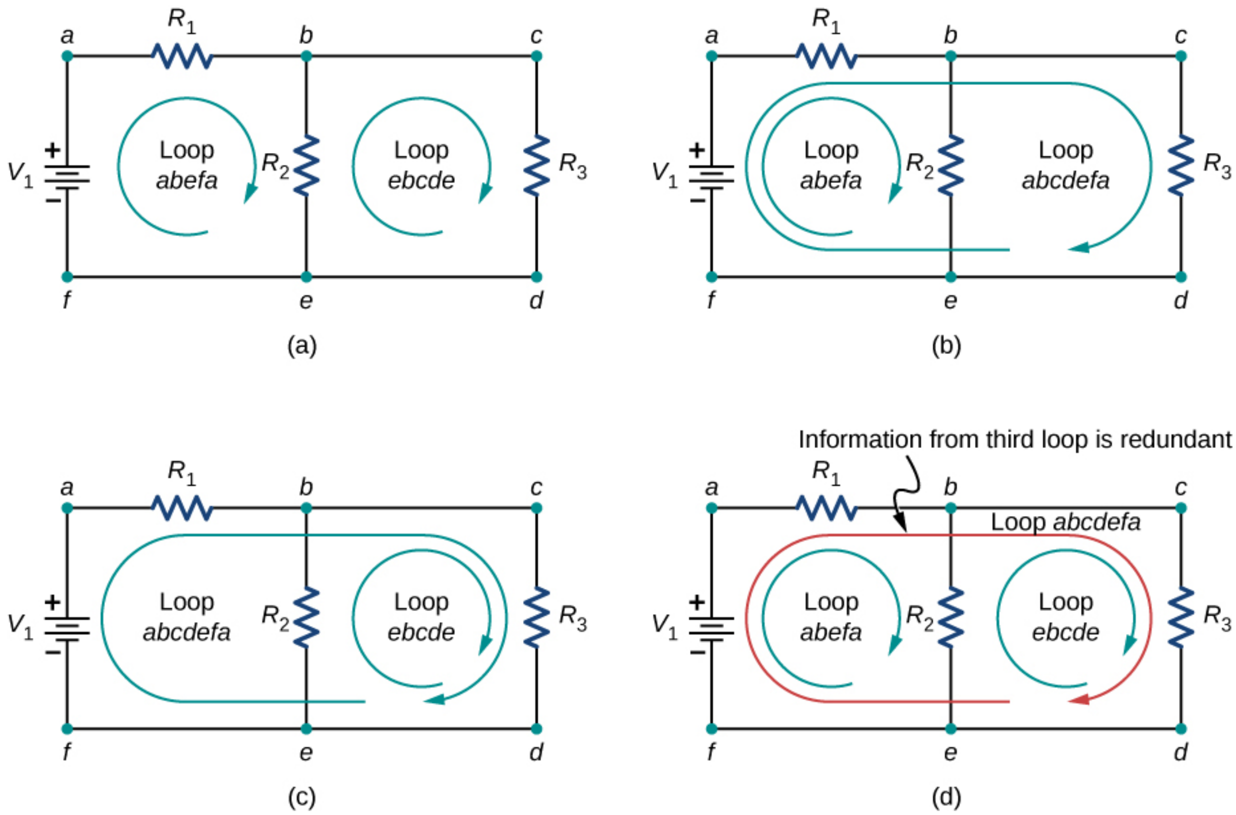
Consider [link]. There are two junctions in this circuit: Junction *b* and Junction *e*. Points *a*, *c*, *d*, and *f* are not junctions, because a junction must have three or more connections. The

equation for Junction b is $I_1 = I_2 + I_3$, and the equation for Junction e is $I_2 + I_3 = I_1$. These are equivalent equations, so it is necessary to keep only one of them.



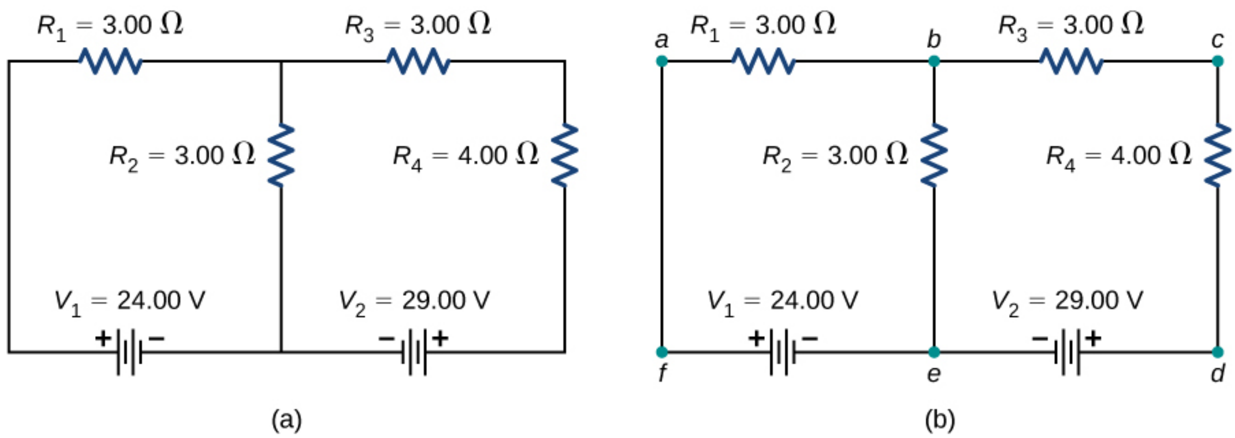
At first glance, this circuit contains two junctions, Junction b and Junction e, but only one should be considered because their junction equations are equivalent.

When choosing the loops in the circuit, you need enough loops so that each component is covered once, without repeating loops. [link] shows four choices for loops to solve a sample circuit; choices (a), (b), and (c) have a sufficient amount of loops to solve the circuit completely. Option (d) reflects more loops than necessary to solve the circuit.



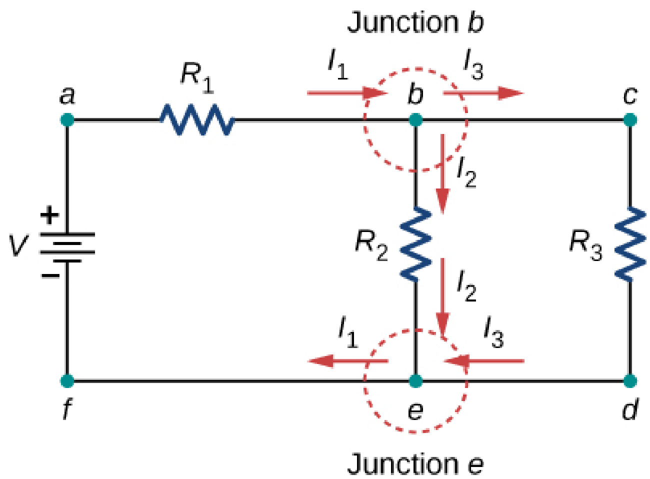
Panels (a)–(c) are sufficient for the analysis of the circuit. In each case, the two loops shown contain all the circuit elements necessary to solve the circuit completely. Panel (d) shows three loops used, which is more than necessary. Any two loops in the system will contain all information needed to solve the circuit. Adding the third loop provides redundant information.

Consider the circuit in [link](a). Let us analyze this circuit to find the current through each resistor. First, label the circuit as shown in part (b).



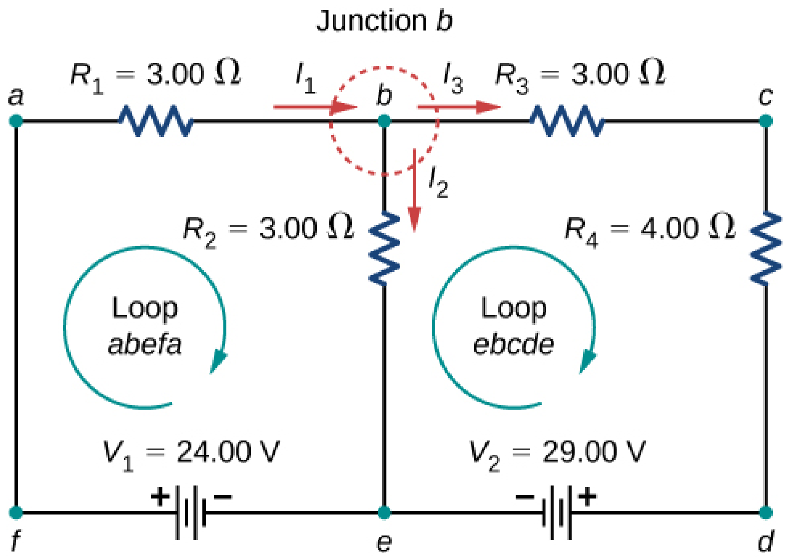
(a) A multi-loop circuit. (b) Label the circuit to help with orientation.

Next, determine the junctions. In this circuit, points *b* and *e* each have three wires connected, making them junctions. Start to apply Kirchhoff's junction rule $\left(\sum I_{\text{in}} = \sum I_{\text{out}}\right)$ by drawing arrows representing the currents and labeling each arrow, as shown in [link](b). Junction *b* shows that $I_1 = I_2 + I_3$ and Junction *e* shows that $I_2 + I_3 = I_1$. Since Junction *e* gives the same information of Junction *b*, it can be disregarded. This circuit has three unknowns, so we need three linearly independent equations to analyze it.



(a) This circuit has two junctions, labeled *b* and *e*, but only node *b* is used in the analysis. (b) Labeled arrows represent the currents into and out of the junctions.

Next we need to choose the loops. In [link], Loop *abefa* includes the voltage source V_1 and resistors R_1 and R_2. The loop starts at point *a*, then travels through points *b*, *e*, and *f*, and then back to point *a*. The second loop, Loop *ebcde*, starts at point *e* and includes resistors R_2 and R_3, and the voltage source V_2.



Choose the loops in the circuit.

Now we can apply Kirchhoff's loop rule, using the map in [link]. Starting at point *a* and moving to point *b*, the resistor R_1 is crossed in the same direction as the current flow I_1, so the potential drop I_1R_1 is subtracted. Moving from point *b* to point *e*, the resistor R_2 is crossed in the same direction as the current flow I_2 so the potential drop I_2R_2 is subtracted. Moving from point *e* to point *f*, the voltage source V_1 is crossed from the negative terminal to the positive terminal, so V_1 is added. There are no components between points *f* and *a*. The sum of the voltage differences must equal zero:

Equation:

$$\text{Loop}\ abefa:\quad -I_1R_1 - I_2R_2 + V_1 = 0 \text{ or } V_1 = I_1R_1 + I_2R_2.$$

Finally, we check loop *ebcde*. We start at point *e* and move to point *b*, crossing R_2 in the opposite direction as the current flow I_2. The potential drop I_2R_2 is added. Next, we cross R_3 and R_4 in the same direction as the current flow I_3 and subtract the potential drops I_3R_3 and I_3R_4. Note that the current is the same through resistors R_3 and R_4, because they are connected in series. Finally, the voltage source is crossed from the positive terminal to the negative terminal, and the voltage source V_2 is subtracted. The sum of these voltage differences equals zero and yields the loop equation

Equation:

$$\text{Loop } ebcde: I_2R_2 - I_3\left(R_3 + R_4\right) - V_2 = 0.$$

We now have three equations, which we can solve for the three unknowns.
Equation:

$$\begin{aligned}
&(1)\ \text{Junction } b: I_1 - I_2 - I_3 = 0. \\
&(2)\ \text{Loop } abefa: \quad I_1R_1 + I_2R_2 = V_1. \\
&(3)\ \text{Loop } ebcde: I_2R_2 - I_3\left(R_3 + R_4\right) = V_2.
\end{aligned}$$

To solve the three equations for the three unknown currents, start by eliminating current I_2. First add Eq. (1) times R_2 to Eq. (2). The result is labeled as Eq. (4):
Equation:

$$(R_1 + R_2)I_1 - R_2I_3 = V_1.$$

$$(4)\ 6\,\Omega I_1 - 3\,\Omega I_3 = 24\,\text{V}.$$

Next, subtract Eq. (3) from Eq. (2). The result is labeled as Eq. (5):
Equation:

$$I_1R_1 + I_3\left(R_3 + R_4\right) = V_1 - V_2.$$

$$(5)\ 3\,\Omega I_1 + 7\,\Omega I_3 = -5\,\text{V}.$$

We can solve Eqs. (4) and (5) for current I_1. Adding seven times Eq. (4) and three times Eq. (5) results in $51\,\Omega I_1 = 153\,\text{V}$, or $I_1 = 3.00\,\text{A}$. Using Eq. (4) results in $I_3 = -2.00\,\text{A}$. Finally, Eq. (1) yields $I_2 = I_1 - I_3 = 5.00\,\text{A}$. One way to check that the solutions are consistent is to check the power supplied by the voltage sources and the power dissipated by the resistors:
Equation:

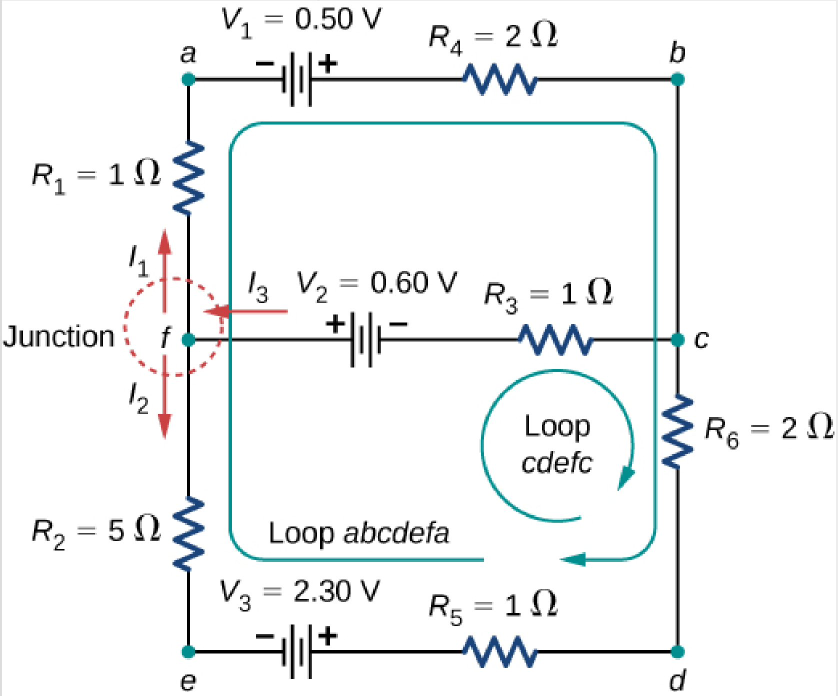
$$\begin{aligned}
&P_{\text{in}} = I_1V_1 + I_3V_2 = 130\,\text{W}, \\
&P_{\text{out}} = I_1^2R_1 + I_2^2R_2 + I_3^2R_3 + I_3^2R_4 = 130\,\text{W}.
\end{aligned}$$

Note that the solution for the current I_3 is negative. This is the correct answer, but suggests that the arrow originally drawn in the junction analysis is the direction opposite of conventional current flow. The power supplied by the second voltage source is 58 W and not −58 W.

Example:

Calculating Current by Using Kirchhoff's Rules

Find the currents flowing in the circuit in [link].



This circuit is combination of series and parallel configurations of resistors and voltage sources. This circuit cannot be analyzed using the techniques discussed in Electromotive Force but can be analyzed using Kirchhoff's rules.

Strategy

This circuit is sufficiently complex that the currents cannot be found using Ohm's law and the series-parallel techniques—it is necessary to use Kirchhoff's rules. Currents have been labeled I_1,I_2, and I_3 in the figure, and assumptions have been made about their directions. Locations on the diagram have been labeled with letters *a* through *h*. In the solution, we apply the junction and loop rules, seeking three independent equations to allow us to solve for the three unknown currents.

Solution

Applying the junction and loop rules yields the following three equations. We have three unknowns, so three equations are required.

Equation:

$$\begin{aligned}
&\text{Junction } c: \ I_1 + I_2 = I_3. \\
&\text{Loop } abcdefa: \ I_1\left(R_1 + R_4\right) - I_2\left(R_2 + R_5 + R_6\right) = V_1 - V_3. \\
&\text{Loop } cdefc: \ I_2\left(R_2 + R_5 + R_6\right) + I_3 R_3 = V_2 + V_3.
\end{aligned}$$

Simplify the equations by placing the unknowns on one side of the equations.
Equation:

$$\begin{aligned}&\text{Junction } c: \; I_1 + I_2 - I_3 = 0.\\&\text{Loop } abcdefa: \; I_1 (3\,\Omega) - I_2 (8\,\Omega) = 0.5\text{ V} - 2.30\text{ V}.\\&\text{Loop } cdefc: \; I_2 (8\,\Omega) + I_3 (1\,\Omega) = 0.6\text{ V} + 2.30\text{ V}.\end{aligned}$$

Simplify the equations. The first loop equation can be simplified by dividing both sides by 3.00. The second loop equation can be simplified by dividing both sides by 6.00.
Equation:

$$\begin{aligned}&\text{Junction } c: \; I_1 + I_2 - I_3 = 0.\\&\text{Loop } abcdefa: \; I_1 (3\,\Omega) - I_2 (8\,\Omega) = -1.8\text{ V}.\\&\text{Loop } cdefc: \; I_2 (8\,\Omega) + I_3 (1\,\Omega) = 2.9\text{ V}.\end{aligned}$$

The results are
Equation:

$$I_1 = 0.20\text{ A}, \quad I_2 = 0.30\text{ A}, \quad I_3 = 0.50\text{ A}.$$

Significance
A method to check the calculations is to compute the power dissipated by the resistors and the power supplied by the voltage sources:
Equation:

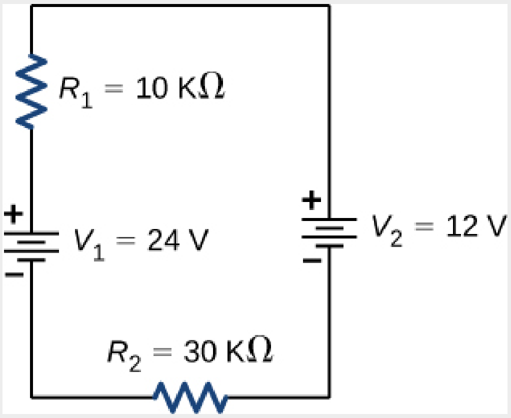
$$\begin{aligned}&P_{R_1} = I_1^2 R_1 = 0.04\text{ W}.\\&P_{R_2} = I_2^2 R_2 = 0.45\text{ W}.\\&P_{R_3} = I_3^2 R_3 = 0.25\text{ W}.\\&P_{R_4} = I_1^2 R_4 = 0.08\text{ W}.\\&P_{R_5} = I_2^2 R_5 = 0.09\text{ W}.\\&P_{R_6} = I_2^2 R_6 = 0.18\text{ W}.\\&P_{\text{dissipated}} = 1.09\text{ W}.\\&P_{\text{source}} = I_1 V_1 + I_2 V_3 + I_3 V_2 = 0.10\text{ W} + 0.69\text{ W} + 0.30\text{ W} = 1.09\text{ W}.\end{aligned}$$

The power supplied equals the power dissipated by the resistors.

Note:
Exercise:

Problem:

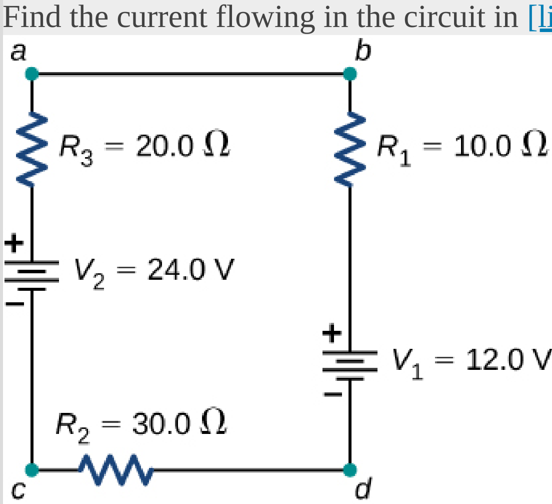
Check Your Understanding In considering the following schematic and the power supplied and consumed by a circuit, will a voltage source always provide power to the circuit, or can a voltage source consume power?



Solution:

The circuit can be analyzed using Kirchhoff's loop rule. The first voltage source supplies power: $P_{\text{in}} = IV_1 = 7.20\,\text{mW}$. The second voltage source consumes power: $P_{\text{out}} = IV_2 + I^2R_1 + I^2R_2 = 7.2\,\text{mW}$.

Example:
Calculating Current by Using Kirchhoff's Rules
Find the current flowing in the circuit in [link].



This circuit consists of three resistors and two batteries connected in series.

Note that the batteries are connected with opposite polarities.

Strategy

This circuit can be analyzed using Kirchhoff's rules. There is only one loop and no nodes. Choose the direction of current flow. For this example, we will use the clockwise direction from point *a* to point *b*. Consider Loop *abcda* and use [link] to write the loop equation. Note that according to [link], battery V_1 will be added and battery V_2 will be subtracted.

Solution

Applying the junction rule yields the following three equations. We have one unknown, so one equation is required:

Equation:

$$\text{Loop } abcda: \; -IR_1 - V_1 - IR_2 + V_2 - IR_3 = 0.$$

Simplify the equations by placing the unknowns on one side of the equations. Use the values given in the figure.

Equation:

$$I\left(R_1 + R_2 + R_3\right) = V_2 - V_1.$$
$$I = \frac{V_2 - V_1}{R_1 + R_2 + R_3} = \frac{24\ \text{V} - 12\ \text{V}}{10.0\ \Omega + 30.0\ \Omega + 10.0\ \Omega} = 0.20\ \text{A}.$$

Significance

The power dissipated or consumed by the circuit equals the power supplied to the circuit, but notice that the current in the battery V_1 is flowing through the battery from the positive terminal to the negative terminal and consumes power.

Equation:

$$\begin{aligned}
P_{R_1} &= I^2 R_1 = 0.40\ \text{W} \\
P_{R_2} &= I^2 R_2 = 1.20\ \text{W} \\
P_{R_3} &= I^2 R_3 = 0.80\ \text{W} \\
P_{V_1} &= IV_1 = 2.40\ \text{W} \\
P_{\text{dissipated}} &= 4.80\ \text{W} \\
P_{\text{source}} &= IV_2 = 4.80\ \text{W}
\end{aligned}$$

The power supplied equals the power dissipated by the resistors and consumed by the battery V_1.

Note:

Exercise:

Problem:

Check Your Understanding When using Kirchhoff's laws, you need to decide which loops to use and the direction of current flow through each loop. In analyzing the circuit in [link], the direction of current flow was chosen to be clockwise, from point *a* to point *b*. How would the results change if the direction of the current was chosen to be counterclockwise, from point *b* to point *a*?

Solution:

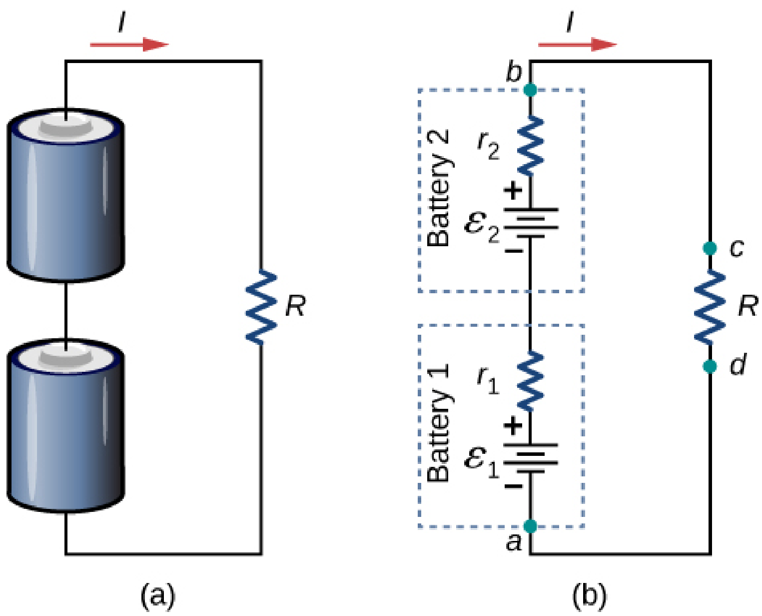
The current calculated would be equal to $I=-0.20\text{ A}$ instead of $I=0.20\text{ A}$. The sum of the power dissipated and the power consumed would still equal the power supplied.

Multiple Voltage Sources

Many devices require more than one battery. Multiple voltage sources, such as batteries, can be connected in series configurations, parallel configurations, or a combination of the two.

In series, the positive terminal of one battery is connected to the negative terminal of another battery. Any number of voltage sources, including batteries, can be connected in series. Two batteries connected in series are shown in [link]. Using Kirchhoff's loop rule for the circuit in part (b) gives the result
Equation:

$$\begin{aligned} &\varepsilon_1 - Ir_1 + \varepsilon_2 - Ir_2 - IR = 0, \\ &\left[(\varepsilon_1 + \varepsilon_2) - I\left(r_1 + r_2\right)\right] - IR = 0. \end{aligned}$$



(a) Two batteries connected in series with a load resistor. (b) The circuit diagram of the two batteries and the load resistor, with each battery modeled as an idealized emf source and an internal resistance.

When voltage sources are in series, their internal resistances can be added together and their emfs can be added together to get the total values. Series connections of voltage sources are common—for example, in flashlights, toys, and other appliances. Usually, the cells are in series in order to produce a larger total emf. In [link], the terminal voltage is

Equation:

$$V_{\text{terminal}} = (\varepsilon_1 - Ir_1) + (\varepsilon_2 - Ir_2) = [(\varepsilon_1 + \varepsilon_2) - I(r_1 + r_2)] = (\varepsilon_1 + \varepsilon_2) + Ir_{\text{eq}}.$$

Note that the same current I is found in each battery because they are connected in series. The disadvantage of series connections of cells is that their internal resistances are additive.

Batteries are connected in series to increase the voltage supplied to the circuit. For instance, an LED flashlight may have two AAA cell batteries, each with a terminal voltage of 1.5 V, to provide 3.0 V to the flashlight.

Any number of batteries can be connected in series. For N batteries in series, the terminal voltage is equal to

Note:
Equation:

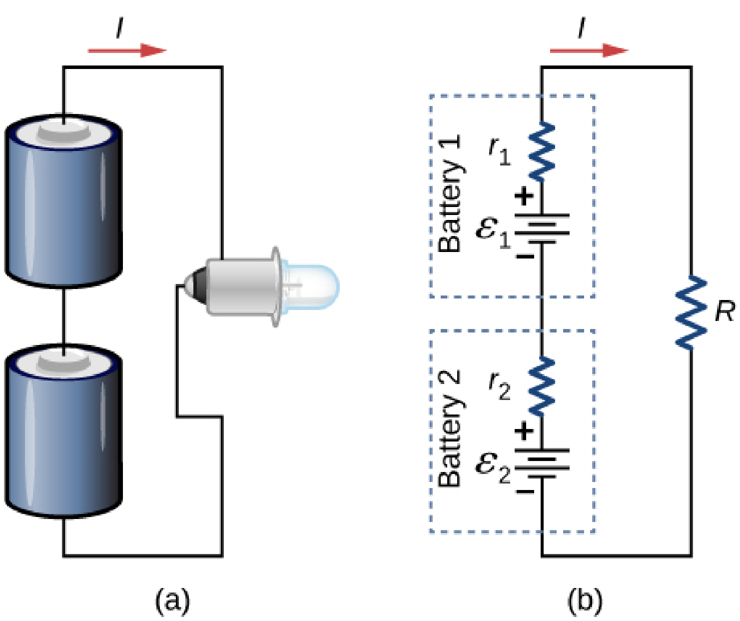
$$V_{\text{terminal}} = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{N-1} + \varepsilon_N) - I(r_1 + r_2 + \cdots + r_{N-1} + r_N) = \sum_{i=1}^{N} \varepsilon_i - Ir_{\text{eq}}$$

where the equivalent resistance is $r_{\text{eq}} = \sum_{i=1}^{N} r_i$.

When a load is placed across voltage sources in series, as in [link], we can find the current:
Equation:

$$\begin{aligned}
&(\varepsilon_1 - Ir_1) + (\varepsilon_2 - Ir_2) = IR, \\
&Ir_1 + Ir_2 + IR = \varepsilon_1 + \varepsilon_2, \\
&I = \frac{\varepsilon_1 + \varepsilon_2}{r_1 + r_2 + R}.
\end{aligned}$$

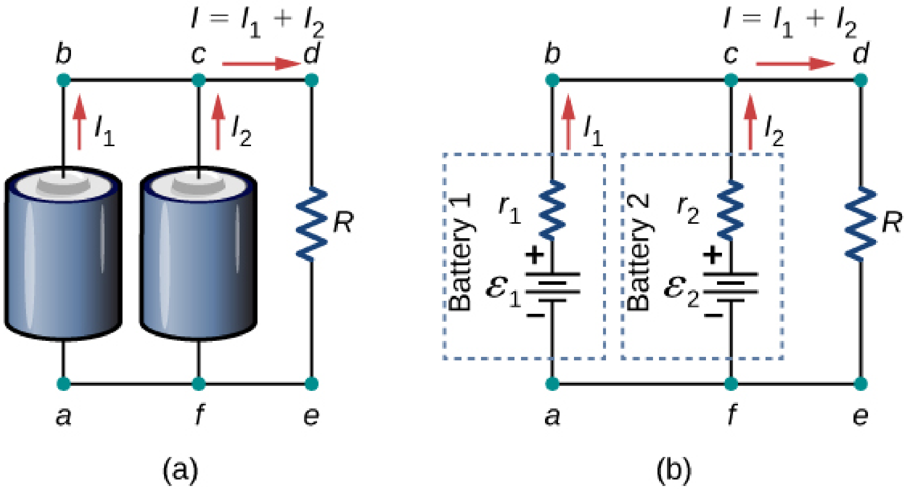
As expected, the internal resistances increase the equivalent resistance.



Two batteries connect in series to an LED bulb, as found in a flashlight.

Voltage sources, such as batteries, can also be connected in parallel. [link] shows two batteries with identical emfs in parallel and connected to a load resistance. When the batteries are

connect in parallel, the positive terminals are connected together and the negative terminals are connected together, and the load resistance is connected to the positive and negative terminals. Normally, voltage sources in parallel have identical emfs. In this simple case, since the voltage sources are in parallel, the total emf is the same as the individual emfs of each battery.



(a) Two batteries connect in parallel to a load resistor. (b) The circuit diagram shows the shows battery as an emf source and an internal resistor. The two emf sources have identical emfs (each labeled by ε) connected in parallel that produce the same emf.

Consider the Kirchhoff analysis of the circuit in [link](b). There are two loops and a node at point *b* and $\varepsilon = \varepsilon_1 = \varepsilon_2$.

Node *b*: $I_1 + I_2 - I = 0$.

Loop *abcfa*: $$\begin{aligned} \varepsilon - I_1 r_1 + I_2 r_2 - \varepsilon &= 0, \\ I_1 r_1 &= I_2 r_2. \end{aligned}$$

Loop *fcdef*: $$\begin{aligned} \varepsilon_2 - I_2 r_2 - IR &= 0, \\ \varepsilon - I_2 r_2 - IR &= 0. \end{aligned}$$

Solving for the current through the load resistor results in $I = \frac{\varepsilon}{r_{\text{eq}} + R}$, where $r_{\text{eq}} = \left(\frac{1}{r_1} + \frac{1}{r_2} \right)^{-1}$. The terminal voltage is equal to the potential drop across the load resistor $IR = \left(\frac{\varepsilon}{r_{\text{eq}} + R} \right)$. The parallel connection reduces the internal resistance and thus can produce a larger current.

Any number of batteries can be connected in parallel. For N batteries in parallel, the terminal voltage is equal to

> **Note:**
> **Equation:**
>
> $$V_{\text{terminal}} = \varepsilon - I\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_{N-1}} + \frac{1}{r_N}\right)^{-1} = \varepsilon - Ir_{\text{eq}}$$

where the equivalent resistance is $r_{\text{eq}} = \left(\sum_{i=1}^{N} \frac{1}{r_i}\right)^{-1}$.

As an example, some diesel trucks use two 12-V batteries in parallel; they produce a total emf of 12 V but can deliver the larger current needed to start a diesel engine.

In summary, the terminal voltage of batteries in series is equal to the sum of the individual emfs minus the sum of the internal resistances times the current. When batteries are connected in parallel, they usually have equal emfs and the terminal voltage is equal to the emf minus the equivalent internal resistance times the current, where the equivalent internal resistance is smaller than the individual internal resistances. Batteries are connected in series to increase the terminal voltage to the load. Batteries are connected in parallel to increase the current to the load.

Solar Cell Arrays

Another example dealing with multiple voltage sources is that of combinations of solar cells—wired in both series and parallel combinations to yield a desired voltage and current. Photovoltaic generation, which is the conversion of sunlight directly into electricity, is based upon the photoelectric effect. The photoelectric effect is beyond the scope of this chapter and is covered in Photons and Matter Waves, but in general, photons hitting the surface of a solar cell create an electric current in the cell.

Most solar cells are made from pure silicon. Most single cells have a voltage output of about 0.5 V, while the current output is a function of the amount of sunlight falling on the cell (the incident solar radiation known as the insolation). Under bright noon sunlight, a current per unit area of about 100 mA/cm^2 of cell surface area is produced by typical single-crystal cells.

Individual solar cells are connected electrically in modules to meet electrical energy needs. They can be wired together in series or in parallel—connected like the batteries discussed earlier. A solar-cell array or module usually consists of between 36 and 72 cells, with a power output of 50 W to 140 W.

Solar cells, like batteries, provide a direct current (dc) voltage. Current from a dc voltage source is unidirectional. Most household appliances need an alternating current (ac) voltage.

Summary

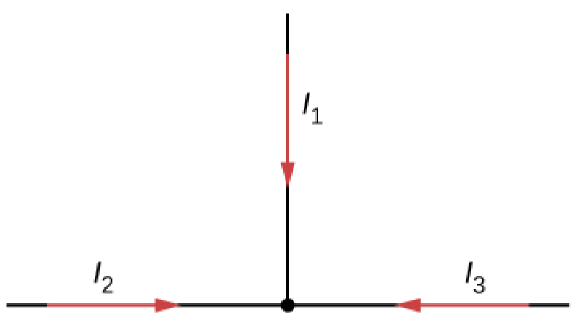
- Kirchhoff's rules can be used to analyze any circuit, simple or complex. The simpler series and parallel connection rules are special cases of Kirchhoff's rules.
- Kirchhoff's first rule, also known as the junction rule, applies to the charge to a junction. Current is the flow of charge; thus, whatever charge flows into the junction must flow out.
- Kirchhoff's second rule, also known as the loop rule, states that the voltage drop around a loop is zero.
- When calculating potential and current using Kirchhoff's rules, a set of conventions must be followed for determining the correct signs of various terms.
- When multiple voltage sources are in series, their internal resistances add together and their emfs add together to get the total values.
- When multiple voltage sources are in parallel, their internal resistances combine to an equivalent resistance that is less than the individual resistance and provides a higher current than a single cell.
- Solar cells can be wired in series or parallel to provide increased voltage or current, respectively.

Conceptual Questions

Exercise:

Problem:

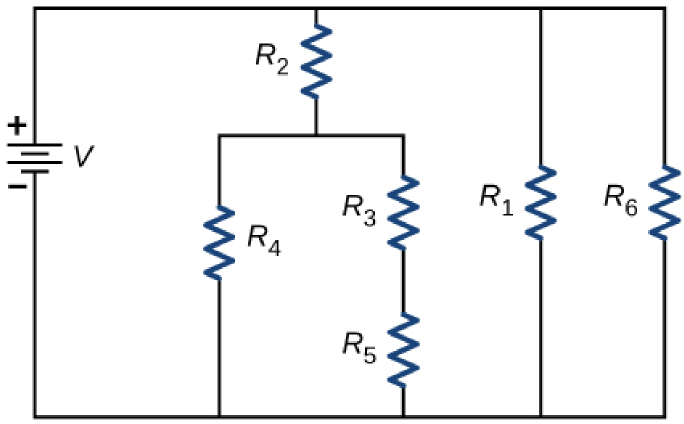
Can all of the currents going into the junction shown below be positive? Explain.



Exercise:

Problem:

Consider the circuit shown below. Does the analysis of the circuit require Kirchhoff's method, or can it be redrawn to simplify the circuit? If it is a circuit of series and parallel connections, what is the equivalent resistance?



Solution:

It can be redrawn.

$$R_{\text{eq}} = \left[\frac{1}{R_6} + \frac{1}{R_1} + \frac{1}{R_2 + \left(\frac{1}{R_4} + \frac{1}{R_3 + R_5} \right)^{-1}} \right]^{-1}$$

Exercise:

Problem:

Do batteries in a circuit always supply power to a circuit, or can they absorb power in a circuit? Give an example.

Exercise:

Problem:

What are the advantages and disadvantages of connecting batteries in series? In parallel?

Solution:

In series the voltages add, but so do the internal resistances, because the internal resistances are in series. In parallel, the terminal voltage is the same, but the equivalent internal resistance is smaller than the smallest individual internal resistance and a higher current can be provided.

Exercise:

Problem:

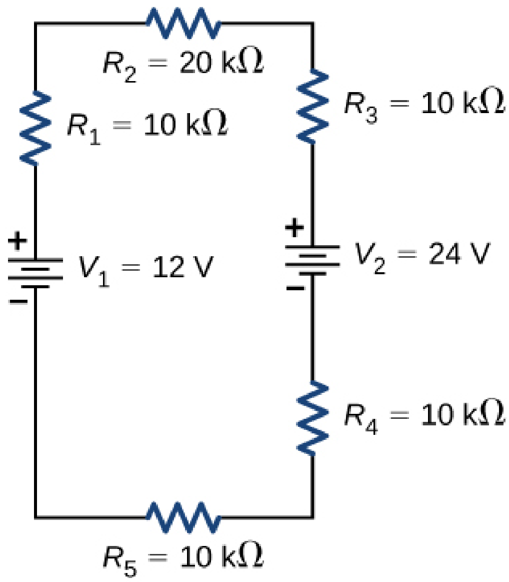
Semi-tractor trucks use four large 12-V batteries. The starter system requires 24 V, while normal operation of the truck's other electrical components utilizes 12 V. How could the four batteries be connected to produce 24 V? To produce 12 V? Why is 24 V better than 12 V for starting the truck's engine (a very heavy load)?

Problems

Exercise:

Problem:

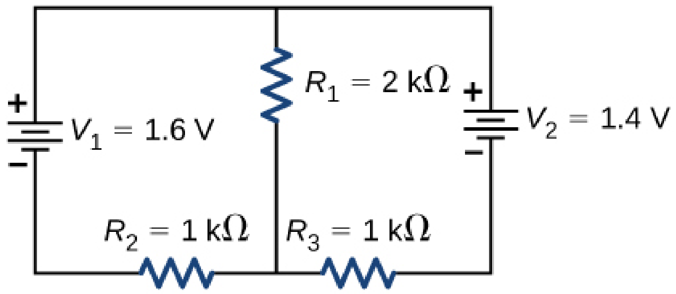
Consider the circuit shown below. (a) Find the voltage across each resistor. (b)What is the power supplied to the circuit and the power dissipated or consumed by the circuit?



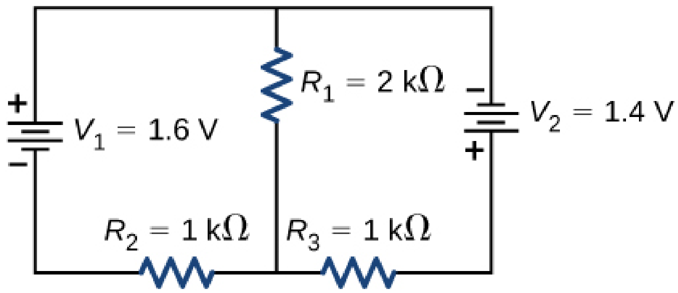
Exercise:

Problem:

Consider the circuits shown below. (a) What is the current through each resistor in part (a)? (b) What is the current through each resistor in part (b)? (c) What is the power dissipated or consumed by each circuit? (d) What is the power supplied to each circuit?



(a)



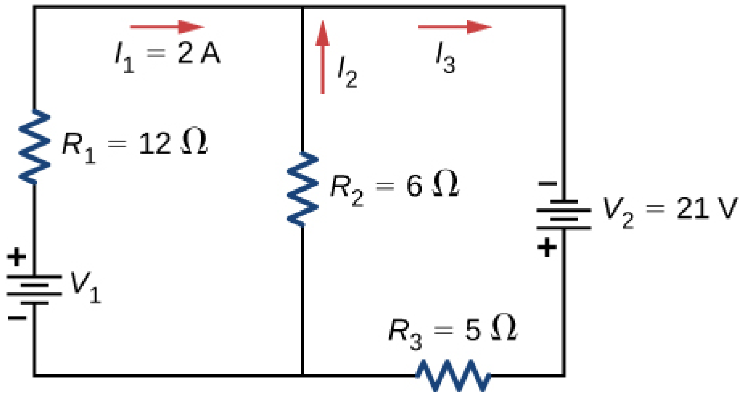
(b)

Solution:

a. $I_1 = 0.6\,\mathrm{mA}, \quad I_2 = 0.4\,\mathrm{mA}, \quad I_3 = 0.2\,\mathrm{mA};$
b. $I_1 = 0.04\,\mathrm{mA}, \quad I_2 = 1.52\,\mathrm{mA}, \quad I_3 = -1.48\,\mathrm{mA};$ c.
$P_{\text{out}} = 0.92\,\mathrm{mW}, \quad P_{\text{out}} = 4.50\,\mathrm{mW};$
d. $P_{\text{in}} = 0.92\,\mathrm{mW}, \quad P_{\text{in}} = 4.50\,\mathrm{mW}$

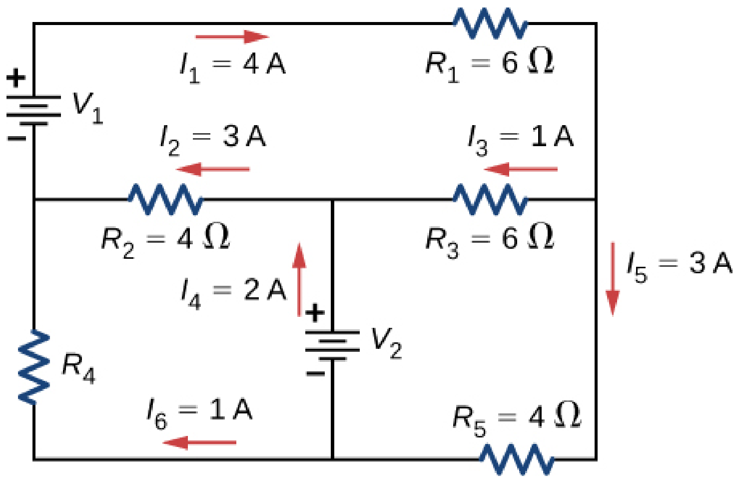
Exercise:

Problem: Consider the circuit shown below. Find $V_1, I_2,$ and I_3.



Exercise:

Problem: Consider the circuit shown below. Find $V_1, V_2,$ and R_4.

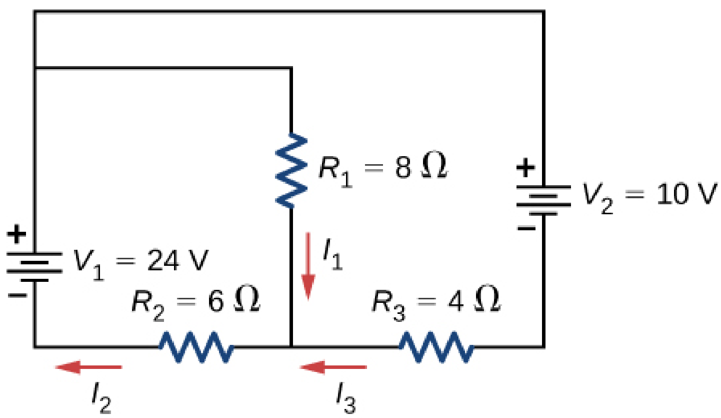


Solution:

$V_1 = 42\ \text{V}, V_2 = 6\ \text{V}, R_4 = 18\ \Omega$

Exercise:

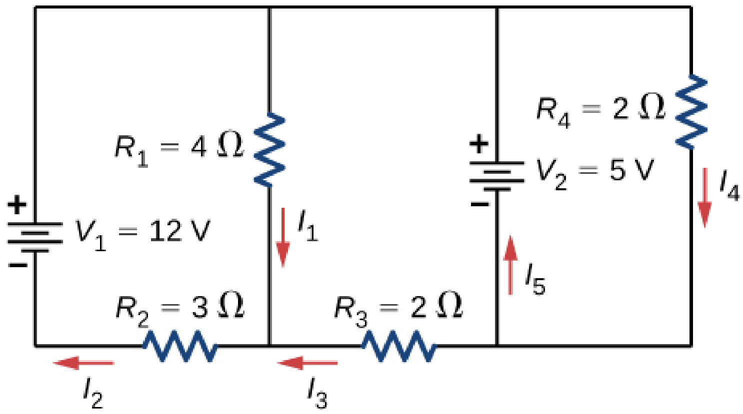
Problem: Consider the circuit shown below. Find $I_1, I_2,$ and I_3.



Exercise:

Problem:

Consider the circuit shown below. (a) Find $I_1, I_2, I_3, I_4,$ and I_5. (b) Find the power supplied by the voltage sources. (c) Find the power dissipated by the resistors.



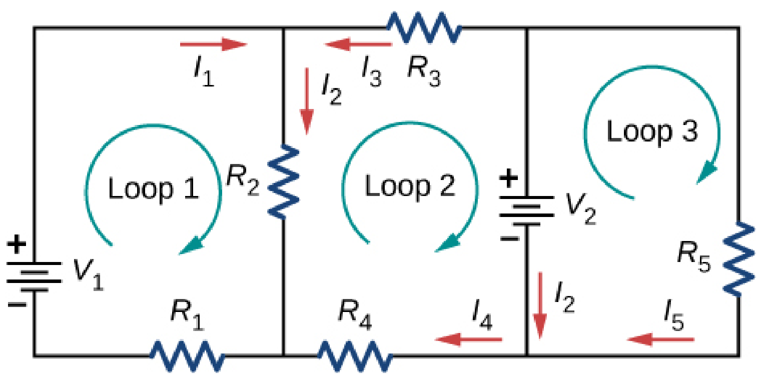
Solution:

a. $I_1 = 1.5\ \text{A}, I_2 = 2\ \text{A}, I_3 = 0.5\ \text{A},\ I_4 = 2.5\ \text{A}, I_5 = 2\ \text{A}$; b.
$P_{\text{in}} = I_2V_1 + I_5V_5 = 34\ \text{W}$;
c. $P_{\text{out}} = I_1^2R_1 + I_2^2R_2 + I_3^2R_3 + I_4^2R_4 = 34\ \text{W}$

Exercise:

Problem:

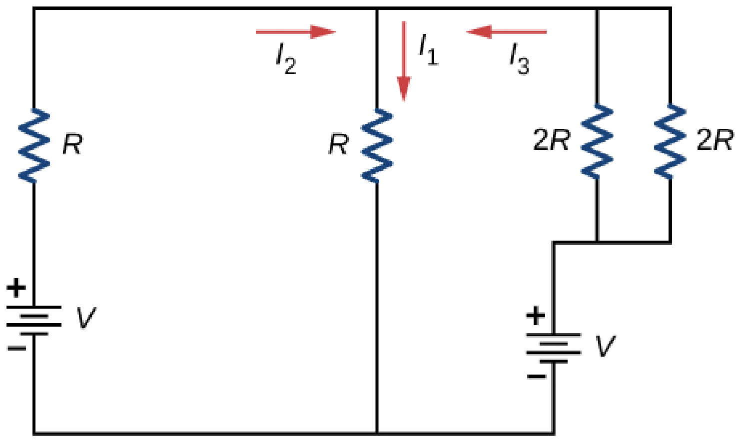
Consider the circuit shown below. Write the three loop equations for the loops shown.



Exercise:

Problem:

Consider the circuit shown below. Write equations for the three currents in terms of R and V.



Solution:

$I_1 = \frac{2}{3}\frac{V}{R}, I_2 = \frac{V}{3R}, I_3 = \frac{V}{3R}$

Exercise:

Problem:

Consider the circuit shown in the preceding problem. Write equations for the power supplied by the voltage sources and the power dissipated by the resistors in terms of R and V.

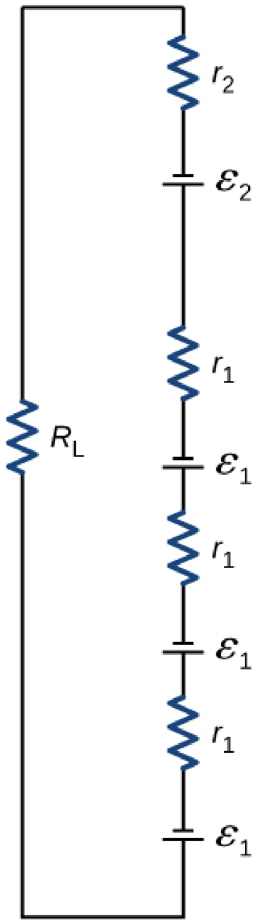
Exercise:

Problem:

A child's electronic toy is supplied by three 1.58-V alkaline cells having internal resistances of $0.0200\ \Omega$ in series with a 1.53-V carbon-zinc dry cell having a $0.100\text{-}\Omega$ internal resistance. The load resistance is $10.0\ \Omega$. (a) Draw a circuit diagram of the toy and its batteries. (b) What current flows? (c) How much power is supplied to the load? (d) What is the internal resistance of the dry cell if it goes bad, resulting in only 0.500 W being supplied to the load?

Solution:

a.

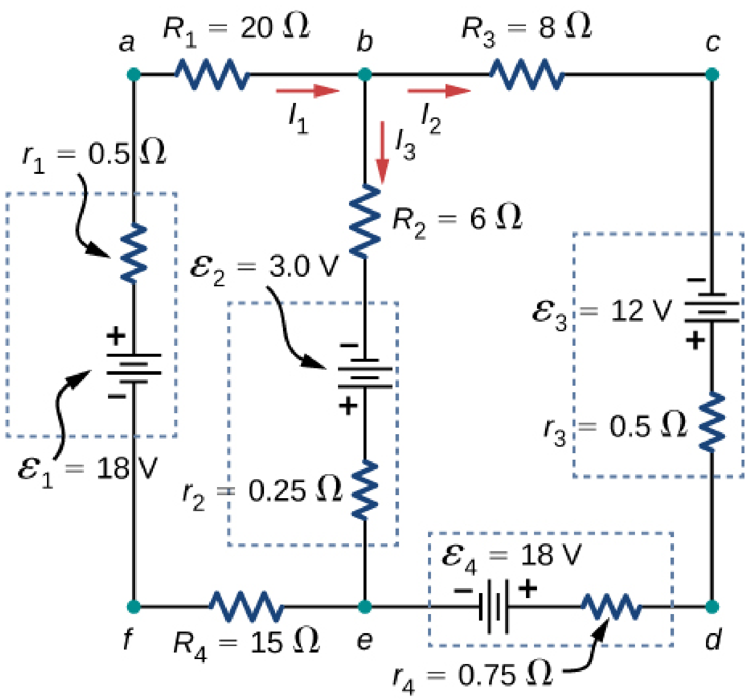


;
b. 0.617 A; c. 3.81 W; d. 18.0 Ω

Exercise:

Problem:

Apply the junction rule to Junction *b* shown below. Is any new information gained by applying the junction rule at *e*?



Exercise:

Problem: Apply the loop rule to Loop *afedcba* in the preceding problem.

Solution:

$$I_1 r_1 - \varepsilon_1 + I_1 R_4 + \varepsilon_4 + I_2 r_4 + I_4 r_3 - \varepsilon_3 + I_2 R_3 + I_1 R_1 = 0$$

Glossary

junction rule
: sum of all currents entering a junction must equal the sum of all currents leaving the junction

Kirchhoff's rules
: set of two rules governing current and changes in potential in an electric circuit

loop rule
: algebraic sum of changes in potential around any closed circuit path (loop) must be zero

Electrical Measuring Instruments
By the end of the section, you will be able to:

- Describe how to connect a voltmeter in a circuit to measure voltage
- Describe how to connect an ammeter in a circuit to measure current
- Describe the use of an ohmmeter

Ohm's law and Kirchhoff's method are useful to analyze and design electrical circuits, providing you with the voltages across, the current through, and the resistance of the components that compose the circuit. To measure these parameters require instruments, and these instruments are described in this section.

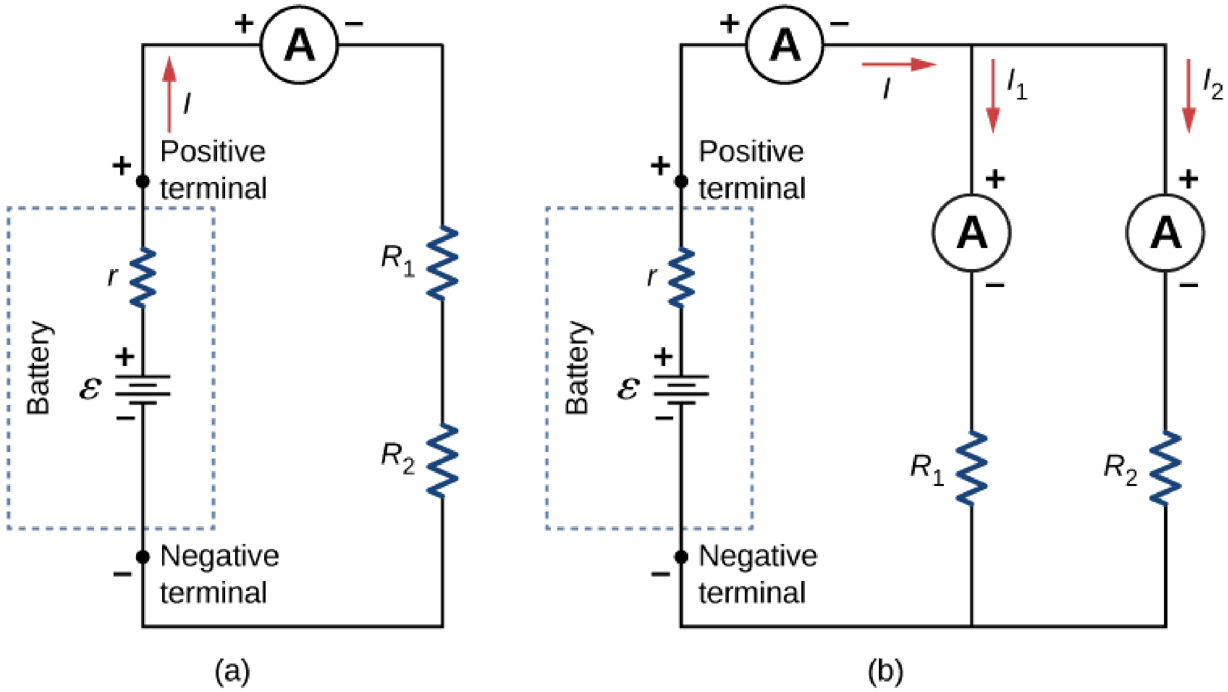
DC Voltmeters and Ammeters

Whereas **voltmeters** measure voltage, **ammeters** measure current. Some of the meters in automobile dashboards, digital cameras, cell phones, and tuner-amplifiers are actually voltmeters or ammeters ([link]). The internal construction of the simplest of these meters and how they are connected to the system they monitor give further insight into applications of series and parallel connections.

The fuel and temperature gauges (far right and far left, respectively) in this 1996 Volkswagen are voltmeters that register the voltage output of “sender” units. These units are proportional to the amount of gasoline in the tank and to the engine temperature. (credit: Christian Giersing)

Measuring Current with an Ammeter

To measure the current through a device or component, the ammeter is placed in series with the device or component. A series connection is used because objects in series have the same current passing through them. (See [link], where the ammeter is represented by the symbol A.)

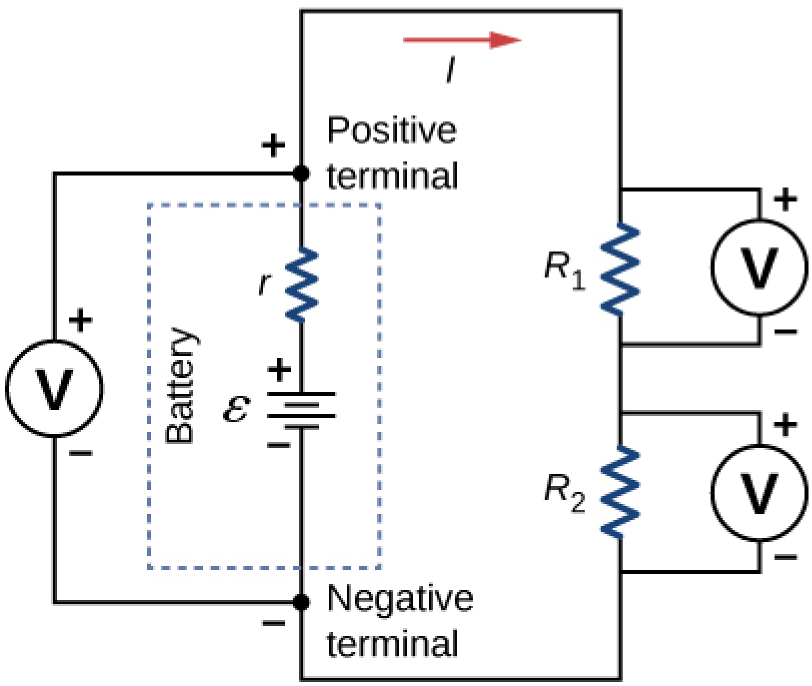


(a) When an ammeter is used to measure the current through two resistors connected in series to a battery, a single ammeter is placed in series with the two resistors because the current is the same through the two resistors in series. (b) When two resistors are connected in parallel with a battery, three meters, or three separate ammeter readings, are necessary to measure the current from the battery and through each resistor. The ammeter is connected in series with the component in question.

Ammeters need to have a very low resistance, a fraction of a milliohm. If the resistance is not negligible, placing the ammeter in the circuit would change the equivalent resistance of the circuit and modify the current that is being measured. Since the current in the circuit travels through the meter, ammeters normally contain a fuse to protect the meter from damage from currents which are too high.

Measuring Voltage with a Voltmeter

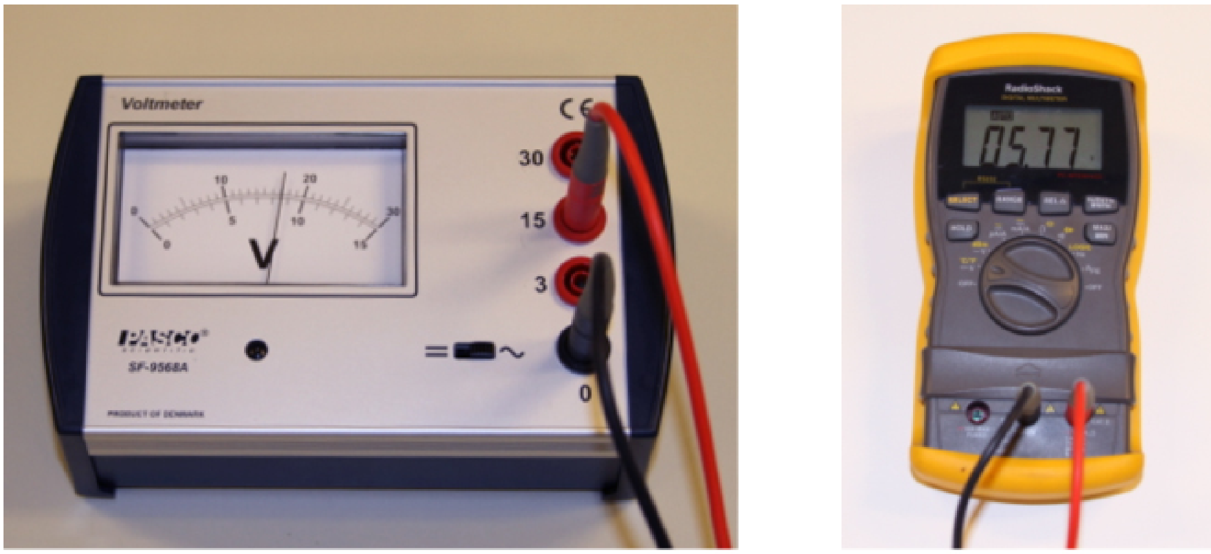
A voltmeter is connected in parallel with whatever device it is measuring. A parallel connection is used because objects in parallel experience the same potential difference. (See [link], where the voltmeter is represented by the symbol V.)



To measure potential differences in this series circuit, the voltmeter (V) is placed in parallel with the voltage source or either of the resistors. Note that terminal voltage is measured between the positive terminal and the negative terminal of the battery or voltage source. It is not possible to connect a voltmeter directly across the emf without including the internal resistance *r* of the battery.

Since voltmeters are connected in parallel, the voltmeter must have a very large resistance. Digital voltmeters convert the analog voltage into a digital

value to display on a digital readout ([link]). Inexpensive voltmeters have resistances on the order of $R_{\text{M}} = 10\,\text{M}\,\Omega$, whereas high-precision voltmeters have resistances on the order of $R_{\text{M}} = 10\,\text{G}\,\Omega$. The value of the resistance may vary, depending on which scale is used on the meter.



(a) (b)

(a) An analog voltmeter uses a galvanometer to measure the voltage. (b) Digital meters use an analog-to-digital converter to measure the voltage. (credit: modification of works by Joseph J. Trout)

Analog and Digital Meters

You may encounter two types of meters in the physics lab: analog and digital. The term ‘analog’ refers to signals or information represented by a continuously variable physical quantity, such as voltage or current. An analog meter uses a galvanometer, which is essentially a coil of wire with a small resistance, in a magnetic field, with a pointer attached that points to a scale. Current flows through the coil, causing the coil to rotate. To use the galvanometer as an ammeter, a small resistance is placed in parallel with the coil. For a voltmeter, a large resistance is placed in series with the coil. A

digital meter uses a component called an analog-to-digital (A to D) converter and expresses the current or voltage as a series of the digits 0 and 1, which are used to run a digital display. Most analog meters have been replaced by digital meters.

Note:
Exercise:

Problem:

Check Your Understanding Digital meters are able to detect smaller currents than analog meters employing galvanometers. How does this explain their ability to measure voltage and current more accurately than analog meters?

Solution:

Since digital meters require less current than analog meters, they alter the circuit less than analog meters. Their resistance as a voltmeter can be far greater than an analog meter, and their resistance as an ammeter can be far less than an analog meter. Consult [link] and [link] and their discussion in the text.

Note:
In this virtual lab simulation, you may construct circuits with resistors, voltage sources, ammeters and voltmeters to test your knowledge of circuit design.

Ohmmeters

An ohmmeter is an instrument used to measure the resistance of a component or device. The operation of the ohmmeter is based on Ohm's

law. Traditional ohmmeters contained an internal voltage source (such as a battery) that would be connected across the component to be tested, producing a current through the component. A galvanometer was then used to measure the current and the resistance was deduced using Ohm's law. Modern digital meters use a constant current source to pass current through the component, and the voltage difference across the component is measured. In either case, the resistance is measured using Ohm's law $(R = V/I)$, where the voltage is known and the current is measured, or the current is known and the voltage is measured.

The component of interest should be isolated from the circuit; otherwise, you will be measuring the equivalent resistance of the circuit. An ohmmeter should never be connected to a "live" circuit, one with a voltage source connected to it and current running through it. Doing so can damage the meter.

Summary

- Voltmeters measure voltage, and ammeters measure current. Analog meters are based on the combination of a resistor and a galvanometer, a device that gives an analog reading of current or voltage. Digital meters are based on analog-to-digital converters and provide a discrete or digital measurement of the current or voltage.
- A voltmeter is placed in parallel with the voltage source to receive full voltage and must have a large resistance to limit its effect on the circuit.
- An ammeter is placed in series to get the full current flowing through a branch and must have a small resistance to limit its effect on the circuit.
- Standard voltmeters and ammeters alter the circuit they are connected to and are thus limited in accuracy.
- Ohmmeters are used to measure resistance. The component in which the resistance is to be measured should be isolated (removed) from the circuit.

Conceptual Questions

Exercise:

Problem:

What would happen if you placed a voltmeter in series with a component to be tested?

Solution:

The voltmeter would put a large resistance in series with the circuit, significantly changing the circuit. It would probably give a reading, but it would be meaningless.

Exercise:

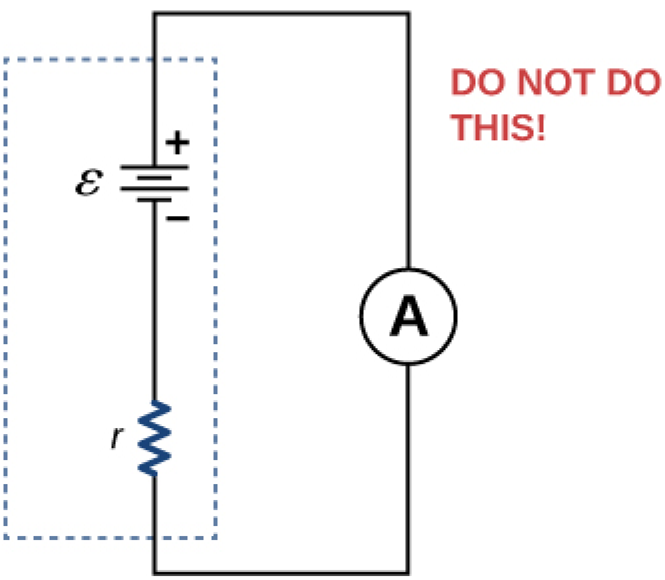
Problem:

What is the basic operation of an ohmmeter as it measures a resistor?

Exercise:

Problem:

Why should you not connect an ammeter directly across a voltage source as shown below?



Solution:

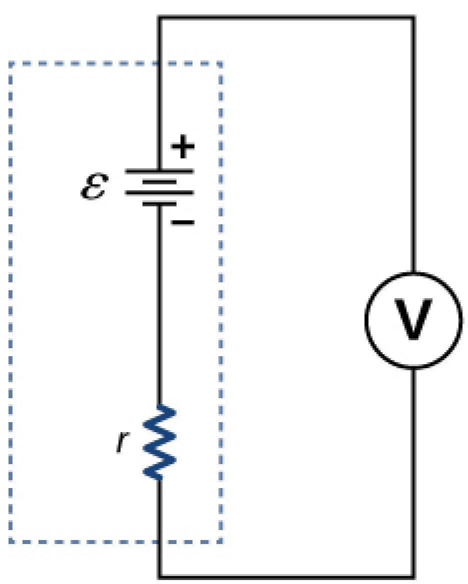
The ammeter has a small resistance; therefore, a large current will be produced and could damage the meter and/or overheat the battery.

Problems

Exercise:

Problem:

Suppose you measure the terminal voltage of a 1.585-V alkaline cell having an internal resistance of $0.100\ \Omega$ by placing a $1.00\text{-k}\Omega$ voltmeter across its terminals (see below). (a) What current flows? (b) Find the terminal voltage. (c) To see how close the measured terminal voltage is to the emf, calculate their ratio.



Glossary

ammeter
: instrument that measures current

voltmeter
instrument that measures voltage

Household Wiring and Electrical Safety
By the end of the section, you will be able to:

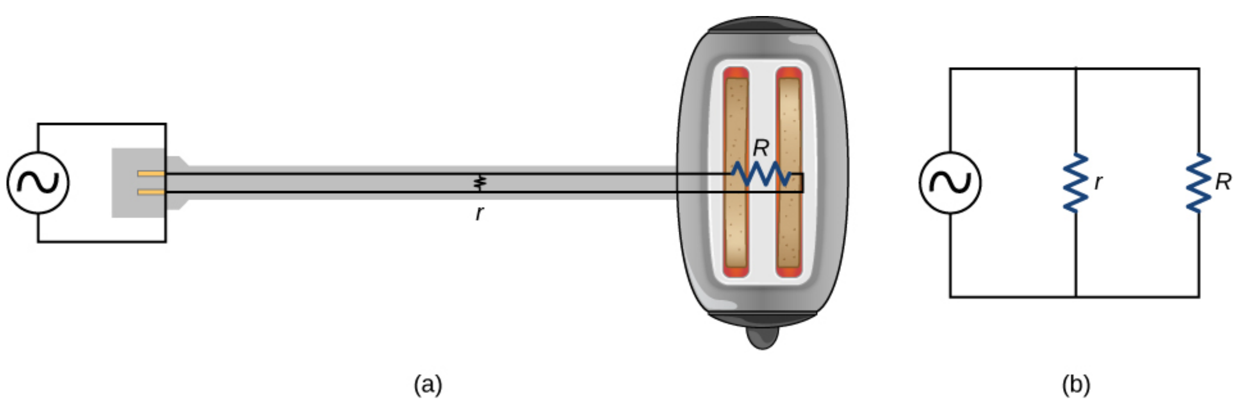
- List the basic concepts involved in house wiring
- Define the terms thermal hazard and shock hazard
- Describe the effects of electrical shock on human physiology and their relationship to the amount of current through the body
- Explain the function of fuses and circuit breakers

Electricity presents two known hazards: thermal and shock. A **thermal hazard** is one in which an excessive electric current causes undesired thermal effects, such as starting a fire in the wall of a house. A **shock hazard** occurs when an electric current passes through a person. Shocks range in severity from painful, but otherwise harmless, to heart-stopping lethality. In this section, we consider these hazards and the various factors affecting them in a quantitative manner. We also examine systems and devices for preventing electrical hazards.

Thermal Hazards

Electric power causes undesired heating effects whenever electric energy is converted into thermal energy at a rate faster than it can be safely dissipated. A classic example of this is the short circuit, a low-resistance path between terminals of a voltage source. An example of a short circuit is shown in [link]. A toaster is plugged into a common household electrical outlet. Insulation on wires leading to an appliance has worn through, allowing the two wires to come into contact, or "short." As a result, thermal energy can quickly raise the temperature of surrounding materials, melting the insulation and perhaps causing a fire.

The circuit diagram shows a symbol that consists of a sine wave enclosed in a circle. This symbol represents an alternating current (ac) voltage source. In an ac voltage source, the voltage oscillates between a positive and negative maximum amplitude. Up to now, we have been considering direct current (dc) voltage sources, but many of the same concepts are applicable to ac circuits.



A short circuit is an undesired low-resistance path across a voltage source. (a) Worn

insulation on the wires of a toaster allow them to come into contact with a low resistance r. Since $P = V^2/r$, thermal power is created so rapidly that the cord melts or burns. (b) A schematic of the short circuit.

Another serious thermal hazard occurs when wires supplying power to an appliance are overloaded. Electrical wires and appliances are often rated for the maximum current they can safely handle. The term “overloaded” refers to a condition where the current exceeds the rated maximum current. As current flows through a wire, the power dissipated in the supply wires is $P = I^2 R_{\mathrm{W}}$, where R_{W} is the resistance of the wires and I is the current flowing through the wires. If either I or R_{W} is too large, the wires overheat. Fuses and circuit breakers are used to limit excessive currents.

Shock Hazards

Electric shock is the physiological reaction or injury caused by an external electric current passing through the body. The effect of an electric shock can be negative or positive. When a current with a magnitude above 300 mA passes through the heart, death may occur. Most electrical shock fatalities occur because a current causes ventricular fibrillation, a massively irregular and often fatal, beating of the heart. On the other hand, a heart attack victim, whose heart is in fibrillation, can be saved by an electric shock from a defibrillator.

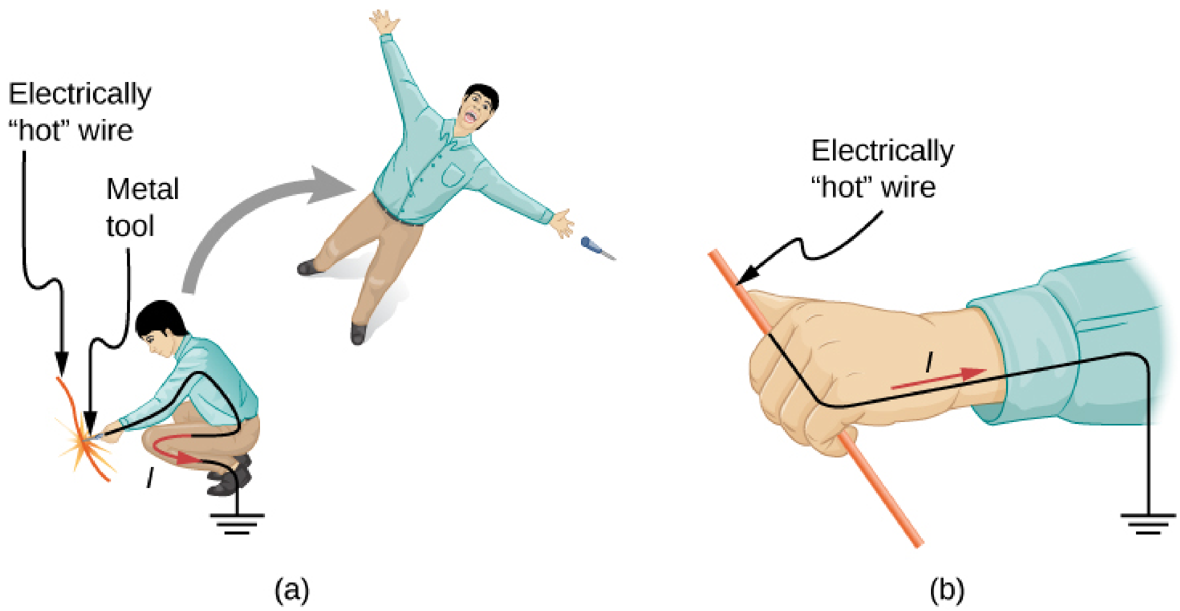
The effects of an undesirable electric shock can vary in severity: a slight sensation at the point of contact, pain, loss of voluntary muscle control, difficulty breathing, heart fibrillation, and possibly death. The loss of voluntary muscle control can cause the victim to not be able to let go of the source of the current.

The major factors upon which the severity of the effects of electrical shock depend are

1. The amount of current I
2. The path taken by the current
3. The duration of the shock
4. The frequency f of the current ($f = 0$ for dc)

Our bodies are relatively good electric conductors due to the body’s water content. A dangerous condition occurs when the body is in contact with a voltage source and “ground.” The term “ground” refers to a large sink or source of electrons, for example, the earth (thus, the name). When there is a direct path to ground, large currents will pass through the parts of the body with the lowest resistance and a direct path to ground. A safety precaution used by many professions is the wearing of insulated shoes. Insulated shoes prohibit a pathway to ground for electrons through the feet by providing a large resistance. Whenever working with high-power tools, or any electric circuit, ensure that you do not provide a pathway for current flow (especially across the heart). A common safety precaution is to work with one hand, reducing the possibility of providing a current path through the heart.

Very small currents pass harmlessly and unfelt through the body. This happens to you regularly without your knowledge. The threshold of sensation is only 1 mA and, although unpleasant, shocks are apparently harmless for currents less than 5 mA. A great number of safety rules take the 5-mA value for the maximum allowed shock. At 5–30 mA and above, the current can stimulate sustained muscular contractions, much as regular nerve impulses do ([link]). Very large currents (above 300 mA) cause the heart and diaphragm of the lung to contract for the duration of the shock. Both the heart and respiration stop. Both often return to normal following the shock.



An electric current can cause muscular contractions with varying effects. (a) The victim is "thrown" backward by involuntary muscle contractions that extend the legs and torso. (b) The victim can't let go of the wire that is stimulating all the muscles in the hand. Those that close the fingers are stronger than those that open them.

Current is the major factor determining shock severity. A larger voltage is more hazardous, but since $I=V/R$, the severity of the shock depends on the combination of voltage and resistance. For example, a person with dry skin has a resistance of about $200\text{ k}\Omega$. If he comes into contact with 120-V ac, a current

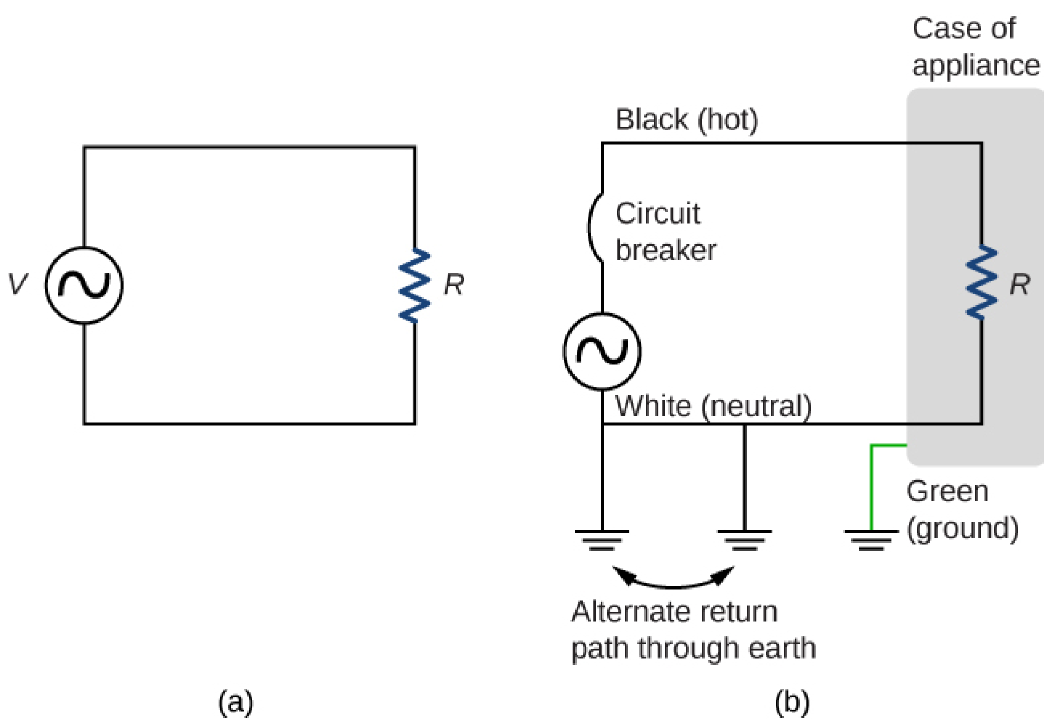
Equation:

$$I = (120\text{ V})/(200\text{ k}\Omega) = 0.6\text{ mA}$$

passes harmlessly through him. The same person soaking wet may have a resistance of $10.0\text{ k}\Omega$ and the same 120 V will produce a current of 12 mA—above the "can't let go" threshold and potentially dangerous.

Electrical Safety: Systems and Devices

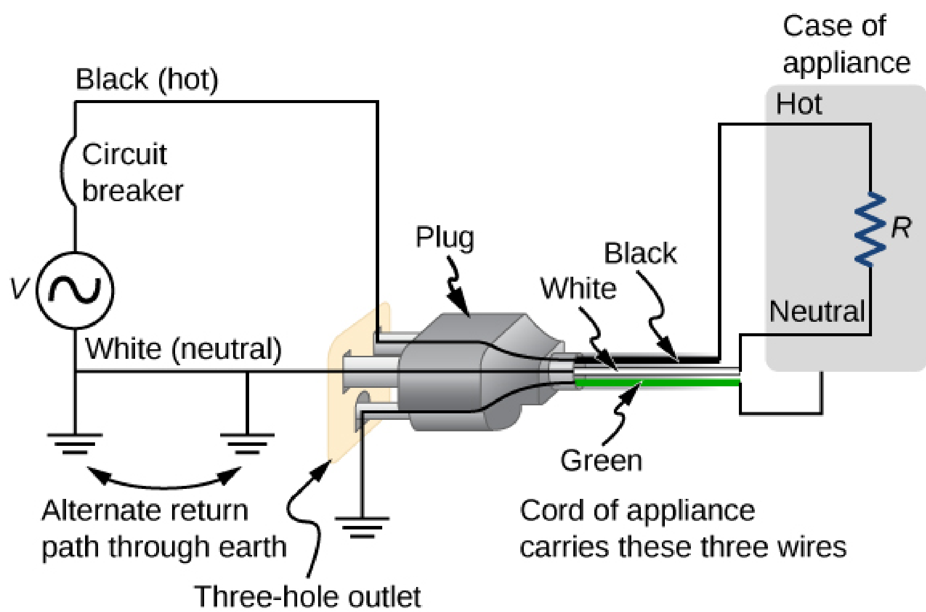
[link](a) shows the schematic for a simple ac circuit with no safety features. This is not how power is distributed in practice. Modern household and industrial wiring requires the **three-wire system**, shown schematically in part (b), which has several safety features, with live, neutral, and ground wires. First is the familiar circuit breaker (or fuse) to prevent thermal overload. Second is a protective case around the appliance, such as a toaster or refrigerator. The case’s safety feature is that it prevents a person from touching exposed wires and coming into electrical contact with the circuit, helping prevent shocks.



(a) Schematic of a simple ac circuit with a voltage source and a single appliance represented by the resistance *R*. There are no safety features in this circuit. (b) The three-wire system connects the neutral wire to ground at the voltage source and user location, forcing it to be at zero volts and supplying an alternative return path for the current through ground. Also grounded to zero volts is the case of the appliance. A circuit breaker or fuse protects against thermal overload and is in series on the active (live/hot) wire.

There are three connections to ground shown in [link](b). Recall that a ground connection is a low-resistance path directly to ground. The two ground connections on the neutral wire force it to be at zero volts relative to ground, giving the wire its name. This wire is therefore safe to

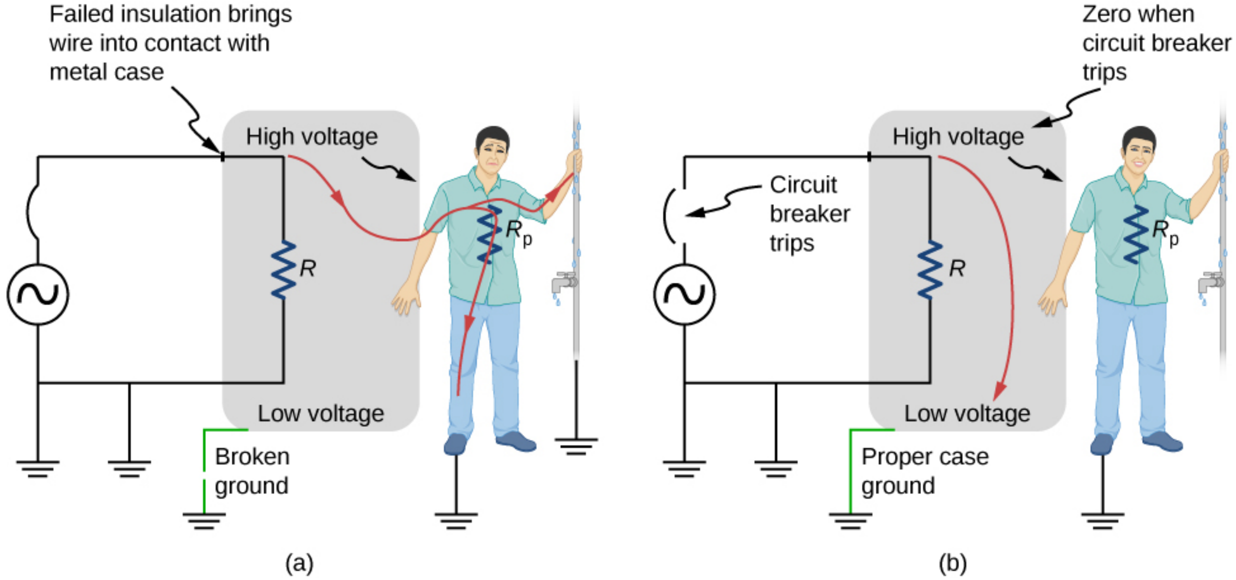
touch even if its insulation, usually white, is missing. The neutral wire is the return path for the current to follow to complete the circuit. Furthermore, the two ground connections supply an alternative path through ground (a good conductor) to complete the circuit. The ground connection closest to the power source could be at the generating plant, whereas the other is at the user's location. The third ground is to the case of the appliance, through the green ground wire, forcing the case, too, to be at zero volts. The live or hot wire (hereafter referred to as "live/hot") supplies voltage and current to operate the appliance. [link] shows a more pictorial version of how the three-wire system is connected through a three-prong plug to an appliance.



The standard three-prong plug can only be inserted in one way, to ensure proper function of the three-wire system.

Insulating plastic is color-coded to identify live/hot, neutral, and ground wires, but these codes vary around the world. It is essential to determine the color code in your region. Striped coatings are sometimes used for the benefit of those who are colorblind.

Grounding the case solves more than one problem. The simplest problem is worn insulation on the live/hot wire that allows it to contact the case, as shown in [link]. Lacking a ground connection, a severe shock is possible. This is particularly dangerous in the kitchen, where a good connection to ground is available through water on the floor or a water faucet. With the ground connection intact, the circuit breaker will trip, forcing repair of the appliance.



Worn insulation allows the live/hot wire to come into direct contact with the metal case of this appliance. (a) The ground connection being broken, the person is severely shocked. The appliance may operate normally in this situation. (b) With a proper ground, the circuit breaker trips, forcing repair of the appliance.

A ground fault circuit interrupter (GFCI) is a safety device found in updated kitchen and bathroom wiring that works based on electromagnetic induction. GFCIs compare the currents in the live/hot and neutral wires. When live/hot and neutral currents are not equal, it is almost always because current in the neutral is less than in the live/hot wire. Then some of the current, called a leakage current, is returning to the voltage source by a path other than through the neutral wire. It is assumed that this path presents a hazard. GFCIs are usually set to interrupt the circuit if the leakage current is greater than 5 mA, the accepted maximum harmless shock. Even if the leakage current goes safely to ground through an intact ground wire, the GFCI will trip, forcing repair of the leakage.

Summary

- The two types of electric hazards are thermal (excessive power) and shock (current through a person). Electrical safety systems and devices are employed to prevent thermal and shock hazards.
- Shock severity is determined by current, path, duration, and ac frequency.
- Circuit breakers and fuses interrupt excessive currents to prevent thermal hazards.
- The three-wire system guards against thermal and shock hazards, utilizing live/hot, neutral, and ground wires, and grounding the neutral wire and case of the appliance.
- A ground fault circuit interrupter (GFCI) prevents shock by detecting the loss of current to unintentional paths.

Key Equations

Terminal voltage of a single voltage source	$V_{\text{terminal}} = \varepsilon - Ir_{\text{eq}}$
Equivalent resistance of a series circuit	$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i$
Equivalent resistance of a parallel circuit	$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R2} + \cdots + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}$
Junction rule	$\sum I_{\text{in}} = \sum I_{\text{out}}$
Loop rule	$\sum V = 0$
Terminal voltage of N voltage sources in series	$V_{\text{terminal}} = \sum_{i=1}^{N} \varepsilon_i - I \sum_{i=1}^{N} r_i = \sum_{i=1}^{N} \varepsilon_i - Ir_{\text{eq}}$
Terminal voltage of N voltage sources in parallel	$V_{\text{terminal}} = \varepsilon - I \sum_{i=1}^{N} \left(\frac{1}{r_i}\right)^{-1} = \varepsilon - Ir_{\text{eq}}$
Charge on a charging capacitor	$q(t) = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right) = Q\left(1 - e^{-\frac{t}{\tau}}\right)$
Time constant	$\tau = RC$
Current during charging of a capacitor	$I = \frac{\varepsilon}{R} e^{-\frac{t}{RC}} = I_o e^{-\frac{t}{RC}}$
Charge on a discharging capacitor	$q(t) = Qe^{-\frac{t}{\tau}}$
Current during discharging of a capacitor	$I(t) = -\frac{Q}{RC} e^{-\frac{t}{\tau}}$

Conceptual Questions

Exercise:

Problem: Why isn’t a short circuit necessarily a shock hazard?

Exercise:

Problem:

We are often advised to not flick electric switches with wet hands, dry your hand first. We are also advised to never throw water on an electric fire. Why?

Solution:

Not only might water drip into the switch and cause a shock, but also the resistance of your body is lower when you are wet.

Problems

Exercise:

Problem:

(a) How much power is dissipated in a short circuit of 240-V ac through a resistance of $0.250\ \Omega$? (b) What current flows?

Exercise:

Problem:

What voltage is involved in a 1.44-kW short circuit through a $0.100\text{-}\Omega$ resistance?

Solution:

12.0 V

Exercise:

Problem:

Find the current through a person and identify the likely effect on her if she touches a 120-V ac source: (a) if she is standing on a rubber mat and offers a total resistance of $300\ \text{k}\Omega$; (b) if she is standing barefoot on wet grass and has a resistance of only $4000\ \text{k}\Omega$.

Exercise:

Problem:

While taking a bath, a person touches the metal case of a radio. The path through the person to the drainpipe and ground has a resistance of $4000\ \Omega$. What is the smallest voltage on the case of the radio that could cause ventricular fibrillation?

Solution:

400 V

Exercise:

Problem:

A man foolishly tries to fish a burning piece of bread from a toaster with a metal butter knife and comes into contact with 120-V ac. He does not even feel it since, luckily, he is wearing rubber-soled shoes. What is the minimum resistance of the path the current follows through the person?

Exercise:

Problem:

(a) During surgery, a current as small as $20.0\ \mu\text{A}$ applied directly to the heart may cause ventricular fibrillation. If the resistance of the exposed heart is $300\ \Omega$, what is the smallest voltage that poses this danger? (b) Does your answer imply that special electrical safety precautions are needed?

Solution:

a. 6.00 mV; b. It would not be necessary to take extra precautions regarding the power coming from the wall. However, it is possible to generate voltages of approximately this value from static charge built up on gloves, for instance, so some precautions are necessary.

Exercise:

Problem:

(a) What is the resistance of a 220-V ac short circuit that generates a peak power of 96.8 kW? (b) What would the average power be if the voltage were 120 V ac?

Exercise:

Problem:

A heart defibrillator passes 10.0 A through a patient's torso for 5.00 ms in an attempt to restore normal beating. (a) How much charge passed? (b) What voltage was applied if 500 J of energy was dissipated? (c) What was the path's resistance? (d) Find the temperature increase caused in the 8.00 kg of affected tissue.

Solution:

a. 5.00×10^{-2} C; b. 10.0 kV; c. $1.00\ \text{k}\Omega$; d. $1.79 \times 10^{-2}\ ^\circ\text{C}$

Exercise:

Problem:

A short circuit in a 120-V appliance cord has a $0.500\text{-}\Omega$ resistance. Calculate the temperature rise of the 2.00 g of surrounding materials, assuming their specific heat capacity is $0.200\ \text{cal/g} \cdot {}^\circ\text{C}$ and that it takes 0.0500 s for a circuit breaker to interrupt the current. Is this likely to be damaging?

Additional Problems

Exercise:

Problem:

A circuit contains a D cell battery, a switch, a $20\text{-}\Omega$ resistor, and four 20-mF capacitors connected in series. (a) What is the equivalent capacitance of the circuit? (b) What is the *RC* time constant? (c) How long before the current decreases to 50% of the initial value once the switch is closed?

Solution:

a. $C_{\text{eq}} = 5.00\ \text{mF}$; b. $\tau = 0.1$ s; c. 0.069 s

Exercise:

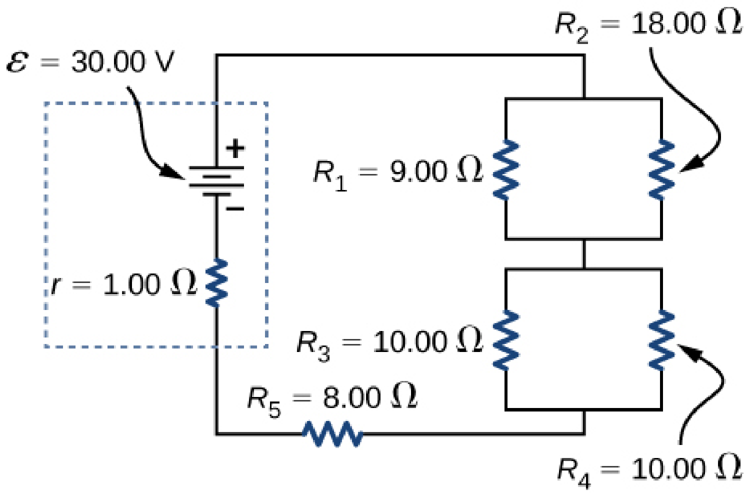
Problem:

A circuit contains a D-cell battery, a switch, a $20\text{-}\Omega$ resistor, and three 20-mF capacitors. The capacitors are connected in parallel, and the parallel connection of capacitors are connected in series with the switch, the resistor and the battery. (a) What is the equivalent capacitance of the circuit? (b) What is the *RC* time constant? (c) How long before the current decreases to 50% of the initial value once the switch is closed?

Exercise:

Problem:

Consider the circuit below. The battery has an emf of $\varepsilon = 30.00\ \text{V}$ and an internal resistance of $r = 1.00\ \Omega$. (a) Find the equivalent resistance of the circuit and the current out of the battery. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the total power supplied by the batteries.



Solution:

a. $R_{\text{eq}} = 20.00\ \Omega$;
b. $I_r = 1.50\ \text{A}, I_1 = 1.00\ \text{A}, I_2 = 0.50\ \text{A}, I_3 = 0.75\ \text{A}, I_4 = 0.75\ \text{A}, I_5 = 1.50\ \text{A}$;
c. $V_r = 1.50\ \text{V}, V_1 = 9.00\ \text{V}, V_2 = 9.00\ \text{V}, V_3 = 7.50\ \text{V}, V_4 = 7.50\ \text{V}, V_5 = 12.00\ \text{V}$;
d.
$P_r = 2.25\ \text{W}, P_1 = 9.00\ \text{W}, P_2 = 4.50\ \text{W}, P_3 = 5.625\ \text{W}, P_4 = 5.625\ \text{W}, P_5 = 18.00\ \text{W}$
;
e. $P = 45.00\ \text{W}$

Exercise:

Problem:

A homemade capacitor is constructed of 2 sheets of aluminum foil with an area of 2.00 square meters, separated by paper, 0.05 mm thick, of the same area and a dielectric constant of 3.7. The homemade capacitor is connected in series with a 100.00-Ω resistor, a switch, and a 6.00-V voltage source. (a) What is the *RC* time constant of the circuit? (b) What is the initial current through the circuit, when the switch is closed? (c) How long does it take the current to reach one third of its initial value?

Exercise:

Problem:

A student makes a homemade resistor from a graphite pencil 5.00 cm long, where the graphite is 0.05 mm in diameter. The resistivity of the graphite is $\rho = 1.38 \times 10^{-5}\ \Omega/\text{m}$. The homemade resistor is place in series with a switch, a 10.00-mF capacitor and a 0.50-V power source. (a) What is the *RC* time constant of the circuit? (b) What is the potential drop across the pencil 1.00 s after the switch is closed?

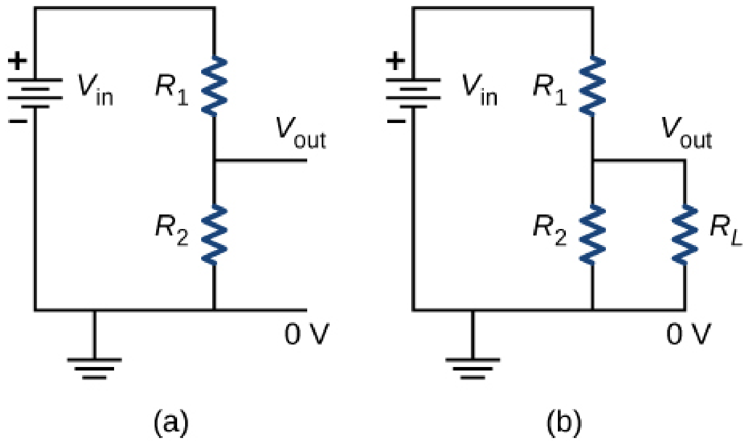
Solution:

a. $\tau = \left(1.38\ \times\ 10^{-5}\Omega\text{m}\left(\frac{5.00 \times 10^{-2}\text{ m}}{3.14\left(\frac{0.05 \times 10^{-3}}{2}\right)^{2}}\right)\right)10\ \times\ 10^{-3}\text{ F} = 3.52\text{ s}$; b. $V = 0.017\text{ A}\left(e^{-\frac{1.00\text{ s}}{3.52\text{ s}}}\right)351.59\ \Omega = 0.122\text{ V}$

Exercise:

Problem:

The rather simple circuit shown below is known as a voltage divider. The symbol consisting of three horizontal lines is represents "ground" and can be defined as the point where the potential is zero. The voltage divider is widely used in circuits and a single voltage source can be used to provide reduced voltage to a load resistor as shown in the second part of the figure. (a) What is the output voltage V_{out} of circuit (a) in terms of $R_1, R_2,$ and V_{in}? (b) What is the output voltage V_{out} of circuit (b) in terms of $R_1, R_2, R_L,$ and V_{in}?



Exercise:

Problem:

Three 300-Ω resistors are connect in series with an AAA battery with a rating of 3 AmpHours. (a) How long can the battery supply the resistors with power? (b) If the resistors are connected in parallel, how long can the battery last?

Solution:

a. $t = \frac{3\text{ A}\cdot\text{h}}{\frac{1.5\text{ V}}{900\ \Omega}} = 1800\text{ h}$; b. $t = \frac{3\text{ A}\cdot\text{h}}{\frac{1.5\text{ V}}{100\ \Omega}} = 200\text{ h}$

Exercise:

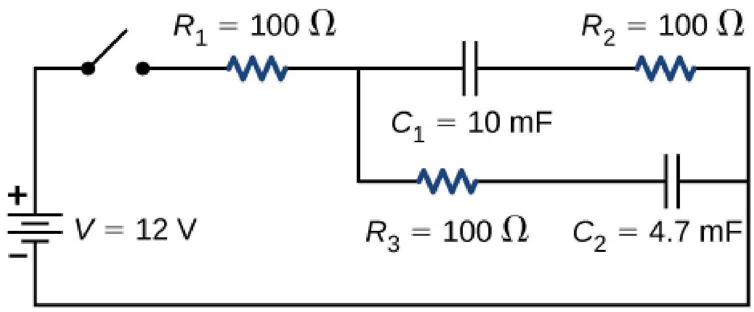
Problem:

Consider a circuit that consists of a real battery with an emf ε and an internal resistance of r connected to a variable resistor R. (a) In order for the terminal voltage of the battery to be equal to the emf of the battery, what should the resistance of the variable resistor be adjusted to? (b) In order to get the maximum current from the battery, what should the resistance of the variable resistor be adjusted to? (c) In order for the maximum power output of the battery to be reached, what should the resistance of the variable resistor be set to?

Exercise:

Problem:

Consider the circuit shown below. What is the energy stored in each capacitor after the switch has been closed for a very long time?



Solution:

$U_1 = C_1V_1^2 = 0.72\text{J},\ \ U_2 = C_2V_2^2 = 0.338\ \text{J}$

Exercise:

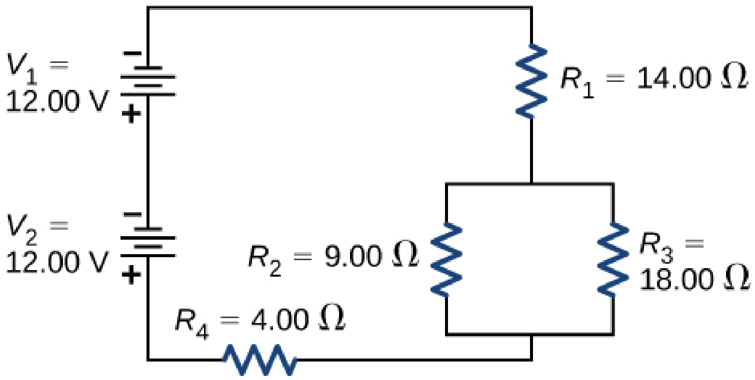
Problem:

Consider a circuit consisting of a battery with an emf ε and an internal resistance of r connected in series with a resistor R and a capacitor C. Show that the total energy supplied by the battery while charging the battery is equal to $\varepsilon^2 C$.

Exercise:

Problem:

Consider the circuit shown below. The terminal voltages of the batteries are shown. (a) Find the equivalent resistance of the circuit and the current out of the battery. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the total power supplied by the batteries.



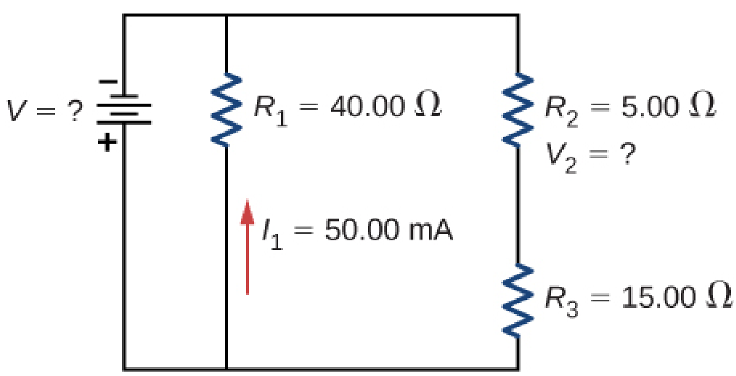
Solution:

a. $R_{\text{eq}} = 24.00\ \Omega$; b. $I_1 = 1.00\ \text{A}, I_2 = 0.67\ \text{A}, I_3 = 0.33\ \text{A}, I_4 = 1.00\ \text{A}$;
c. $V_1 = 14.00\ \text{V}, V_2 = 6.00\ \text{V}, V_3 = 6.00\ \text{V}, V_4 = 4.00\ \text{V}$;
d. $P_1 = 14.00\ \text{W}, P_2 = 4.04\ \text{W}, P_3 = 1.96\ \text{W}, P_4 = 4.00\ \text{W}$; e. $P = 24.00\ \text{W}$

Exercise:

Problem:

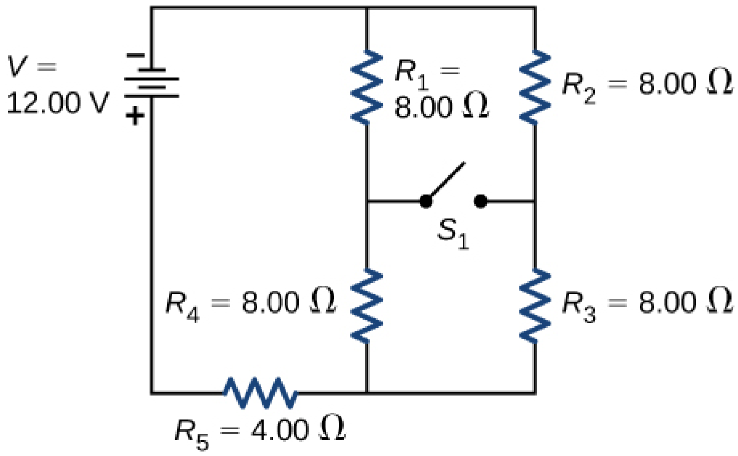
Consider the circuit shown below. (a) What is the terminal voltage of the battery? (b) What is the potential drop across resistor R_2?



Exercise:

Problem:

Consider the circuit shown below. (a)Determine the equivalent resistance and the current from the battery with switch S_1 open. (b) Determine the equivalent resistance and the current from the battery with switch S_1 closed.



Solution:

a. $R_{\text{eq}} = 12.00\ \Omega, I = 1.00\ \text{A}$; b. $R_{\text{eq}} = 12.00\ \Omega, I = 1.00\ \text{A}$

Exercise:

Problem:

Two resistors, one having a resistance of $145\ \Omega$, are connected in parallel to produce a total resistance of $150\ \Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Exercise:

Problem:

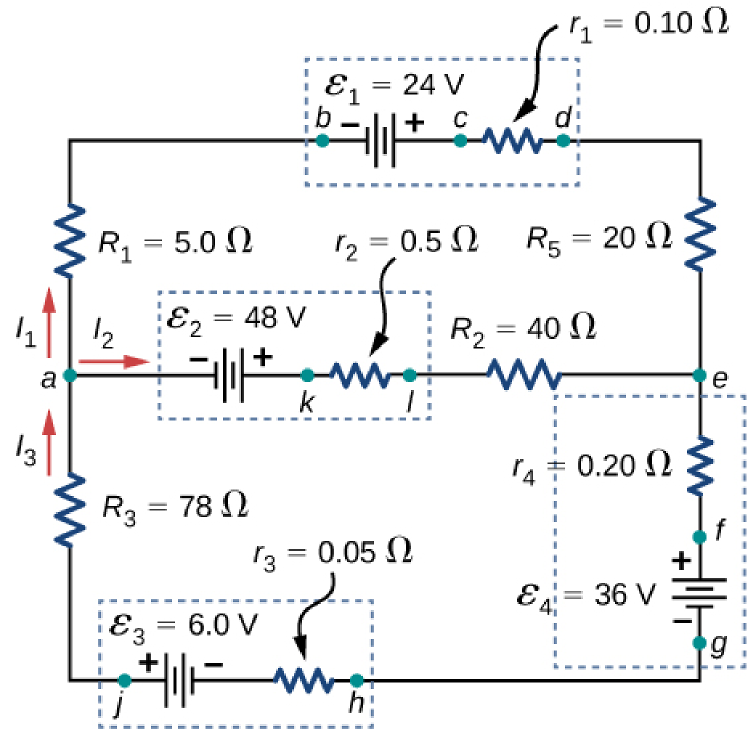
Two resistors, one having a resistance of $900\ \text{k}\Omega$, are connected in series to produce a total resistance of $0.500\ \text{M}\Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Solution:

a. $-400\ \text{k}\Omega$; b. You cannot have negative resistance. c. The assumption that $R_{\text{eq}} < R_1$ is unreasonable. Series resistance is always greater than any of the individual resistances.

Exercise:

Problem: Apply the junction rule at point *a* shown below.



Exercise:

Problem: Apply the loop rule to Loop *akledcba* in the preceding problem.

Solution:

$$E_2 - I_2 r_2 - I_2 R_2 + I_1 R_5 + I_1 r_1 - E_1 + I_1 R_1 = 0$$

Exercise:

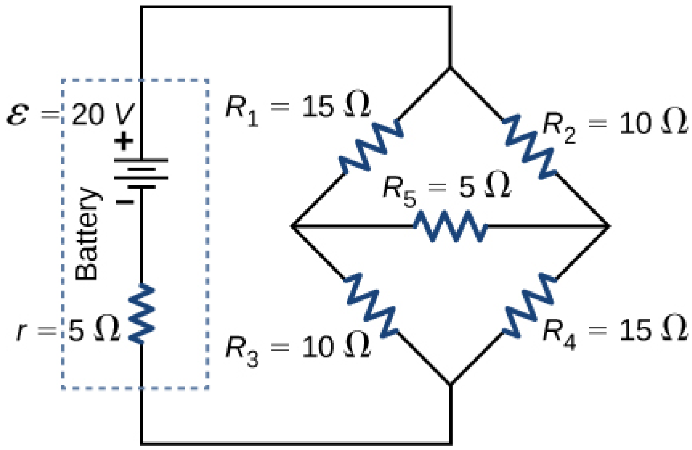
Problem:

Find the currents flowing in the circuit in the preceding problem. Explicitly show how you follow the steps in the [Problem-Solving Strategy: Series and Parallel Resistors](#).

Exercise:

Problem:

Consider the circuit shown below. (a) Find the current through each resistor. (b) Check the calculations by analyzing the power in the circuit.



Solution:

a. $I = 1.17\ \text{A}, I_1 = 0.50\ \text{A}, I_2 = 0.67\ \text{A}, I_3 = 0.67\ \text{A}, I_4 = 0.50\ \text{A}, I_5 = 0.17\ \text{A}$;
b. $P_{\text{output}} = 23.4\ \text{W}, P_{\text{input}} = 23.4\ \text{W}$

Exercise:

Problem:

A flashing lamp in a Christmas earring is based on an *RC* discharge of a capacitor through its resistance. The effective duration of the flash is 0.250 s, during which it produces an average 0.500 W from an average 3.00 V. (a) What energy does it dissipate? (b) How much charge moves through the lamp? (c) Find the capacitance. (d) What is the resistance of the lamp? (Since average values are given for some quantities, the shape of the pulse profile is not needed.)

Exercise:

Problem:

A 160-μF capacitor charged to 450 V is discharged through a $31.2\text{-k}\Omega$ resistor. (a) Find the time constant. (b) Calculate the temperature increase of the resistor, given that its mass is 2.50 g and its specific heat is $1.67\ \text{kJ/kg} \cdot {}^{\circ}\text{C}$, noting that most of the thermal energy is retained in the short time of the discharge. (c) Calculate the new resistance, assuming it is pure carbon. (d) Does this change in resistance seem significant?

Solution:

a. 4.99 s; b. $3.87\ {}^{\circ}\text{C}$; c. $3.11\ \times\ 10^{4}\ \Omega$; d. No, this change does not seem significant. It probably would not be noticed.

Challenge Problems

Exercise:

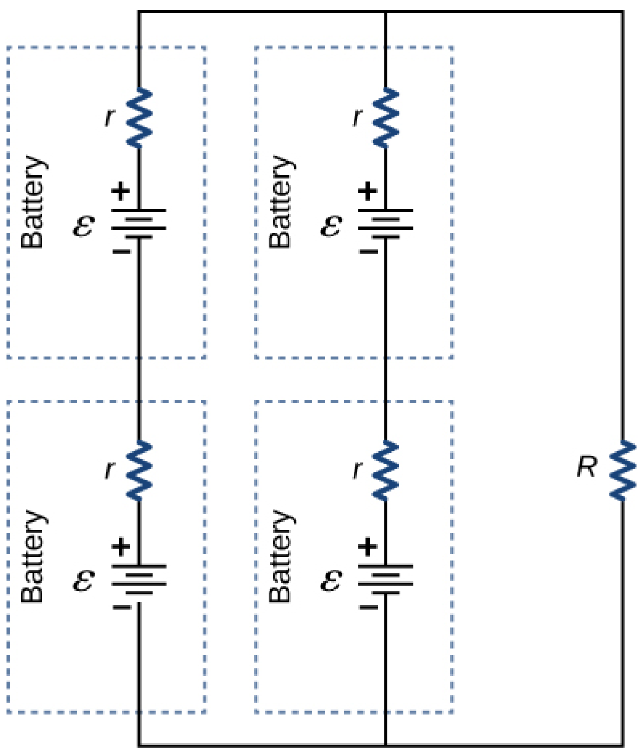
Problem:

Some camera flashes use flash tubes that require a high voltage. They obtain a high voltage by charging capacitors in parallel and then internally changing the connections of the capacitors to place them in series. Consider a circuit that uses four AAA batteries connected in series to charge six 10-mF capacitors through an equivalent resistance of $100\ \Omega$. The connections are then switched internally to place the capacitors in series. The capacitors discharge through a lamp with a resistance of $100\ \Omega$. (a) What is the *RC* time constant and the initial current out of the batteries while they are connected in parallel? (b) How long does it take for the capacitors to charge to 90% of the terminal voltages of the batteries? (c) What is the *RC* time constant and the initial current of the capacitors connected in series assuming it discharges at 90% of full charge? (d) How long does it take the current to decrease to 10% of the initial value?

Exercise:

Problem:

Consider the circuit shown below. Each battery has an emf of 1.50 V and an internal resistance of $1.00\ \Omega$. (a) What is the current through the external resistor, which has a resistance of 10.00 ohms? (b) What is the terminal voltage of each battery?



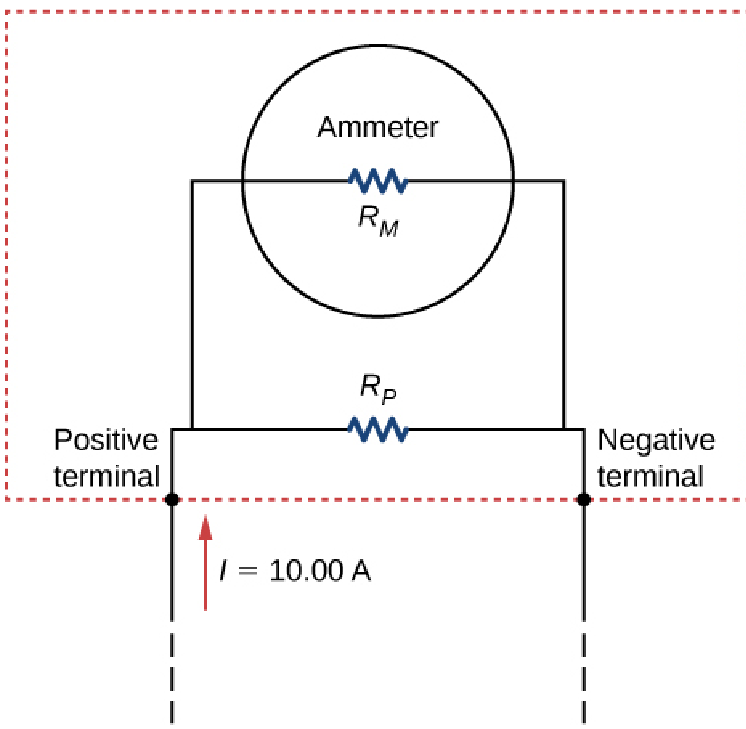
Solution:

a. 0.273 A; b. $V_T = 1.36$ V

Exercise:

Problem:

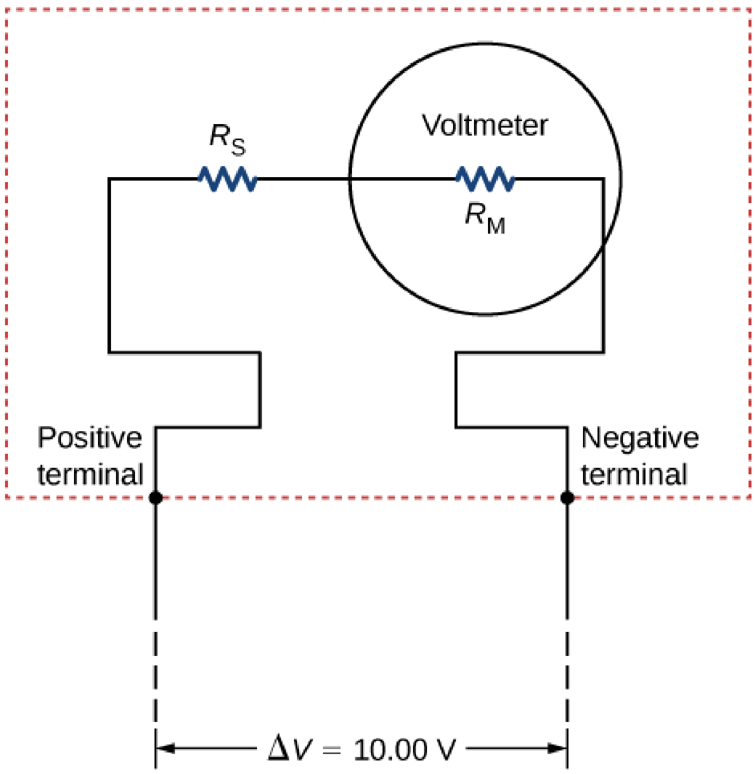
Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the pointer turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make an ammeter if a resistor is placed in parallel with the galvanometer. Consider a galvanometer that has a resistance of $25.00\ \Omega$ and gives a full scale reading when a $50\text{-}\mu\text{A}$ current runs through it. The galvanometer is to be used to make an ammeter that has a full scale reading of 10.00 A, as shown below. Recall that an ammeter is connected in series with the circuit of interest, so all 10 A must run through the meter. (a) What is the current through the parallel resistor in the meter? (b) What is the voltage across the parallel resistor? (c) What is the resistance of the series resistor?



Exercise:

Problem:

Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the point turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make a voltmeter if a resistor is placed in series with the galvanometer. Consider a galvanometer that has a resistance of $25.00\ \Omega$ and gives a full scale reading when a $50\text{-}\mu\text{A}$ current runs through it. The galvanometer is to be used to make an voltmeter that has a full scale reading of 10.00 V, as shown below. Recall that a voltmeter is connected in parallel with the component of interest, so the meter must have a high resistance or it will change the current running through the component. (a) What is the potential drop across the series resistor in the meter? (b) What is the resistance of the parallel resistor?

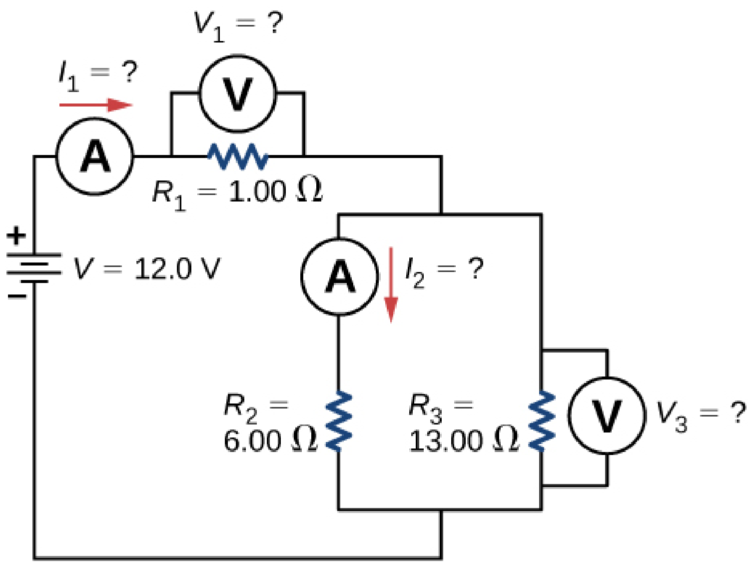


Solution:

a. $V_s = V - I_M R_M = 9.99875\ \text{V}$; b. $R_S = \frac{V_P}{I_M} = 199.975\ \text{k}\Omega$

Exercise:

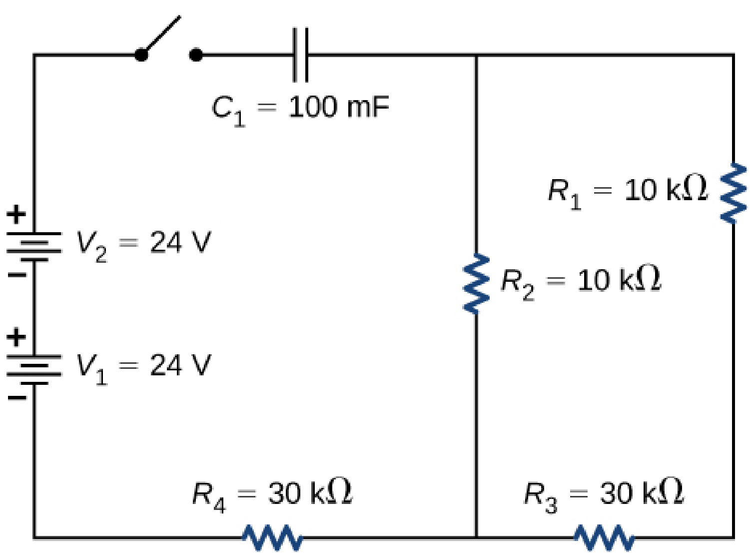
Problem: Consider the circuit shown below. Find $I_1, V_1, I_2,$ and V_3.



Exercise:

Problem:

Consider the circuit below. (a) What is the *RC* time constant of the circuit? (b) What is the initial current in the circuit once the switch is closed? (c) How much time passes between the instant the switch is closed and the time the current has reached half of the initial current?



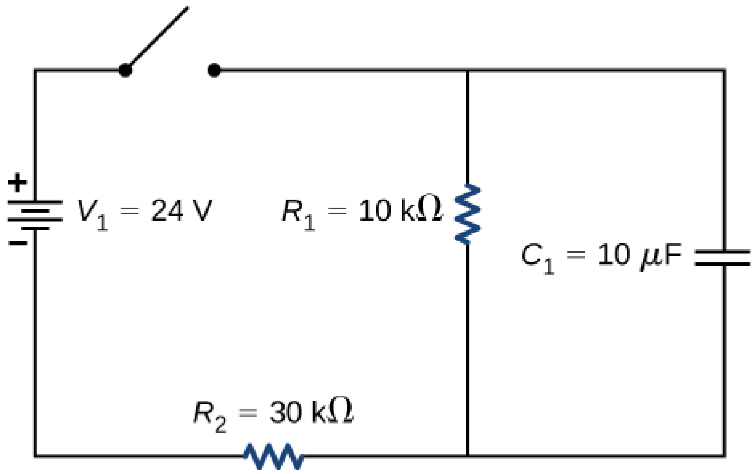
Solution:

a. $\tau = 3800$ s; b. 1.26 mA; c. $t = 2633.96$ s

Exercise:

Problem:

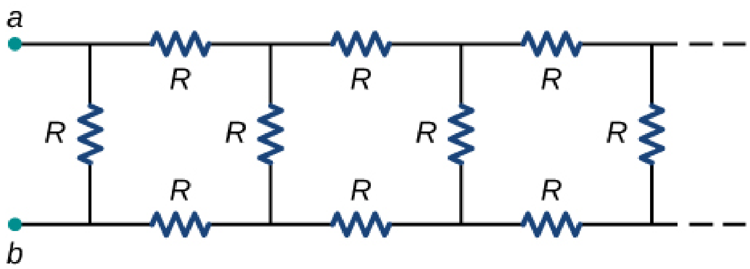
Consider the circuit below. (a) What is the initial current through resistor R_2 when the switch is closed? (b) What is the current through resistor R_2 when the capacitor is fully charged, long after the switch is closed? (c) What happens if the switch is opened after it has been closed for some time? (d) If the switch has been closed for a time period long enough for the capacitor to become fully charged, and then the switch is opened, how long before the current through resistor R_1 reaches half of its initial value?



Exercise:

Problem:

Consider the infinitely long chain of resistors shown below. What is the resistance between terminals *a* and *b*?



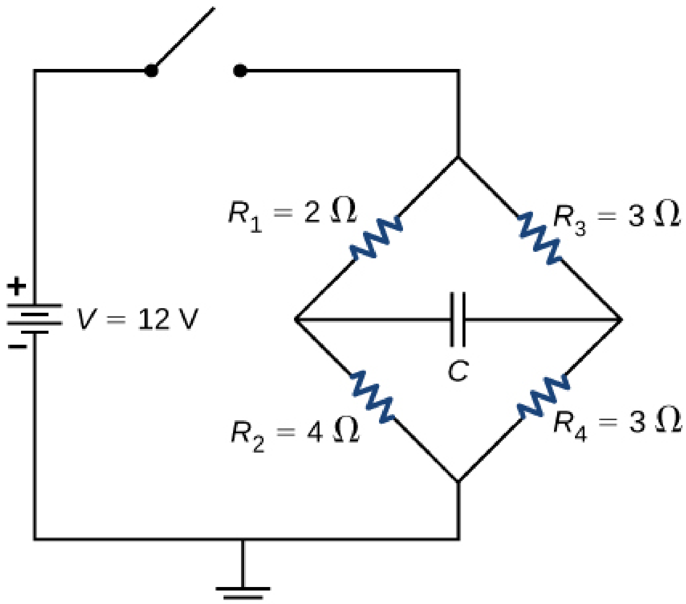
Solution:

$R_{\text{eq}} = \left(\sqrt{3}-1\right)R$

Exercise:

Problem:

Consider the circuit below. The capacitor has a capacitance of 10 mF. The switch is closed and after a long time the capacitor is fully charged. (a) What is the current through each resistor a long time after the switch is closed? (b) What is the voltage across each resistor a long time after the switch is closed? (c) What is the voltage across the capacitor a long time after the switch is closed? (d) What is the charge on the capacitor a long time after the switch is closed? (e) The switch is then opened. The capacitor discharges through the resistors. How long from the time before the current drops to one fifth of the initial value?



Exercise:

Problem:

A 120-V immersion heater consists of a coil of wire that is placed in a cup to boil the water. The heater can boil one cup of $20.00\ ^\circ\text{C}$ water in 180.00 seconds. You buy one to use in your dorm room, but you are worried that you will overload the circuit and trip the 15.00-A, 120-V circuit breaker, which supplies your dorm room. In your dorm room, you have four 100.00-W incandescent lamps and a 1500.00-W space heater. (a) What is the power rating of the immersion heater? (b) Will it trip the breaker when everything is turned on? (c) If it you replace the incandescent bulbs with 18.00-W LED, will the breaker trip when everything is turned on?

Solution:

a. $P_{\text{imheater}} = \dfrac{1\text{cup}\left(\frac{0.000237\ \text{m}^3}{\text{cup}}\right)\left(\frac{1000\ \text{kg}}{\text{m}^3}\right)\left(4186\frac{\text{J}}{\text{kg}\ ^\circ\text{C}}\right)(100\ ^\circ\text{C} - 20\ ^\circ\text{C})}{180.00\ \text{s}} \approx 441\ \text{W};$

b. $I = \frac{441\ \text{W}}{120\ \text{V}} + 4\left(\frac{100\ \text{W}}{120\ \text{V}}\right) + \frac{1500\ \text{W}}{120\ \text{V}} = 19.51\ \text{A}$; Yes, the breaker will trip.

c. $I = \frac{441\ \text{W}}{120\ \text{V}} + 4\left(\frac{18\ \text{W}}{120\ \text{V}}\right) + \frac{1500\ \text{W}}{120\ \text{V}} = 16.78\ \text{A}$; Yes, the breaker will trip.

Exercise:

Problem:

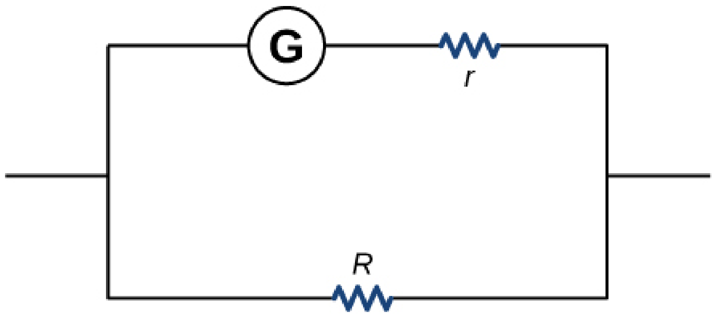
Find the resistance that must be placed in series with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as a voltmeter with a 3000-V full-scale reading. Include a circuit diagram with your solution.

Exercise:

Problem:

Find the resistance that must be placed in parallel with a $60.0\text{-}\Omega$ galvanometer having a 1.00-mA sensitivity (the same as the one discussed in the text) to allow it to be used as an ammeter with a 25.0-A full-scale reading. Include a circuit diagram with your solution.

Solution:



,
$2.40 \times 10^{-3}\,\Omega$

Glossary

shock hazard
: hazard in which an electric current passes through a person

thermal hazard
: hazard in which an excessive electric current causes undesired thermal effects

three-wire system
: wiring system used at present for safety reasons, with live, neutral, and ground wires

Introduction
class="introduction"

The tree-like branch patterns in this clear Plexiglas® block are known as a Lichtenberg figure, named for the German physicist Georg Christof Lichtenberg (1742–1799), who was the first to study these patterns. The “branches” are created by the dielectric breakdown produced by a strong electric field. (credit: modification of work

by Bert Hickman)

Capacitors are important components of electrical circuits in many electronic devices, including pacemakers, cell phones, and computers. In this chapter, we study their properties, and, over the next few chapters, we examine their function in combination with other circuit elements. By themselves, capacitors are often used to store electrical energy and release it when needed; with other circuit components, capacitors often act as part of a filter that allows some electrical signals to pass while blocking others. You can see why capacitors are considered one of the fundamental components of electrical circuits.

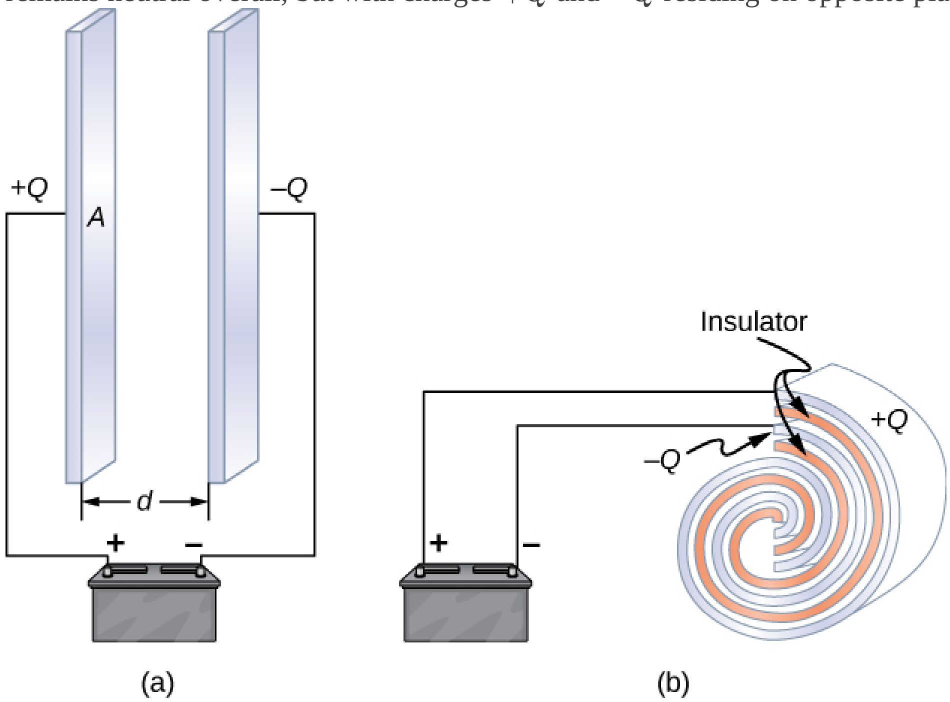
Capacitors and Capacitance

By the end of this section, you will be able to:

- Explain the concepts of a capacitor and its capacitance
- Describe how to evaluate the capacitance of a system of conductors

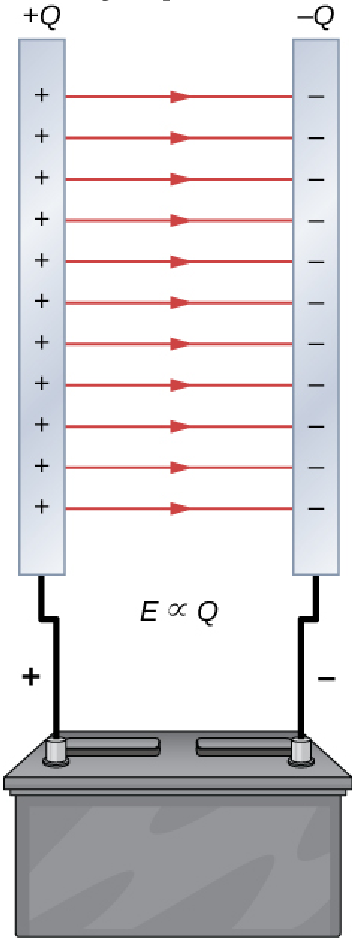
A **capacitor** is a device used to store electrical charge and electrical energy. It consists of at least two electrical conductors separated by a distance. (Note that such electrical conductors are sometimes referred to as “electrodes,” but more correctly, they are “capacitor plates.”) The space between capacitors may simply be a vacuum, and, in that case, a capacitor is then known as a “vacuum capacitor.” However, the space is usually filled with an insulating material known as a **dielectric**. (You will learn more about dielectrics in the sections on dielectrics later in this chapter.) The amount of storage in a capacitor is determined by a property called *capacitance*, which you will learn more about a bit later in this section.

Capacitors have applications ranging from filtering static from radio reception to energy storage in heart defibrillators. Typically, commercial capacitors have two conducting parts close to one another but not touching, such as those in [link]. Most of the time, a dielectric is used between the two plates. When battery terminals are connected to an initially uncharged capacitor, the battery potential moves a small amount of charge of magnitude Q from the positive plate to the negative plate. The capacitor remains neutral overall, but with charges $+Q$ and $-Q$ residing on opposite plates.



Both capacitors shown here were initially uncharged before being connected to a battery. They now have charges of $+Q$ and $-Q$ (respectively) on their plates. (a) A parallel-plate capacitor consists of two plates of opposite charge with area A separated by distance d. (b) A rolled capacitor has a dielectric material between its two conducting sheets (plates).

A system composed of two identical parallel-conducting plates separated by a distance is called a **parallel-plate capacitor** ([link]). The magnitude of the electrical field in the space between the parallel plates is $E = \sigma / \varepsilon_0$, where σ denotes the surface charge density on one plate (recall that σ is the charge Q per the surface area A). Thus, the magnitude of the field is directly proportional to Q.



The charge separation in a capacitor shows that the charges remain on the surfaces of the capacitor plates. Electrical field lines in a parallel-plate capacitor begin with positive charges and end with negative charges. The magnitude of the electrical field in the space between the plates is in direct proportion to the amount of charge on the capacitor.

Capacitors with different physical characteristics (such as shape and size of their plates) store different amounts of charge for the same applied voltage V across their plates. The **capacitance** C of a capacitor is defined as the ratio of the maximum charge Q that can be stored in a capacitor to the applied voltage V across its plates. In other words, capacitance is the largest amount of charge per volt that can be stored on the device:

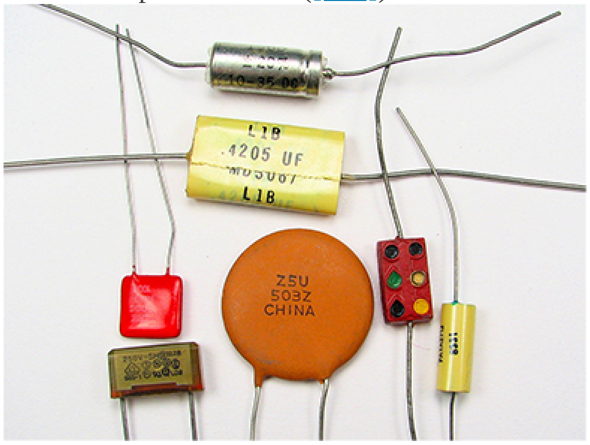
Note:
Equation:

$$C = \frac{Q}{V}.$$

The SI unit of capacitance is the farad (F), named after Michael Faraday (1791–1867). Since capacitance is the charge per unit voltage, one farad is one coulomb per one volt, or
Equation:

$$1\,\mathrm{F} = \frac{1\,\mathrm{C}}{1\,\mathrm{V}}.$$

By definition, a 1.0-F capacitor is able to store 1.0 C of charge (a very large amount of charge) when the potential difference between its plates is only 1.0 V. One farad is therefore a very large capacitance. Typical capacitance values range from picofarads $(1\,\mathrm{pF} = 10^{-12}\,\mathrm{F})$ to millifarads $(1\,\mathrm{mF} = 10^{-3}\,\mathrm{F})$, which also includes microfarads ($1\,\mu\mathrm{F} = 10^{-6}\,\mathrm{F}$). Capacitors can be produced in various shapes and sizes ([link]).



These are some typical capacitors used in electronic devices. A capacitor’s size is not necessarily related to its capacitance value. (credit: Windell Oskay)

Calculation of Capacitance

We can calculate the capacitance of a pair of conductors with the standard approach that follows.

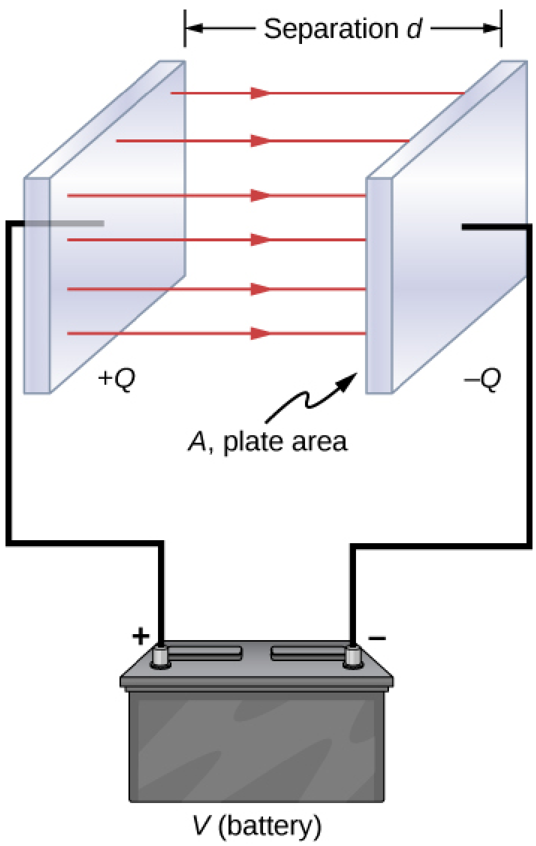
Note:
Problem-Solving Strategy: Calculating Capacitance

1. Assume that the capacitor has a charge Q.
2. Determine the electrical field $\vec{\mathbf{E}}$ between the conductors. If symmetry is present in the arrangement of conductors, you may be able to use Gauss's law for this calculation.
3. Find the potential difference between the conductors from
   Equation:
   $$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}},$$
   where the path of integration leads from one conductor to the other. The magnitude of the potential difference is then $V = |V_B - V_A|$.
4. With V known, obtain the capacitance directly from [link].

To show how this procedure works, we now calculate the capacitances of parallel-plate, spherical, and cylindrical capacitors. In all cases, we assume vacuum capacitors (empty capacitors) with no dielectric substance in the space between conductors.

Parallel-Plate Capacitor

The parallel-plate capacitor ([link]) has two identical conducting plates, each having a surface area A, separated by a distance d. When a voltage V is applied to the capacitor, it stores a charge Q, as shown. We can see how its capacitance may depend on A and d by considering characteristics of the Coulomb force. We know that force between the charges increases with charge values and decreases with the distance between them. We should expect that the bigger the plates are, the more charge they can store. Thus, C should be greater for a larger value of A. Similarly, the closer the plates are together, the greater the attraction of the opposite charges on them. Therefore, C should be greater for a smaller d.



In a parallel-plate capacitor with plates separated by a distance *d*, each plate has the same surface area *A*.

We define the surface charge density σ on the plates as

Equation:

$$\sigma = \frac{Q}{A}.$$

We know from previous chapters that when *d* is small, the electrical field between the plates is fairly uniform (ignoring edge effects) and that its magnitude is given by

Equation:

$$E = \frac{\sigma}{\varepsilon_0},$$

where the constant ε_0 is the permittivity of free space, $\varepsilon_0 = 8.85\ \times\ 10^{-12}\ \mathrm{F/m}$. The SI unit of F/m is equivalent to $\mathrm{C^2/N \cdot m^2}$. Since the electrical field $\vec{\mathbf{E}}$ between the plates is uniform, the potential difference between the plates is

Equation:

$$V = Ed = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A}.$$

Therefore [link] gives the capacitance of a parallel-plate capacitor as

Note:
Equation:

$$C = \frac{Q}{V} = \frac{Q}{Qd/\varepsilon_0 A} = \varepsilon_0 \frac{A}{d}.$$

Notice from this equation that capacitance is a function *only of the geometry* and what material fills the space between the plates (in this case, vacuum) of this capacitor. In fact, this is true not only for a parallel-plate capacitor, but for all capacitors: The capacitance is independent of Q or V. If the charge changes, the potential changes correspondingly so that Q/V remains constant.

Example:
Capacitance and Charge Stored in a Parallel-Plate Capacitor
(a) What is the capacitance of an empty parallel-plate capacitor with metal plates that each have an area of $1.00\ \text{m}^2$, separated by 1.00 mm? (b) How much charge is stored in this capacitor if a voltage of 3.00×10^3 V is applied to it?
Strategy
Finding the capacitance C is a straightforward application of [link]. Once we find C, we can find the charge stored by using [link].
Solution

a. Entering the given values into [link] yields
   Equation:

$$C = \varepsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12} \frac{\text{F}}{\text{m}}\right) \frac{1.00\ \text{m}^2}{1.00 \times 10^{-3}\ \text{m}} = 8.85 \times 10^{-9}\ \text{F} = 8.85\ \text{nF}.$$

   This small capacitance value indicates how difficult it is to make a device with a large capacitance.
b. Inverting [link] and entering the known values into this equation gives
   Equation:

$$Q = CV = (8.85 \times 10^{-9}\ \text{F})(3.00 \times 10^3\ \text{V}) = 26.6\ \mu\text{C}.$$

Significance

This charge is only slightly greater than those found in typical static electricity applications. Since air breaks down (becomes conductive) at an electrical field strength of about 3.0 MV/m, no more charge can be stored on this capacitor by increasing the voltage.

Example:
A 1-F Parallel-Plate Capacitor
Suppose you wish to construct a parallel-plate capacitor with a capacitance of 1.0 F. What area must you use for each plate if the plates are separated by 1.0 mm?
Solution
Rearranging [link], we obtain
Equation:

$$A = \frac{Cd}{\varepsilon_0} = \frac{(1.0\,\mathrm{F})(1.0\,\times\,10^{-3}\,\mathrm{m})}{8.85\,\times\,10^{-12}\,\mathrm{F/m}} = 1.1\,\times\,10^{8}\,\mathrm{m}^2.$$

Each square plate would have to be 10 km across. It used to be a common prank to ask a student to go to the laboratory stockroom and request a 1-F parallel-plate capacitor, until stockroom attendants got tired of the joke.

Note:
Exercise:

Problem:

Check Your Understanding The capacitance of a parallel-plate capacitor is 2.0 pF. If the area of each plate is $2.4\,\mathrm{cm}^2$, what is the plate separation?

Solution:

$1.1\,\times\,10^{-3}\,\mathrm{m}$

Note:
Exercise:

Problem: **Check Your Understanding** Verify that σ/V and ε_0/d have the same physical units.

Spherical Capacitor

A spherical capacitor is another set of conductors whose capacitance can be easily determined ([link]). It consists of two concentric conducting spherical shells of radii R_1 (inner shell) and R_2 (outer shell). The shells are given equal and opposite charges $+Q$ and $-Q$, respectively. From symmetry, the electrical field between the shells is directed radially outward. We can obtain the magnitude of the

field by applying Gauss's law over a spherical Gaussian surface of radius *r* concentric with the shells. The enclosed charge is $+Q$; therefore we have

Equation:

$$\oint_S \vec{\mathbf{E}} \cdot \widehat{\mathbf{n}} dA = E(4\pi r^2) = \frac{Q}{\varepsilon_0}.$$

Thus, the electrical field between the conductors is

Equation:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}.$$

We substitute this $\vec{\mathbf{E}}$ into [link] and integrate along a radial path between the shells:

Equation:

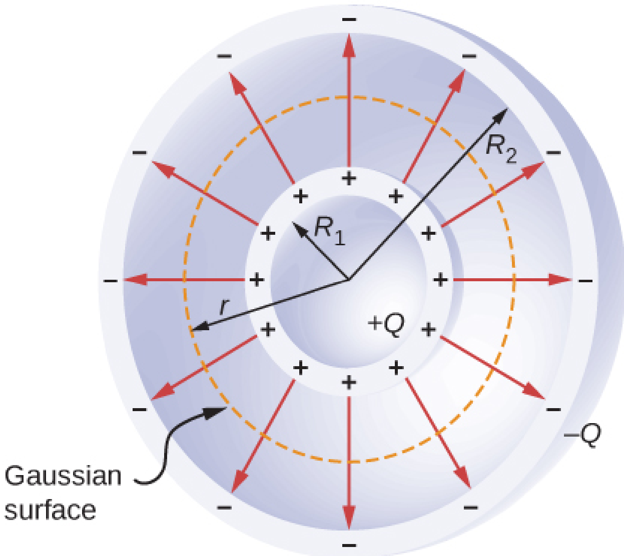
$$V = \int_{R_1}^{R_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \int_{R_1}^{R_2} \left(\frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \right) \cdot (\hat{\mathbf{r}}\, dr) = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right).$$

In this equation, the potential difference between the plates is $V = -(V_2 - V_1) = V_1 - V_2$. We substitute this result into [link] to find the capacitance of a spherical capacitor:

Note:

Equation:

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 \frac{R_1 R_2}{R_2 - R_1}.$$



A spherical capacitor consists of two concentric conducting spheres. Note that the charges on a conductor reside on its surface.

Example:

Capacitance of an Isolated Sphere

Calculate the capacitance of a single isolated conducting sphere of radius R_1 and compare it with [link] in the limit as $R_2 \to \infty$.

Strategy

We assume that the charge on the sphere is Q, and so we follow the four steps outlined earlier. We also assume the other conductor to be a concentric hollow sphere of infinite radius.

Solution

On the outside of an isolated conducting sphere, the electrical field is given by [link]. The magnitude of the potential difference between the surface of an isolated sphere and infinity is

Equation:

$$V = \int_{R_1}^{+\infty} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{+\infty} \frac{1}{r^2} \hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}\, dr) = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{+\infty} \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R_1}.$$

The capacitance of an isolated sphere is therefore

Equation:

$$C = \frac{Q}{V} = Q \frac{4\pi\varepsilon_0 R_1}{Q} = 4\pi\varepsilon_0 R_1.$$

Significance

The same result can be obtained by taking the limit of [link] as $R_2 \to \infty$. A single isolated sphere is therefore equivalent to a spherical capacitor whose outer shell has an infinitely large radius.

Note:
Exercise:

Problem:

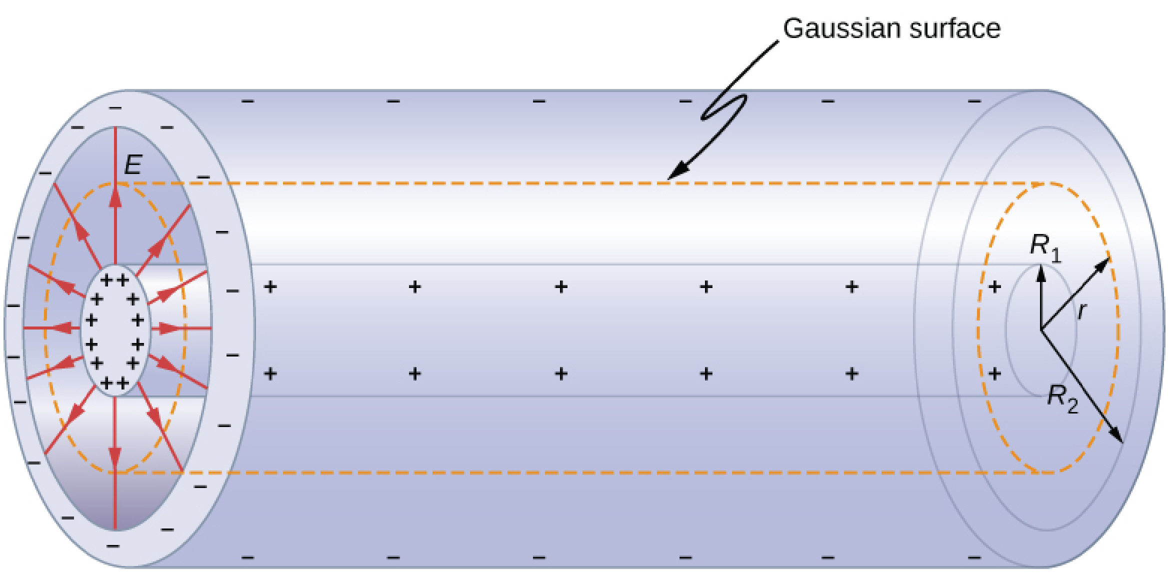
Check Your Understanding The radius of the outer sphere of a spherical capacitor is five times the radius of its inner shell. What are the dimensions of this capacitor if its capacitance is 5.00 pF?

Solution:

3.59 cm, 17.98 cm

Cylindrical Capacitor

A cylindrical capacitor consists of two concentric, conducting cylinders ([link]). The inner cylinder, of radius R_1, may either be a shell or be completely solid. The outer cylinder is a shell of inner radius R_2. We assume that the length of each cylinder is l and that the excess charges $+Q$ and $-Q$ reside on the inner and outer cylinders, respectively.



A cylindrical capacitor consists of two concentric, conducting cylinders. Here, the charge on the outer surface of the inner cylinder is positive (indicated by $+$) and the charge on the inner surface of the outer cylinder is negative (indicated by $-$).

With edge effects ignored, the electrical field between the conductors is directed radially outward from the common axis of the cylinders. Using the Gaussian surface shown in [link], we have
Equation:

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = E(2\pi r l) = \frac{Q}{\varepsilon_0}.$$

Therefore, the electrical field between the cylinders is

Equation:

$$\vec{\mathbf{E}} = \frac{1}{2\pi\varepsilon_0} \frac{Q}{r\, l} \hat{\mathbf{r}}.$$

Here $\hat{\mathbf{r}}$ is the unit radial vector along the radius of the cylinder. We can substitute into [link] and find the potential difference between the cylinders:

Equation:

$$V = \int_{R_1}^{R_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}_p = \frac{Q}{2\pi\varepsilon_0\, l} \int_{R_1}^{R_2} \frac{1}{r} \hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}\, dr) = \frac{Q}{2\pi\varepsilon_0\, l} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0\, l} \ln r|_{R_1}^{R_2} = \frac{Q}{2\pi\varepsilon_0\, l} \ln \frac{R_2}{R_1}.$$

Thus, the capacitance of a cylindrical capacitor is

Note:

Equation:

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0\, l}{\ln(R_2/R_1)}.$$

As in other cases, this capacitance depends only on the geometry of the conductor arrangement. An important application of [link] is the determination of the capacitance per unit length of a *coaxial cable*, which is commonly used to transmit time-varying electrical signals. A coaxial cable consists of two concentric, cylindrical conductors separated by an insulating material. (Here, we assume a vacuum between the conductors, but the physics is qualitatively almost the same when the space between the conductors is filled by a dielectric.) This configuration shields the electrical signal propagating down the inner conductor from stray electrical fields external to the cable. Current flows in opposite directions in the inner and the outer conductors, with the outer conductor usually grounded. Now, from [link], the capacitance per unit length of the coaxial cable is given by

Equation:

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(R_2/R_1)}.$$

In practical applications, it is important to select specific values of C/l. This can be accomplished with appropriate choices of radii of the conductors and of the insulating material between them.

Note:
Exercise:

Problem:

Check Your Understanding When a cylindrical capacitor is given a charge of 0.500 nC, a potential difference of 20.0 V is measured between the cylinders. (a) What is the capacitance of this system? (b) If the cylinders are 1.0 m long, what is the ratio of their radii?

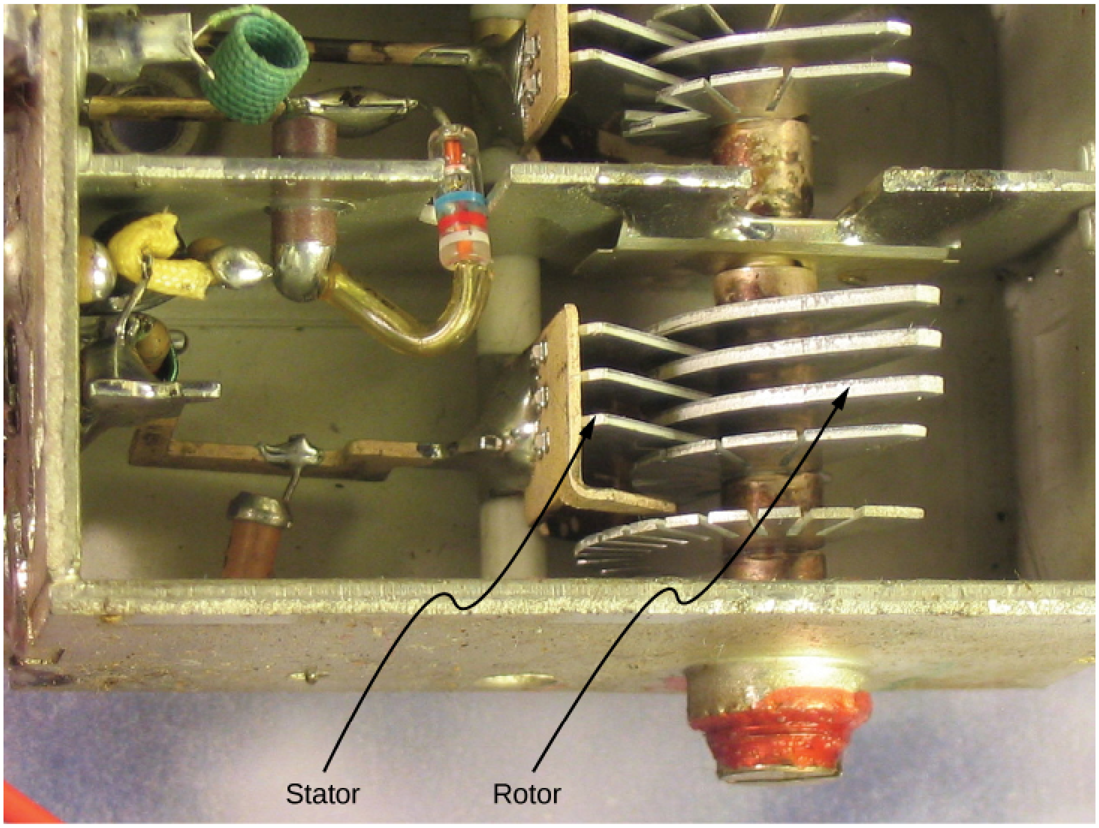
Solution:

a. 25.0 pF; b. 9.2

Several types of practical capacitors are shown in [link]. Common capacitors are often made of two small pieces of metal foil separated by two small pieces of insulation (see [link](b)). The metal foil and insulation are encased in a protective coating, and two metal leads are used for connecting the foils to an external circuit. Some common insulating materials are mica, ceramic, paper, and Teflon™ non-stick coating.

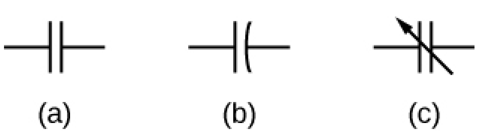
Another popular type of capacitor is an electrolytic capacitor. It consists of an oxidized metal in a conducting paste. The main advantage of an electrolytic capacitor is its high capacitance relative to other common types of capacitors. For example, capacitance of one type of aluminum electrolytic capacitor can be as high as 1.0 F. However, you must be careful when using an electrolytic capacitor in a circuit, because it only functions correctly when the metal foil is at a higher potential than the conducting paste. When reverse polarization occurs, electrolytic action destroys the oxide film. This type of capacitor cannot be connected across an alternating current source, because half of the time, ac voltage would have the wrong polarity, as an alternating current reverses its polarity (see Alternating-Current Circuts on alternating-current circuits).

A variable air capacitor ([link]) has two sets of parallel plates. One set of plates is fixed (indicated as “stator”), and the other set of plates is attached to a shaft that can be rotated (indicated as “rotor”). By turning the shaft, the cross-sectional area in the overlap of the plates can be changed; therefore, the capacitance of this system can be tuned to a desired value. Capacitor tuning has applications in any type of radio transmission and in receiving radio signals from electronic devices. Any time you tune your car radio to your favorite station, think of capacitance.



In a variable air capacitor, capacitance can be tuned by changing the effective area of the plates. (credit: modification of work by Robbie Sproule)

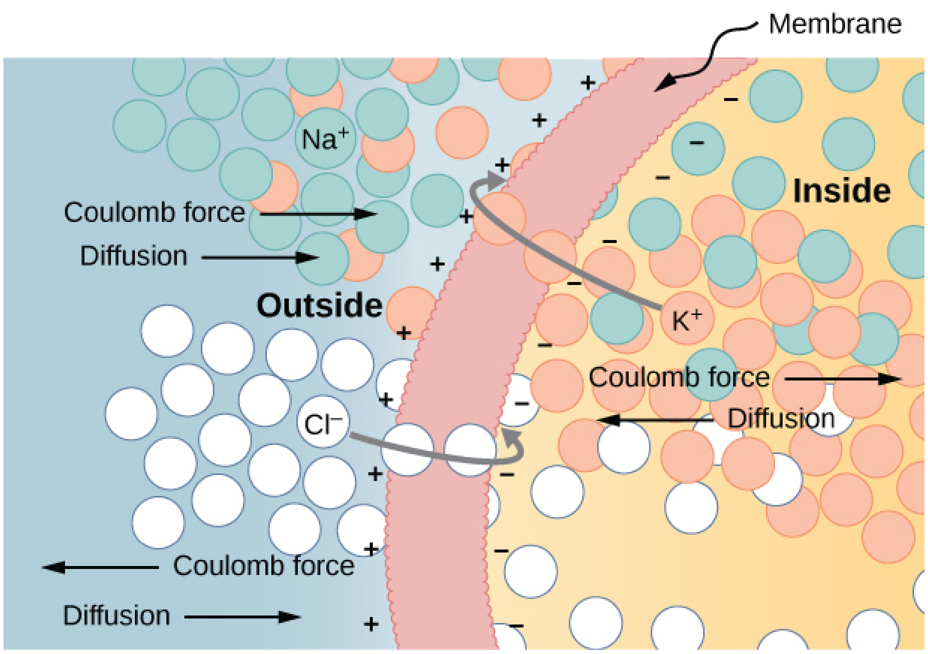
The symbols shown in [link] are circuit representations of various types of capacitors. We generally use the symbol shown in [link](a). The symbol in [link](c) represents a variable-capacitance capacitor. Notice the similarity of these symbols to the symmetry of a parallel-plate capacitor. An electrolytic capacitor is represented by the symbol in part [link](b), where the curved plate indicates the negative terminal.



This shows three different circuit representations of capacitors. The symbol in (a) is the most commonly used one. The symbol in (b) represents an electrolytic capacitor. The symbol in (c) represents a variable-capacitance capacitor.

An interesting applied example of a capacitor model comes from cell biology and deals with the electrical potential in the plasma membrane of a living cell ([link]). Cell membranes separate cells from their surroundings but allow some selected ions to pass in or out of the cell. The potential difference across a membrane is about 70 mV. The cell membrane may be 7 to 10 nm thick. Treating the cell membrane as a nano-sized capacitor, the estimate of the smallest electrical field strength across its 'plates' yields the value $E=\frac{V}{d}=\frac{70\times10^{-3}\text{V}}{10\times10^{-9}\text{m}}=7\times10^{6}\text{ V/m}>3\,\text{MV/m}.$

This magnitude of electrical field is great enough to create an electrical spark in the air.



The semipermeable membrane of a biological cell has different concentrations of ions on its interior surface than on its exterior. Diffusion moves the K^+ (potassium) and Cl^- (chloride) ions in the directions shown, until the Coulomb force halts further transfer. In this way, the exterior of the membrane acquires a positive charge and its interior surface acquires a negative charge, creating a potential difference across the membrane. The membrane is normally impermeable to Na+ (sodium ions).

Note:
Visit the PhET Explorations: Capacitor Lab to explore how a capacitor works. Change the size of the plates and add a dielectric to see the effect on capacitance. Change the voltage and see charges built

up on the plates. Observe the electrical field in the capacitor. Measure the voltage and the electrical field.

Summary

- A capacitor is a device that stores an electrical charge and electrical energy. The amount of charge a vacuum capacitor can store depends on two major factors: the voltage applied and the capacitor’s physical characteristics, such as its size and geometry.
- The capacitance of a capacitor is a parameter that tells us how much charge can be stored in the capacitor per unit potential difference between its plates. Capacitance of a system of conductors depends only on the geometry of their arrangement and physical properties of the insulating material that fills the space between the conductors. The unit of capacitance is the farad, where $1\,\mathrm{F} = 1\,\mathrm{C}/1\,\mathrm{V}$.

Conceptual Questions

Exercise:

Problem:

Does the capacitance of a device depend on the applied voltage? Does the capacitance of a device depend on the charge residing on it?

Solution:

no; yes

Exercise:

Problem:

Would you place the plates of a parallel-plate capacitor closer together or farther apart to increase their capacitance?

Exercise:

Problem: The value of the capacitance is zero if the plates are not charged. True or false?

Solution:

false

Exercise:

Problem:

If the plates of a capacitor have different areas, will they acquire the same charge when the capacitor is connected across a battery?

Exercise:

Problem:

Does the capacitance of a spherical capacitor depend on which sphere is charged positively or negatively?

Solution:

no

Problems

Exercise:

Problem: What charge is stored in a $180.0\text{-}\mu\text{F}$ capacitor when 120.0 V is applied to it?

Solution:

21.6 mC

Exercise:

Problem: Find the charge stored when 5.50 V is applied to an 8.00-pF capacitor.

Exercise:

Problem: Calculate the voltage applied to a $2.00\text{-}\mu\text{F}$ capacitor when it holds $3.10\ \mu\text{C}$ of charge.

Solution:

1.55 V

Exercise:

Problem: What voltage must be applied to an 8.00-nF capacitor to store 0.160 mC of charge?

Exercise:

Problem: What capacitance is needed to store $3.00\ \mu\text{C}$ of charge at a voltage of 120 V?

Solution:

25.0 nF

Exercise:

Problem:

What is the capacitance of a large Van de Graaff generator's terminal, given that it stores 8.00 mC of charge at a voltage of 12.0 MV?

Exercise:

Problem:

The plates of an empty parallel-plate capacitor of capacitance 5.0 pF are 2.0 mm apart. What is the area of each plate?

Solution:

$1.1 \times 10^{-3} \text{m}^2$

Exercise:

Problem:

A 60.0-pF vacuum capacitor has a plate area of 0.010 m^2. What is the separation between its plates?

Exercise:

Problem:

A set of parallel plates has a capacitance of $5.0 \mu\text{F}$. How much charge must be added to the plates to increase the potential difference between them by 100 V?

Solution:

500 μC

Exercise:

Problem:

Consider Earth to be a spherical conductor of radius 6400 km and calculate its capacitance.

Exercise:

Problem:

If the capacitance per unit length of a cylindrical capacitor is 20 pF/m, what is the ratio of the radii of the two cylinders?

Solution:

1:16

Exercise:

Problem:

An empty parallel-plate capacitor has a capacitance of $20 \mu\text{F}$. How much charge must leak off its plates before the voltage across them is reduced by 100 V?

Glossary

capacitance
: amount of charge stored per unit volt

capacitor
: device that stores electrical charge and electrical energy

dielectric
: insulating material used to fill the space between two plates

parallel-plate capacitor
: system of two identical parallel conducting plates separated by a distance

Capacitors in Series and in Parallel
By the end of this section, you will be able to:

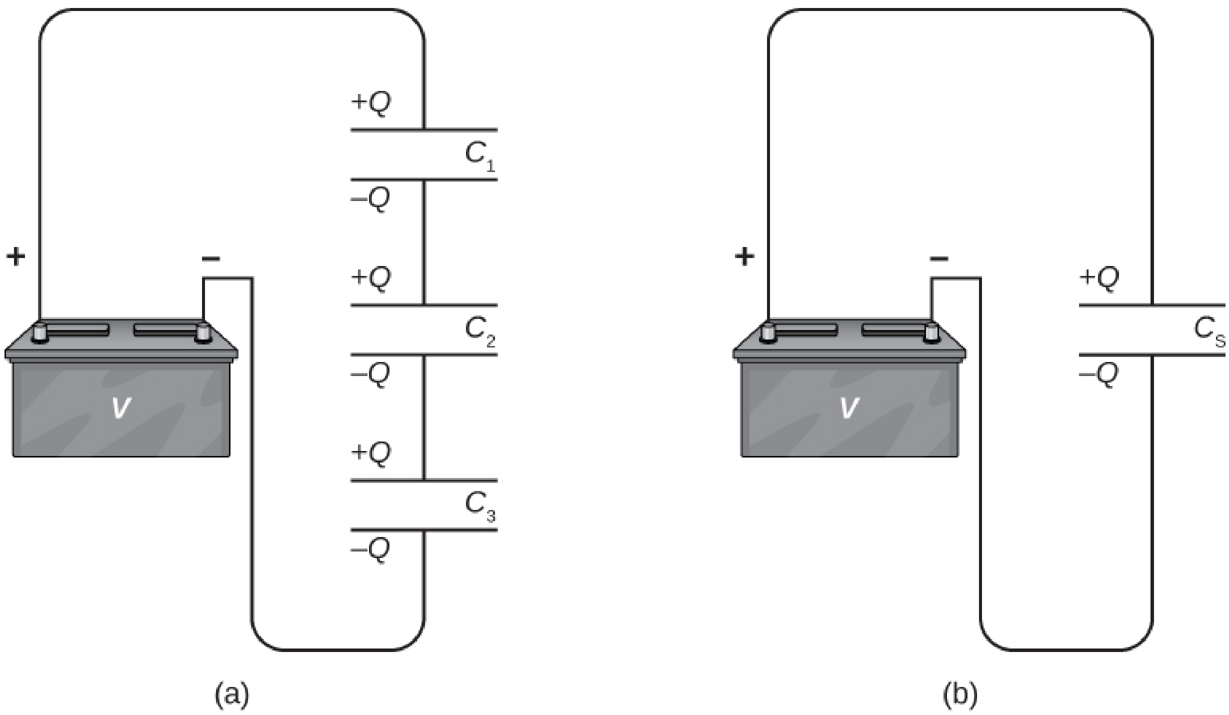
- Explain how to determine the equivalent capacitance of capacitors in series and in parallel combinations
- Compute the potential difference across the plates and the charge on the plates for a capacitor in a network and determine the net capacitance of a network of capacitors

Several capacitors can be connected together to be used in a variety of applications. Multiple connections of capacitors behave as a single equivalent capacitor. The total capacitance of this equivalent single capacitor depends both on the individual capacitors and how they are connected. Capacitors can be arranged in two simple and common types of connections, known as *series* and *parallel*, for which we can easily calculate the total capacitance. These two basic combinations, series and parallel, can also be used as part of more complex connections.

The Series Combination of Capacitors

[link] illustrates a series combination of three capacitors, arranged in a row within the circuit. As for any capacitor, the capacitance of the combination is related to the charge and voltage by using [link]. When this series combination is connected to a battery with voltage V, each of the capacitors acquires an identical charge Q. To explain, first note that the charge on the plate connected to the positive terminal of the battery is $+Q$ and the charge on the plate connected to the negative terminal is $-Q$. Charges are then induced on the other plates so that the sum of the charges on all plates, and the sum of charges on any pair of capacitor plates, is zero. However, the potential drop $V_1 = Q/C_1$ on one capacitor may be different from the potential drop $V_2 = Q/C_2$ on another capacitor, because, generally, the capacitors may have different capacitances. The series combination of two or three capacitors resembles a single capacitor with a smaller capacitance. Generally, any number of capacitors connected in series is equivalent to one capacitor whose capacitance (called the *equivalent capacitance*) is smaller than the smallest of the capacitances in the series combination. Charge on this equivalent capacitor is the same as the charge on any capacitor in a series combination: That is, *all capacitors of a series combination have the same charge*. This occurs due to

the conservation of charge in the circuit. When a charge Q in a series circuit is removed from a plate of the first capacitor (which we denote as $-Q$), it must be placed on a plate of the second capacitor (which we denote as $+Q$), and so on.



(a) Three capacitors are connected in series. The magnitude of the charge on each plate is Q. (b) The network of capacitors in (a) is equivalent to one capacitor that has a smaller capacitance than any of the individual capacitances in (a), and the charge on its plates is Q.

We can find an expression for the total (equivalent) capacitance by considering the voltages across the individual capacitors. The potentials across capacitors 1, 2, and 3 are, respectively, $V_1 = Q/C_1$, $V_2 = Q/C_2$, and $V_3 = Q/C_3$. These potentials must sum up to the voltage of the battery, giving the following potential balance:

Equation:

$$V = V_1 + V_2 + V_3.$$

Potential V is measured across an equivalent capacitor that holds charge Q and has an equivalent capacitance C_S. Entering the expressions for V_1, V_2, and V_3, we get
Equation:

$$\frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}.$$

Canceling the charge Q, we obtain an expression containing the equivalent capacitance, C_S, of three capacitors connected in series:
Equation:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

This expression can be generalized to any number of capacitors in a series network.

Note:
Series Combination
For capacitors connected in a **series combination**, the reciprocal of the equivalent capacitance is the sum of reciprocals of individual capacitances:
Equation:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots.$$

Example:
Equivalent Capacitance of a Series Network
Find the total capacitance for three capacitors connected in series, given their individual capacitances are $1.000\ \mu\text{F}$, $5.000\ \mu\text{F}$, and $8.000\ \mu\text{F}$.

Strategy
Because there are only three capacitors in this network, we can find the equivalent capacitance by using [link] with three terms.
Solution
We enter the given capacitances into [link]:
Equation:

$$\begin{array}{ccc}\frac{1}{C_\text{S}} & = & \frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\\ & = & \frac{1}{1.000\ \mu\text{F}}+\frac{1}{5.000\ \mu\text{F}}+\frac{1}{8.000\ \mu\text{F}}\\ \frac{1}{C_\text{S}} & = & \frac{1.325}{\mu\text{F}}.\end{array}$$

Now we invert this result and obtain $C_\text{S}=\frac{\mu\text{F}}{1.325}=0.755\ \mu\text{F}.$
Significance
Note that in a series network of capacitors, the equivalent capacitance is always less than the smallest individual capacitance in the network.

The Parallel Combination of Capacitors

A parallel combination of three capacitors, with one plate of each capacitor connected to one side of the circuit and the other plate connected to the other side, is illustrated in [link](a). Since the capacitors are connected in parallel, *they all have the same voltage V across their plates*. However, each capacitor in the parallel network may store a different charge. To find the equivalent capacitance C_P of the parallel network, we note that the total charge Q stored by the network is the sum of all the individual charges:
Equation:

$$Q=Q_1+Q_2+Q_3.$$

On the left-hand side of this equation, we use the relation $Q=C_\text{P}V$, which holds for the entire network. On the right-hand side of the equation, we use the relations $Q_1=C_1V,Q_2=C_2V$, and $Q_3=C_3V$ for the three capacitors in the network. In this way we obtain
Equation:

$$C_{\text{P}}V = C_1V + C_2V + C_3V.$$

This equation, when simplified, is the expression for the equivalent capacitance of the parallel network of three capacitors:

Equation:

$$C_{\text{P}} = C_1 + C_2 + C_3.$$

This expression is easily generalized to any number of capacitors connected in parallel in the network.

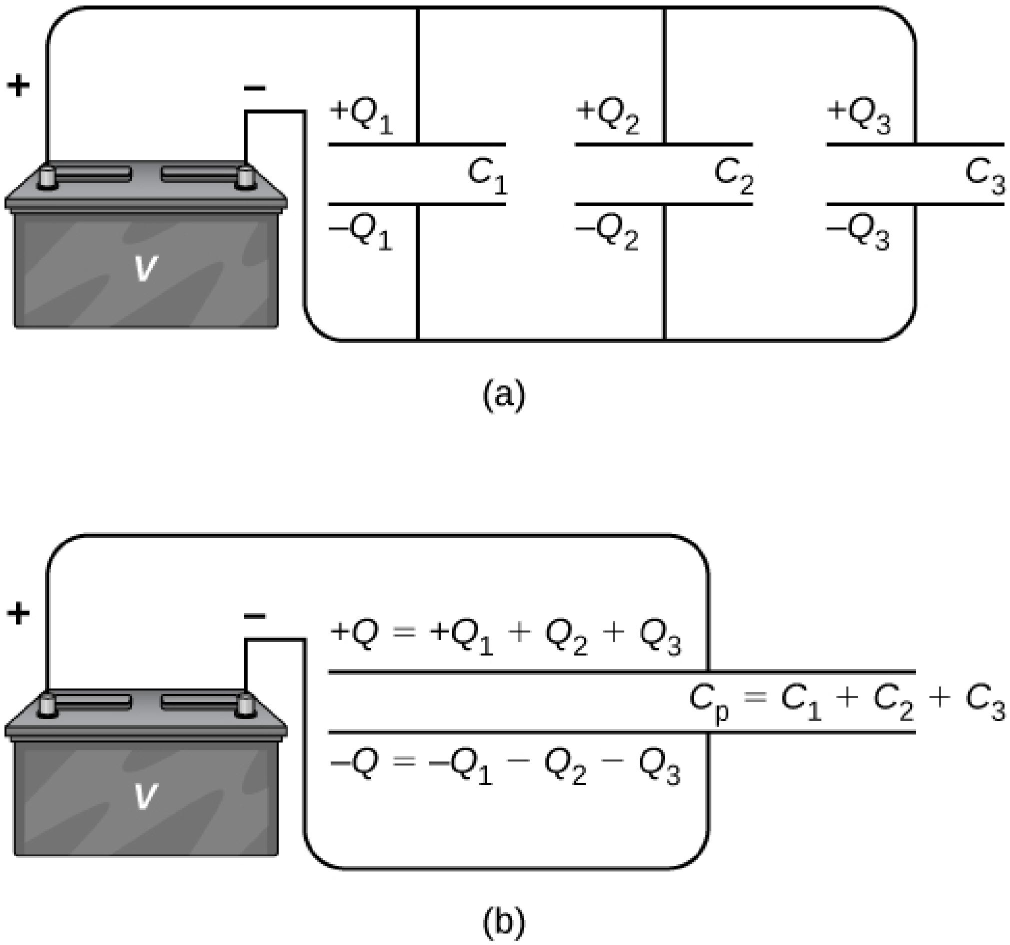
Note:

Parallel Combination

For capacitors connected in a **parallel combination**, the equivalent (net) capacitance is the sum of all individual capacitances in the network,

Equation:

$$C_{\text{P}} = C_1 + C_2 + C_3 + \cdots.$$



(a) Three capacitors are connected in parallel. Each capacitor is connected directly to the battery. (b) The charge on the equivalent capacitor is the sum of the charges on the individual capacitors.

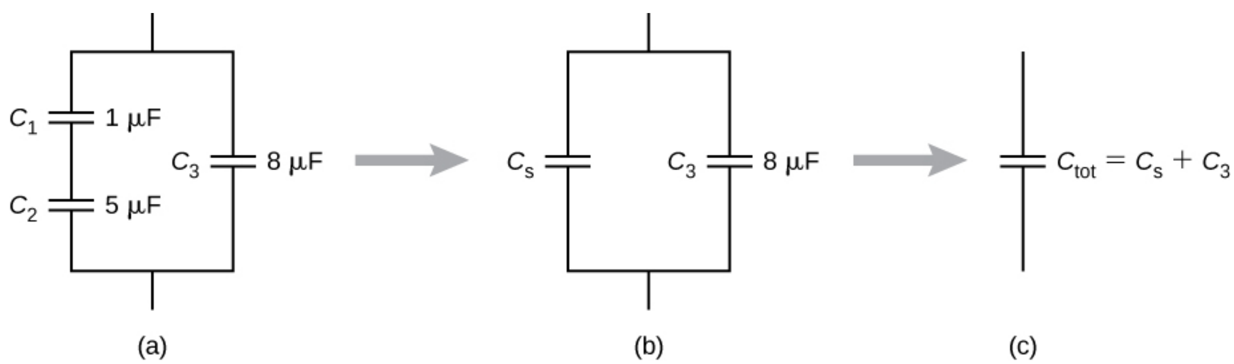
Example:
Equivalent Capacitance of a Parallel Network
Find the net capacitance for three capacitors connected in parallel, given their individual capacitances are $1.0\ \mu\text{F}, 5.0\ \mu\text{F},$ and $8.0\ \mu\text{F}.$
Strategy
Because there are only three capacitors in this network, we can find the equivalent capacitance by using [link] with three terms.

Solution
Entering the given capacitances into [link] yields
Equation:

$$\begin{aligned} C_{\text{P}} &= C_1 + C_2 + C_3 = 1.0\,\mu\text{F} + 5.0\,\mu\text{F} + 8.0\,\mu\text{F} \\ C_{\text{P}} &= 14.0\,\mu\text{F}. \end{aligned}$$

Significance
Note that in a parallel network of capacitors, the equivalent capacitance is always larger than any of the individual capacitances in the network.

Capacitor networks are usually some combination of series and parallel connections, as shown in [link]. To find the net capacitance of such combinations, we identify parts that contain only series or only parallel connections, and find their equivalent capacitances. We repeat this process until we can determine the equivalent capacitance of the entire network. The following example illustrates this process.



(a) This circuit contains both series and parallel connections of capacitors. (b) C_1 and C_2 are in series; their equivalent capacitance is C_{S}. (c) The equivalent capacitance C_{S} is connected in parallel with C_3. Thus, the equivalent capacitance of the entire network is the sum of C_{S} and C_3.

Example:

Equivalent Capacitance of a Network

Find the total capacitance of the combination of capacitors shown in [link]. Assume the capacitances are known to three decimal places $(C_1 = 1.000\ \mu\text{F}, C_2 = 5.000\ \mu\text{F}, C_3 = 8.000\ \mu\text{F})$. Round your answer to three decimal places.

Strategy

We first identify which capacitors are in series and which are in parallel. Capacitors C_1 and C_2 are in series. Their combination, labeled C_S, is in parallel with C_3.

Solution

Since C_1 and C_2 are in series, their equivalent capacitance C_S is obtained with [link]:

Equation:

$$\frac{1}{C_\text{S}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1.000\ \mu\text{F}} + \frac{1}{5.000\ \mu\text{F}} = \frac{1.200}{\mu\text{F}} \Rightarrow C_\text{S} = 0.833\ \mu\text{F}.$$

Capacitance C_S is connected in parallel with the third capacitance C_3, so we use [link] to find the equivalent capacitance C of the entire network:

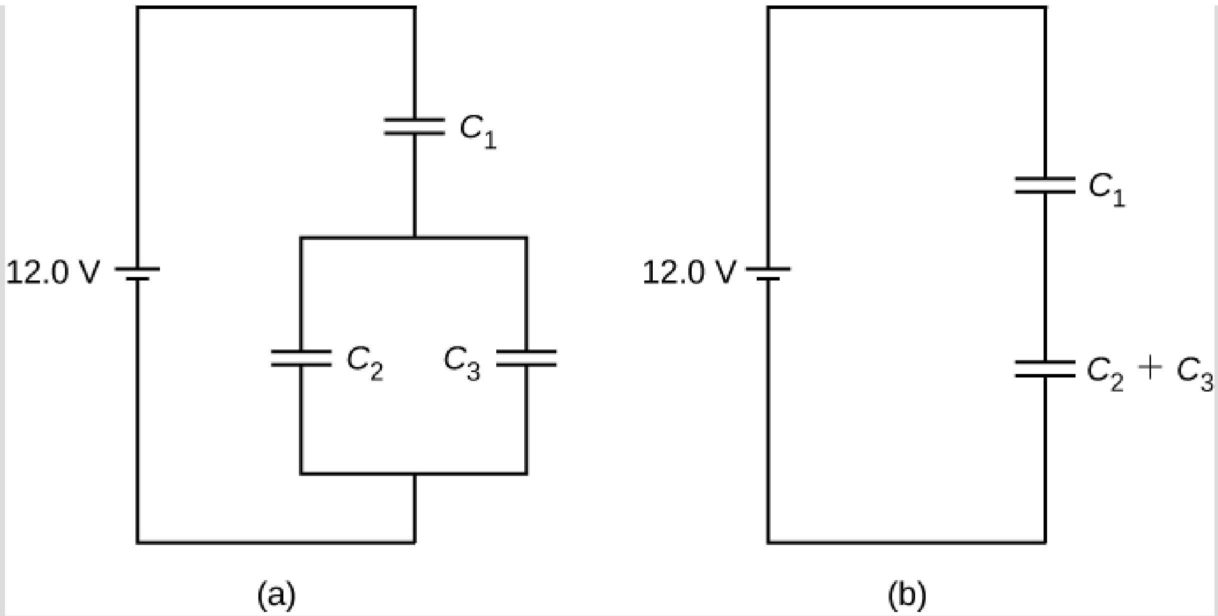
Equation:

$$C = C_\text{S} + C_3 = 0.833\ \mu\text{F} + 8.000\ \mu\text{F} = 8.833\ \mu\text{F}.$$

Example:

Network of Capacitors

Determine the net capacitance C of the capacitor combination shown in [link] when the capacitances are $C_1 = 12.0\ \mu\text{F}, C_2 = 2.0\ \mu\text{F}$, and $C_3 = 4.0\ \mu\text{F}$. When a 12.0-V potential difference is maintained across the combination, find the charge and the voltage across each capacitor.



(a) A capacitor combination. (b) An equivalent two-capacitor combination.

Strategy
We first compute the net capacitance C_{23} of the parallel connection C_2 and C_3. Then C is the net capacitance of the series connection C_1 and C_{23}. We use the relation $C = Q/V$ to find the charges Q_1,Q_2, and Q_3, and the voltages V_1, V_2, and V_3, across capacitors 1, 2, and 3, respectively.
Solution
The equivalent capacitance for C_2 and C_3 is
Equation:

$$C_{23} = C_2 + C_3 = 2.0\,\mu\text{F} + 4.0\,\mu\text{F} = 6.0\,\mu\text{F}.$$

The entire three-capacitor combination is equivalent to two capacitors in series,
Equation:

$$\frac{1}{C} = \frac{1}{12.0\,\mu\text{F}} + \frac{1}{6.0\,\mu\text{F}} = \frac{1}{4.0\,\mu\text{F}} \Rightarrow C = 4.0\,\mu\text{F}.$$

Consider the equivalent two-capacitor combination in [link](b). Since the capacitors are in series, they have the same charge, $Q_1 = Q_{23}$. Also, the

capacitors share the 12.0-V potential difference, so
Equation:

$$12.0\text{ V} = V_1 + V_{23} = \frac{Q_1}{C_1} + \frac{Q_{23}}{C_{23}} = \frac{Q_1}{12.0\ \mu\text{F}} + \frac{Q_1}{6.0\ \mu\text{F}} \Rightarrow Q_1 = 48.0\ \mu\text{C}.$$

Now the potential difference across capacitor 1 is
Equation:

$$V_1 = \frac{Q_1}{C_1} = \frac{48.0\ \mu\text{C}}{12.0\ \mu\text{F}} = 4.0\text{ V}.$$

Because capacitors 2 and 3 are connected in parallel, they are at the same potential difference:
Equation:

$$V_2 = V_3 = 12.0\text{ V} - 4.0\text{ V} = 8.0\text{ V}.$$

Hence, the charges on these two capacitors are, respectively,
Equation:

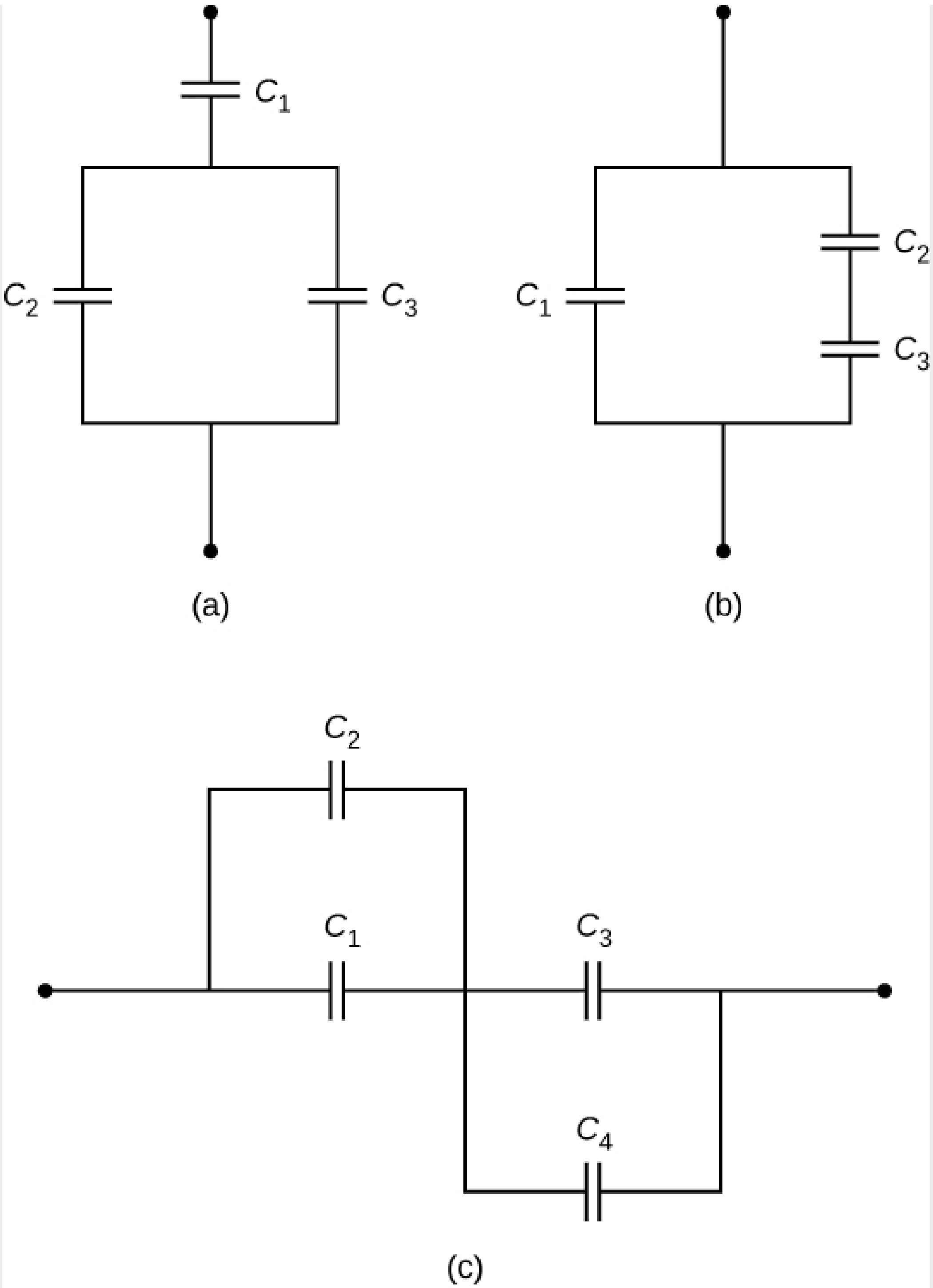
$$\begin{aligned} Q_2 &= C_2V_2 = (2.0\ \mu\text{F})(8.0\text{ V}) = 16.0\ \mu\text{C}, \\ Q_3 &= C_3V_3 = (4.0\ \mu\text{F})(8.0\text{ V}) = 32.0\ \mu\text{C}. \end{aligned}$$

Significance
As expected, the net charge on the parallel combination of C_2 and C_3 is $Q_{23} = Q_2 + Q_3 = 48.0\ \mu\text{C}$.

Note:
Exercise:

Problem:

Check Your Understanding Determine the net capacitance C of each network of capacitors shown below. Assume that $C_1 = 1.0\text{ pF}$, $C_2 = 2.0\text{ pF}$, $C_3 = 4.0\text{ pF}$, and $C_4 = 5.0\text{ pF}$. Find the charge on each capacitor, assuming there is a potential difference of 12.0 V across each network.



Solution:

a. $C = 0.86\,\text{pF}$, $Q_1 = 10\,\text{pC}$, $Q_2 = 3.4\,\text{pC}$, $Q_3 = 6.8\,\text{pC}$;
b. $C = 2.3\,\text{pF}$, $Q_1 = 12\,\text{pC}$, $Q_2 = Q_3 = 16\,\text{pC}$;

c. $C = 2.3\,\text{pF}, Q_1 = 9.0\,\text{pC}, Q_2 = 18\,\text{pC}, Q_3 = 12\,\text{pC}, Q_4 = 15\,\text{pC}$

Summary

- When several capacitors are connected in a series combination, the reciprocal of the equivalent capacitance is the sum of the reciprocals of the individual capacitances.
- When several capacitors are connected in a parallel combination, the equivalent capacitance is the sum of the individual capacitances.
- When a network of capacitors contains a combination of series and parallel connections, we identify the series and parallel networks, and compute their equivalent capacitances step by step until the entire network becomes reduced to one equivalent capacitance.

Conceptual Questions

Exercise:

Problem:

If you wish to store a large amount of charge in a capacitor bank, would you connect capacitors in series or in parallel? Explain.

Exercise:

Problem:

What is the maximum capacitance you can get by connecting three 1.0-μF capacitors? What is the minimum capacitance?

Solution:

$3.0\ \mu\text{F}, 0.33\ \mu\text{F}$

Problems

Exercise:

Problem:

A 4.00-pF is connected in series with an 8.00-pF capacitor and a 400-V potential difference is applied across the pair. (a) What is the charge on each capacitor? (b) What is the voltage across each capacitor?

Solution:

a. 1.07 nC; b. 267 V, 133 V

Exercise:

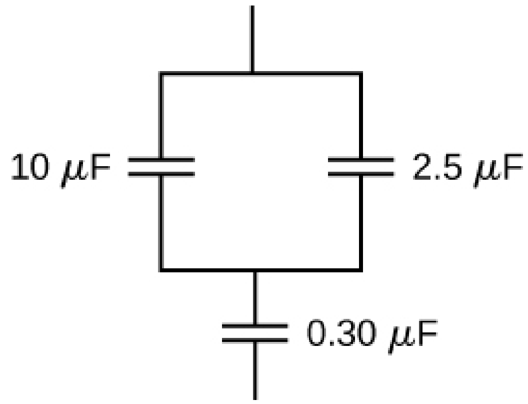
Problem:

Three capacitors, with capacitances of $C_1 = 2.0\ \mu\text{F}$,$C_2 = 3.0\ \mu\text{F}$, and $C_3 = 6.0\ \mu\text{F}$, respectively, are connected in parallel. A 500-V potential difference is applied across the combination. Determine the voltage across each capacitor and the charge on each capacitor.

Exercise:

Problem:

Find the total capacitance of this combination of series and parallel capacitors shown below.



Solution:

$0.29\ \mu\text{F}$

Exercise:

Problem:

Suppose you need a capacitor bank with a total capacitance of 0.750 F but you have only 1.50-mF capacitors at your disposal. What is the smallest number of capacitors you could connect together to achieve your goal, and how would you connect them?

Solution:

500 capacitors; connected in parallel

Exercise:

Problem:

What total capacitances can you make by connecting a $5.00\text{-}\mu\text{F}$ and a $8.00\text{-}\mu\text{F}$ capacitor?

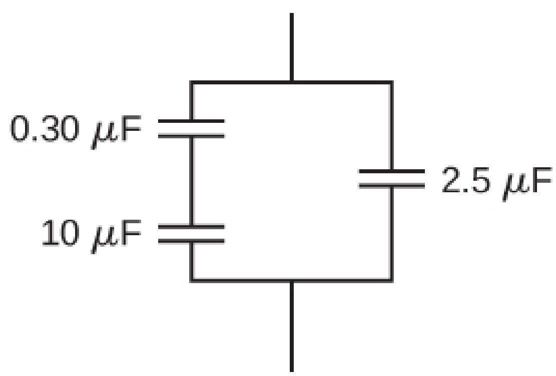
Solution:

$3.08\ \mu\text{F}$ (series) and $13.0\ \mu\text{F}$ (parallel)

Exercise:

Problem:

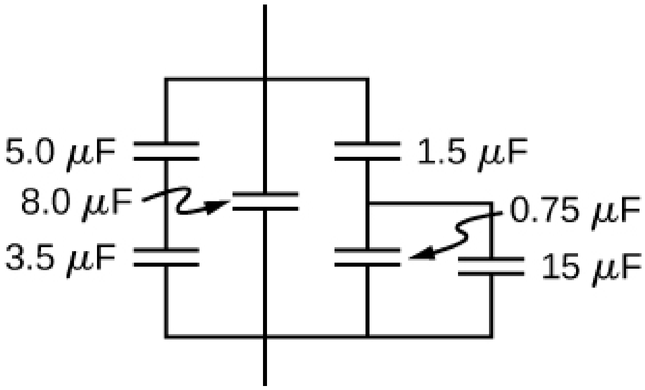
Find the equivalent capacitance of the combination of series and parallel capacitors shown below.



Exercise:

Problem:

Find the net capacitance of the combination of series and parallel capacitors shown below.



Solution:

11.4 μF

Exercise:

Problem:

A 40-pF capacitor is charged to a potential difference of 500 V. Its terminals are then connected to those of an uncharged 10-pF capacitor. Calculate: (a) the original charge on the 40-pF capacitor; (b) the charge on each capacitor after the connection is made; and (c) the potential difference across the plates of each capacitor after the connection.

Exercise:

Problem:

A 2.0-μF capacitor and a 4.0-μF capacitor are connected in series across a 1.0-kV potential. The charged capacitors are then disconnected from the source and connected to each other with terminals of like sign together. Find the charge on each capacitor and the voltage across each capacitor.

Solution:

0.89 mC; 1.78 mC; 444 V

Glossary

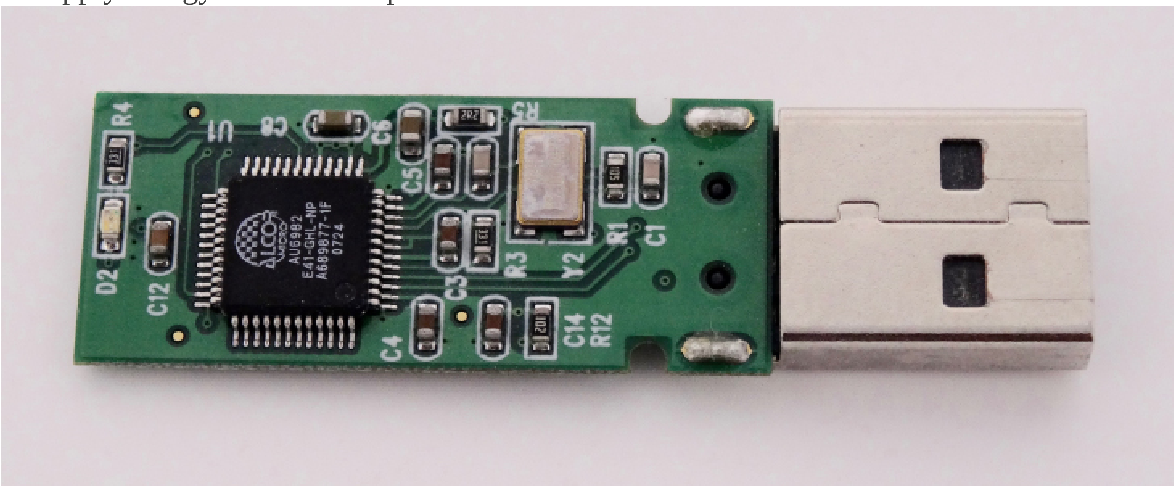
parallel combination
: components in a circuit arranged with one side of each component connected to one side of the circuit and the other sides of the components connected to the other side of the circuit

series combination
: components in a circuit arranged in a row one after the other in a circuit

Energy Stored in a Capacitor
By the end of this section, you will be able to:

- Explain how energy is stored in a capacitor
- Use energy relations to determine the energy stored in a capacitor network

Most of us have seen dramatizations of medical personnel using a defibrillator to pass an electrical current through a patient's heart to get it to beat normally. Often realistic in detail, the person applying the shock directs another person to "make it 400 joules this time." The energy delivered by the defibrillator is stored in a capacitor and can be adjusted to fit the situation. SI units of joules are often employed. Less dramatic is the use of capacitors in microelectronics to supply energy when batteries are charged ([link]). Capacitors are also used to supply energy for flash lamps on cameras.



The capacitors on the circuit board for an electronic device follow a labeling convention that identifies each one with a code that begins with the letter "C." (credit: Windell Oskay)

The energy U_C stored in a capacitor is electrostatic potential energy and is thus related to the charge Q and voltage V between the capacitor plates. A charged capacitor stores energy in the electrical field between its plates. As the capacitor is being charged, the electrical field builds up. When a charged capacitor is disconnected from a battery, its energy remains in the field in the space between its plates.

To gain insight into how this energy may be expressed (in terms of Q and V), consider a charged, empty, parallel-plate capacitor; that is, a capacitor without a dielectric but with a vacuum between its plates. The space between its plates has a volume Ad, and it is filled with a uniform electrostatic field E. The total energy U_C of the capacitor is contained within this space. The **energy density** u_E in this space is simply U_C divided by the volume Ad. If we know the energy density, the energy can be found as $U_C = u_E(Ad)$. We will learn in Electromagnetic Waves (after completing the study of Maxwell's equations) that the energy

density u_E in a region of free space occupied by an electrical field E depends only on the magnitude of the field and is

Note:
Equation:

$$u_E = \frac{1}{2}\varepsilon_0 E^2.$$

If we multiply the energy density by the volume between the plates, we obtain the amount of energy stored between the plates of a parallel-plate capacitor:
$U_C = u_E(Ad) = \frac{1}{2}\varepsilon_0 E^2 Ad = \frac{1}{2}\varepsilon_0 \frac{V^2}{d^2} Ad = \frac{1}{2}V^2 \varepsilon_0 \frac{A}{d} = \frac{1}{2}V^2 C.$

In this derivation, we used the fact that the electrical field between the plates is uniform so that $E = V/d$ and $C = \varepsilon_0 A/d$. Because $C = Q/V$, we can express this result in other equivalent forms:

Note:
Equation:

$$U_C = \frac{1}{2}V^2 C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV.$$

The expression in [link] for the energy stored in a parallel-plate capacitor is generally valid for all types of capacitors. To see this, consider any uncharged capacitor (not necessarily a parallel-plate type). At some instant, we connect it across a battery, giving it a potential difference $V = q/C$ between its plates. Initially, the charge on the plates is $Q = 0$. As the capacitor is being charged, the charge gradually builds up on its plates, and after some time, it reaches the value Q. To move an infinitesimal charge dq from the negative plate to the positive plate (from a lower to a higher potential), the amount of work dW that must be done on dq is $dW = Vdq = \frac{q}{C}dq$.

This work becomes the energy stored in the electrical field of the capacitor. In order to charge the capacitor to a charge Q, the total work required is
Equation:

$$W=\int_0^{W(Q)} dW=\int_0^Q \frac{q}{C}\,dq=\frac{1}{2}\,\frac{Q^2}{C}.$$

Since the geometry of the capacitor has not been specified, this equation holds for any type of capacitor. The total work W needed to charge a capacitor is the electrical potential energy U_C stored in it, or $U_C=W$. When the charge is expressed in coulombs, potential is expressed in volts, and the capacitance is expressed in farads, this relation gives the energy in joules.

Knowing that the energy stored in a capacitor is $U_C=Q^2/(2C)$, we can now find the energy density u_E stored in a vacuum between the plates of a charged parallel-plate capacitor. We just have to divide U_C by the volume Ad of space between its plates and take into account that for a parallel-plate capacitor, we have $E=\sigma/\varepsilon_0$ and $C=\varepsilon_0 A/d$. Therefore, we obtain

Equation:

$$u_E=\frac{U_C}{Ad}=\frac{1}{2}\,\frac{Q^2}{C}\,\frac{1}{Ad}=\frac{1}{2}\,\frac{Q^2}{\varepsilon_0 A/d}\,\frac{1}{Ad}=\frac{1}{2}\,\frac{1}{\varepsilon_0}\left(\frac{Q}{A}\right)^2=\frac{\sigma^2}{2\varepsilon_0}=\frac{(E\varepsilon_0)^2}{2\varepsilon_0}=\frac{\varepsilon_0}{2}E^2.$$

We see that this expression for the density of energy stored in a parallel-plate capacitor is in accordance with the general relation expressed in [link]. We could repeat this calculation for either a spherical capacitor or a cylindrical capacitor—or other capacitors—and in all cases, we would end up with the general relation given by [link].

Example:

Energy Stored in a Capacitor

Calculate the energy stored in the capacitor network in [link](a) when the capacitors are fully charged and when the capacitances are $C_1=12.0\,\mu\text{F}$, $C_2=2.0\,\mu\text{F}$, and $C_3=4.0\,\mu\text{F}$, respectively.

Strategy

We use [link] to find the energy U_1, U_2, and U_3 stored in capacitors 1, 2, and 3, respectively. The total energy is the sum of all these energies.

Solution

We identify $C_1=12.0\,\mu\text{F}$ and $V_1=4.0\text{ V}$, $C_2=2.0\,\mu\text{F}$ and $V_2=8.0\text{ V}$, $C_3=4.0\,\mu\text{F}$ and $V_3=8.0\text{ V}$. The energies stored in these capacitors are

Equation:

$$\begin{aligned}U_1&=\tfrac{1}{2}C_1V_1^2=\tfrac{1}{2}(12.0\,\mu\text{F})(4.0\text{ V})^2=96\,\mu\text{J},\\ U_2&=\tfrac{1}{2}C_2V_2^2=\tfrac{1}{2}(2.0\,\mu\text{F})(8.0\text{ V})^2=64\,\mu\text{J},\\ U_3&=\tfrac{1}{2}C_3V_3^2=\tfrac{1}{2}(4.0\,\mu\text{F})(8.0\text{ V})^2=130\,\mu\text{J}.\end{aligned}$$

The total energy stored in this network is

Equation:

$$U_C = U_1 + U_2 + U_3 = 96\,\mu\text{J} + 64\,\mu\text{J} + 130\,\mu\text{J} = 0.29\,\text{mJ}.$$

Significance
We can verify this result by calculating the energy stored in the single 4.0-μF capacitor, which is found to be equivalent to the entire network. The voltage across the network is 12.0 V. The total energy obtained in this way agrees with our previously obtained result, $U_C = \frac{1}{2}CV^2 = \frac{1}{2}(4.0\,\mu\text{F})(12.0\,\text{V})^2 = 0.29\,\text{mJ}.$

Note:
Exercise:

Problem:

Check Your Understanding The potential difference across a 5.0-pF capacitor is 0.40 V. (a) What is the energy stored in this capacitor? (b) The potential difference is now increased to 1.20 V. By what factor is the stored energy increased?

Solution:

a.4.0×10^{-13}J; b. 9 times

In a cardiac emergency, a portable electronic device known as an automated external defibrillator (AED) can be a lifesaver. A **defibrillator** ([link]) delivers a large charge in a short burst, or a shock, to a person’s heart to correct abnormal heart rhythm (an arrhythmia). A heart attack can arise from the onset of fast, irregular beating of the heart—called cardiac or ventricular fibrillation. Applying a large shock of electrical energy can terminate the arrhythmia and allow the body’s natural pacemaker to resume its normal rhythm. Today, it is common for ambulances to carry AEDs. AEDs are also found in many public places. These are designed to be used by lay persons. The device automatically diagnoses the patient’s heart rhythm and then applies the shock with appropriate energy and waveform. CPR (cardiopulmonary resuscitation) is recommended in many cases before using a defibrillator.

Automated external defibrillators are found in many public places. These portable units provide verbal instructions for use in the important first few minutes for a person suffering a cardiac attack. (credit: Owain Davies)

Example:

Capacitance of a Heart Defibrillator

A heart defibrillator delivers $4.00 \times 10^2 \text{J}$ of energy by discharging a capacitor initially at 1.00×10^4 V. What is its capacitance?

Strategy

We are given U_C and V, and we are asked to find the capacitance C. We solve [link] for C and substitute.

Solution

Solving this expression for C and entering the given values yields

$$C = 2\frac{U_C}{V^2} = 2\frac{4.00 \times 10^2 \text{ J}}{(1.00 \times 10^4 \text{ V})^2} = 8.00\ \mu\text{F}.$$

Summary

- Capacitors are used to supply energy to a variety of devices, including defibrillators, microelectronics such as calculators, and flash lamps.
- The energy stored in a capacitor is the work required to charge the capacitor, beginning with no charge on its plates. The energy is stored in the electrical field in the space between the capacitor plates. It depends on the amount of electrical charge on the plates and on the potential difference between the plates.
- The energy stored in a capacitor network is the sum of the energies stored on individual capacitors in the network. It can be computed as the energy stored in the equivalent capacitor of the network.

Conceptual Questions

Exercise:

Problem:

If you wish to store a large amount of energy in a capacitor bank, would you connect capacitors in series or parallel? Explain.

Problems

Exercise:

Problem:

How much energy is stored in an $8.00\text{-}\mu F$ capacitor whose plates are at a potential difference of 6.00 V?

Exercise:

Problem:

A capacitor has a charge of $2.5\ \mu C$ when connected to a 6.0-V battery. How much energy is stored in this capacitor?

Solution:

$7.5\ \mu J$

Exercise:

Problem:

How much energy is stored in the electrical field of a metal sphere of radius 2.0 m that is kept at a 10.0-V potential?

Exercise:

Problem:

(a) What is the energy stored in the $10.0\text{-}\mu\text{F}$ capacitor of a heart defibrillator charged to 9.00×10^3 V ? (b) Find the amount of the stored charge.

Solution:

a. 405 J; b. 90.0 mC

Exercise:

Problem:

In open-heart surgery, a much smaller amount of energy will defibrillate the heart. (a) What voltage is applied to the $8.00\text{-}\mu\text{F}$ capacitor of a heart defibrillator that stores 40.0 J of energy? (b) Find the amount of the stored charge.

Exercise:

Problem:

A $165\text{-}\mu\text{F}$ capacitor is used in conjunction with a dc motor. How much energy is stored in it when 119 V is applied?

Solution:

1.17 J

Exercise:

Problem:

Suppose you have a 9.00-V battery, a $2.00\text{-}\mu\text{F}$ capacitor, and a $7.40\text{-}\mu\text{F}$ capacitor. (a) Find the charge and energy stored if the capacitors are connected to the battery in series. (b) Do the same for a parallel connection.

Exercise:

Problem:

An anxious physicist worries that the two metal shelves of a wood frame bookcase might obtain a high voltage if charged by static electricity, perhaps produced by friction. (a) What is the capacitance of the empty shelves if they have area $1.00 \times 10^2\ \text{m}^2$ and are 0.200 m apart? (b) What is the voltage between them if opposite charges of magnitude 2.00 nC are placed on them? (c) To show that this voltage poses a small hazard, calculate the energy stored. (d) The actual shelves have an area 100 times smaller than these hypothetical shelves. Are his fears justified?

Solution:

a. 4.43×10^{-9} F; b. 0.453 V; c. 4.53×10^{-10} J; d. no

Exercise:

Problem:

A parallel-plate capacitor is made of two square plates 25 cm on a side and 1.0 mm apart. The capacitor is connected to a 50.0-V battery. With the battery still connected, the plates are pulled apart to a separation of 2.00 mm. What are the energies stored in the capacitor before and after the plates are pulled farther apart? Why does the energy decrease even though work is done in separating the plates?

Exercise:

Problem:

Suppose that the capacitance of a variable capacitor can be manually changed from 100 pF to 800 pF by turning a dial, connected to one set of plates by a shaft, from 0° to 180°. With the dial set at 180° (corresponding to $C = 800\ \text{pF}$), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to 0°. If friction is negligible, how much work is required to turn the dial from 180° to 0° ?

Solution:

0.7 mJ

Glossary

energy density
: energy stored in a capacitor divided by the volume between the plates

Capacitor with a Dielectric
By the end of this section, you will be able to:

- Describe the effects a dielectric in a capacitor has on capacitance and other properties
- Calculate the capacitance of a capacitor containing a dielectric

As we discussed earlier, an insulating material placed between the plates of a capacitor is called a dielectric. Inserting a dielectric between the plates of a capacitor affects its capacitance. To see why, let's consider an experiment described in [link]. Initially, a capacitor with capacitance C_0 when there is air between its plates is charged by a battery to voltage V_0. When the capacitor is fully charged, the battery is disconnected. A charge Q_0 then resides on the plates, and the potential difference between the plates is measured to be V_0. Now, suppose we insert a dielectric that *totally* fills the gap between the plates. If we monitor the voltage, we find that the voltmeter reading has dropped to a *smaller* value *V*. We write this new voltage value as a fraction of the original voltage V_0, with a positive number κ, $\kappa > 1$:
Equation:

$$V = \frac{1}{\kappa} V_0.$$

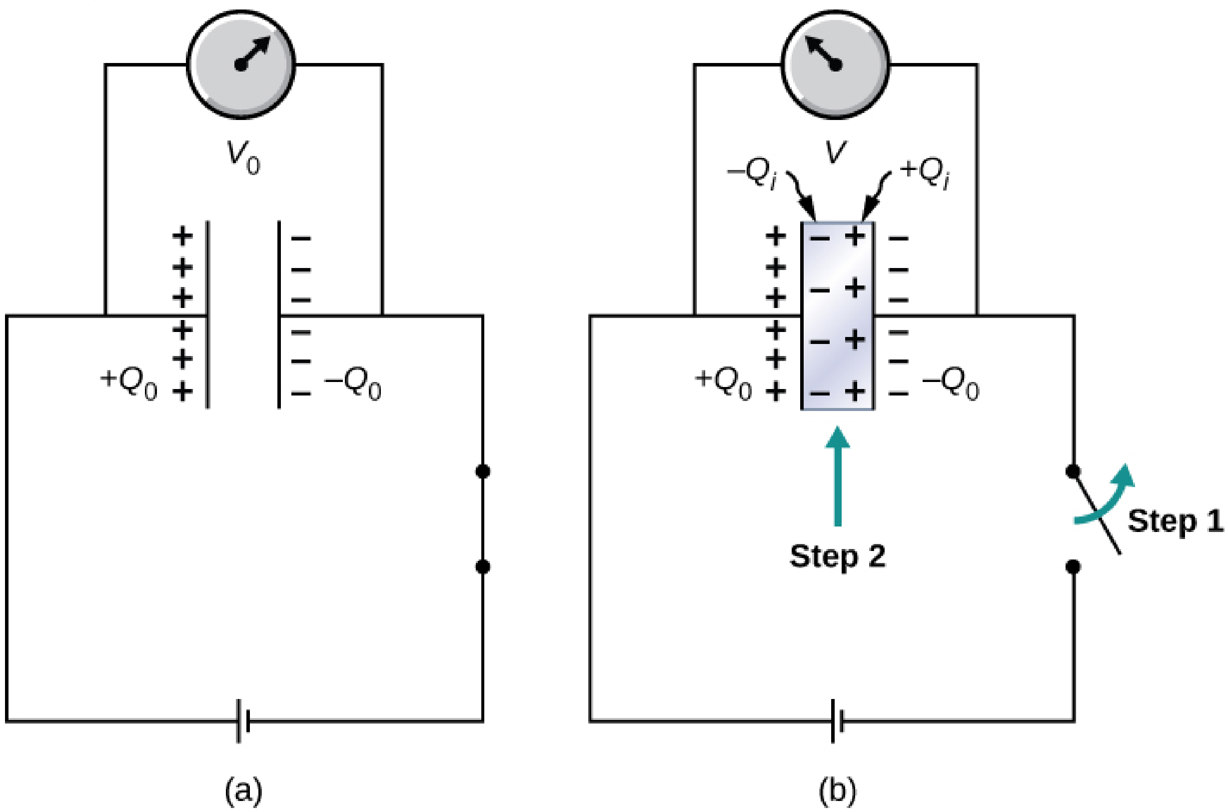
The constant κ in this equation is called the **dielectric constant** of the material between the plates, and its value is characteristic for the material. A detailed explanation for why the dielectric reduces the voltage is given in the next section. Different materials have different dielectric constants (a table of values for typical materials is provided in the next section). Once the battery becomes disconnected, there is no path for a charge to flow to the battery from the capacitor plates. Hence, the insertion of the dielectric has no effect on the charge on the plate, which remains at a value of Q_0. Therefore, we find that the capacitance of the capacitor with a dielectric is

Note:

Equation:

$$C = \frac{Q_0}{V} = \frac{Q_0}{V_0/\kappa} = \kappa\frac{Q_0}{V_0} = \kappa C_0.$$

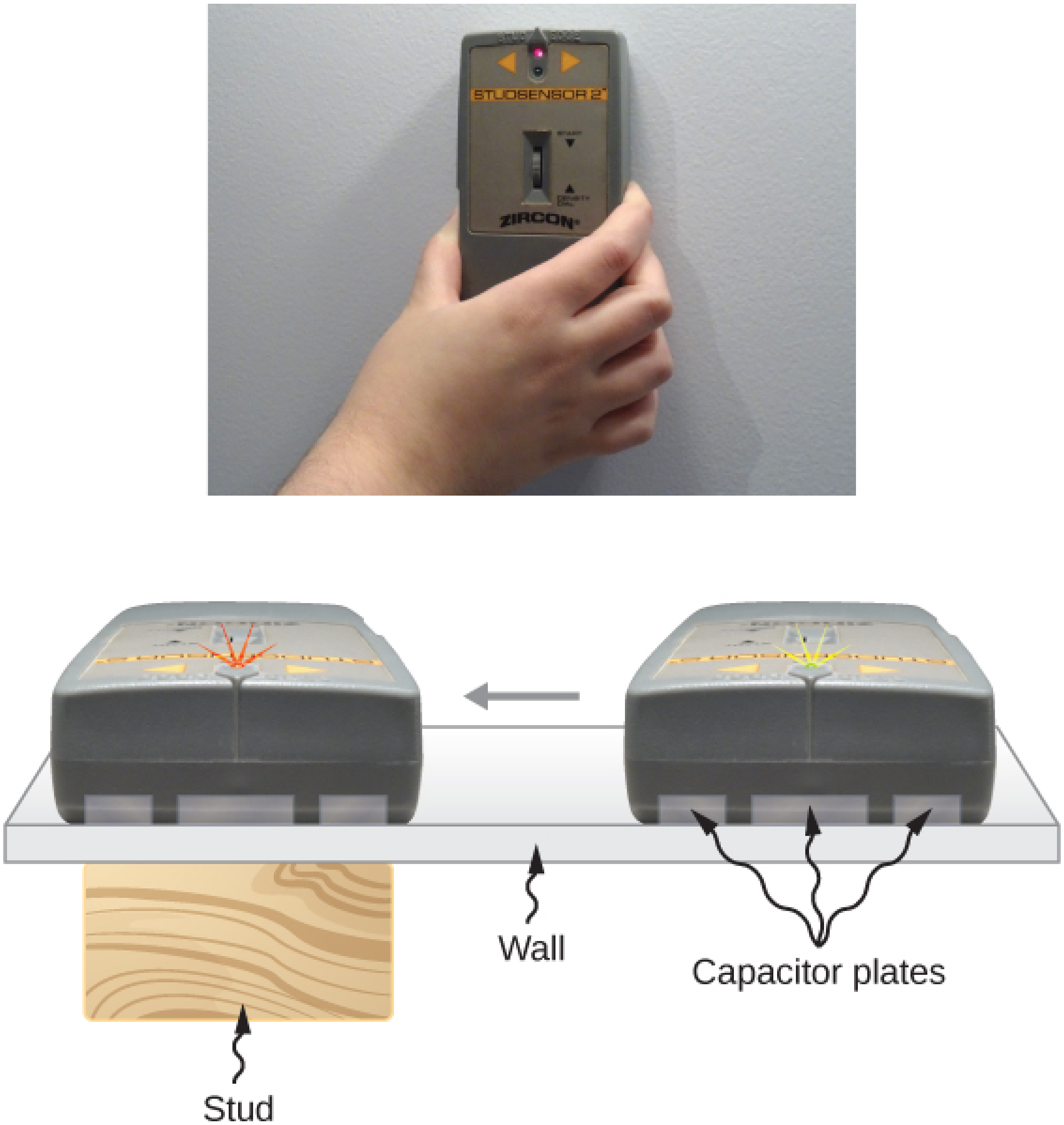
This equation tells us that the *capacitance* C_0 *of an empty (vacuum) capacitor can be increased by a factor of* κ *when we insert a dielectric material to completely fill the space between its plates*. Note that [link] can also be used for an empty capacitor by setting $\kappa = 1$. In other words, we can say that the dielectric constant of the vacuum is 1, which is a reference value.



(a) When fully charged, a vacuum capacitor has a voltage V_0 and charge Q_0 (the charges remain on plate's inner surfaces; the schematic indicates the sign of charge on each plate). (b) In step 1, the battery is disconnected. Then, in step 2, a dielectric (that is electrically neutral) is inserted into the charged capacitor. When the voltage across the

capacitor is now measured, it is found that the voltage value has decreased to $V = V_0/\kappa$. The schematic indicates the sign of the induced charge that is now present on the surfaces of the dielectric material between the plates.

The principle expressed by [link] is widely used in the construction industry ([link]). Metal plates in an electronic stud finder act effectively as a capacitor. You place a stud finder with its flat side on the wall and move it continually in the horizontal direction. When the finder moves over a wooden stud, the capacitance of its plates changes, because wood has a different dielectric constant than a gypsum wall. This change triggers a signal in a circuit, and thus the stud is detected.



An electronic stud finder is used to detect wooden studs behind drywall. (credit top: modification of work by Jane Whitney)

The electrical energy stored by a capacitor is also affected by the presence of a dielectric. When the energy stored in an empty capacitor is U_0, the energy U stored in a capacitor with a dielectric is smaller by a factor of κ,

Note:
Equation:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q_0^2}{\kappa C_0} = \frac{1}{\kappa}U_0.$$

As a dielectric material sample is brought near an empty charged capacitor, the sample reacts to the electrical field of the charges on the capacitor plates. Just as we learned in Electric Charges and Fields on electrostatics, there will be the induced charges on the surface of the sample; however, they are not free charges like in a conductor, because a perfect insulator does not have freely moving charges. These induced charges on the dielectric surface are of an opposite sign to the free charges on the plates of the capacitor, and so they are attracted by the free charges on the plates. Consequently, the dielectric is “pulled” into the gap, and the work to polarize the dielectric material between the plates is done at the expense of the stored electrical energy, which is reduced, in accordance with [link].

Example:
Inserting a Dielectric into an Isolated Capacitor
An empty 20.0-pF capacitor is charged to a potential difference of 40.0 V. The charging battery is then disconnected, and a piece of Teflon™ with a dielectric constant of 2.1 is inserted to completely fill the space between the capacitor plates (see [link]). What are the values of (a) the capacitance, (b) the charge of the plate, (c) the potential difference between the plates, and (d) the energy stored in the capacitor with and without dielectric?
Strategy
We identify the original capacitance $C_0 = 20.0\text{ pF}$ and the original potential difference $V_0 = 40.0\text{ V}$ between the plates. We combine [link] with other relations involving capacitance and substitute.
Solution

a. The capacitance increases to

Equation:

$$C = \kappa C_0 = 2.1(20.0\,\text{pF}) = 42.0\,\text{pF}.$$

b. Without dielectric, the charge on the plates is

Equation:

$$Q_0 = C_0 V_0 = (20.0\,\text{pF})(40.0\,\text{V}) = 0.8\,\text{nC}.$$

Since the battery is disconnected before the dielectric is inserted, the plate charge is unaffected by the dielectric and remains at 0.8 nC.

c. With the dielectric, the potential difference becomes

Equation:

$$V = \frac{1}{\kappa}V_0 = \frac{1}{2.1}40.0\,\text{V} = 19.0\,\text{V}.$$

d. The stored energy without the dielectric is

Equation:

$$U_0 = \frac{1}{2}C_0 V_0^2 = \frac{1}{2}(20.0\,\text{pF})(40.0\,\text{V})^2 = 16.0\,\text{nJ}.$$

With the dielectric inserted, we use [link] to find that the stored energy decreases to

Equation:

$$U = \frac{1}{\kappa}U_0 = \frac{1}{2.1}16.0\,\text{nJ} = 7.6\,\text{nJ}.$$

Significance

Notice that the effect of a dielectric on the capacitance of a capacitor is a drastic increase of its capacitance. This effect is far more profound than a mere change in the geometry of a capacitor.

Note:
Exercise:

Problem:

Check Your Understanding When a dielectric is inserted into an isolated and charged capacitor, the stored energy decreases to 33% of its original value. (a) What is the dielectric constant? (b) How does the capacitance change?

Solution:

a. 3.0; b. $C = 3.0\,C_0$

Summary

- The capacitance of an empty capacitor is increased by a factor of κ when the space between its plates is completely filled by a dielectric with dielectric constant κ.
- Each dielectric material has its specific dielectric constant.
- The energy stored in an empty isolated capacitor is decreased by a factor of κ when the space between its plates is completely filled with a dielectric with dielectric constant κ.

Conceptual Questions

Exercise:

Problem:

Discuss what would happen if a conducting slab rather than a dielectric were inserted into the gap between the capacitor plates.

Solution:

answers may vary

Exercise:

Problem:

Discuss how the energy stored in an empty but charged capacitor changes when a dielectric is inserted if (a) the capacitor is isolated so that its charge does not change; (b) the capacitor remains connected to a battery so that the potential difference between its plates does not change.

Problems

Exercise:

Problem:

Show that for a given dielectric material, the maximum energy a parallel-plate capacitor can store is directly proportional to the volume of dielectric.

Exercise:

Problem:

An air-filled capacitor is made from two flat parallel plates 1.0 mm apart. The inside area of each plate is $8.0\ \text{cm}^2$. (a) What is the capacitance of this set of plates? (b) If the region between the plates is filled with a material whose dielectric constant is 6.0, what is the new capacitance?

Solution:

a. 7.1 pF; b. 42 pF

Exercise:

Problem:

A capacitor is made from two concentric spheres, one with radius 5.00 cm, the other with radius 8.00 cm. (a) What is the capacitance of this set of conductors? (b) If the region between the conductors is filled with a material whose dielectric constant is 6.00, what is the capacitance of the system?

Exercise:

Problem:

A parallel-plate capacitor has charge of magnitude $9.00\ \mu\mathrm{C}$ on each plate and capacitance $3.00\ \mu\mathrm{F}$ when there is air between the plates. The plates are separated by 2.00 mm. With the charge on the plates kept constant, a dielectric with $\kappa = 5$ is inserted between the plates, completely filling the volume between the plates. (a) What is the potential difference between the plates of the capacitor, before and after the dielectric has been inserted? (b) What is the electrical field at the point midway between the plates before and after the dielectric is inserted?

Solution:

a. before 3.00 V; after 0.600 V; b. before 1500 V/m; after 300 V/m

Exercise:

Problem:

Some cell walls in the human body have a layer of negative charge on the inside surface. Suppose that the surface charge densities are $\pm 0.50 \times 10^{-3} \mathrm{C/m^2}$, the cell wall is $5.0 \times 10^{-9} \mathrm{m}$ thick, and the cell wall material has a dielectric constant of $\kappa = 5.4$. (a) Find the magnitude of the electric field in the wall between two charge layers. (b) Find the potential difference between the inside and the outside of the cell. Which is at higher potential? (c) A typical cell in the human body has volume $10^{-16} \mathrm{m^3}$. Estimate the total electrical field energy stored in the wall of a cell of this size when assuming that the cell is spherical. (*Hint*: Calculate the volume of the cell wall.)

Exercise:

Problem:

A parallel-plate capacitor with only air between its plates is charged by connecting the capacitor to a battery. The capacitor is then disconnected from the battery, without any of the charge leaving the plates. (a) A voltmeter reads 45.0 V when placed across the capacitor. When a dielectric is inserted between the plates, completely filling the space, the voltmeter reads 11.5 V. What is the dielectric constant of the material? (b) What will the voltmeter read if the dielectric is now pulled away out so it fills only one-third of the space between the plates?

Solution:

a. 3.91; b. 22.8 V

Glossary

dielectric constant
: factor by which capacitance increases when a dielectric is inserted between the plates of a capacitor

RC Circuits
By the end of the section, you will be able to:

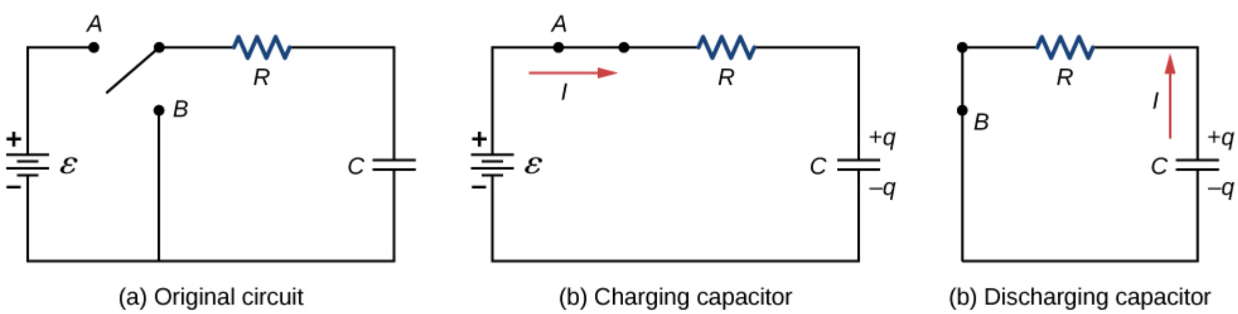
- Describe the charging process of a capacitor
- Describe the discharging process of a capacitor
- List some applications of RC circuits

When you use a flash camera, it takes a few seconds to charge the capacitor that powers the flash. The light flash discharges the capacitor in a tiny fraction of a second. Why does charging take longer than discharging? This question and several other phenomena that involve charging and discharging capacitors are discussed in this module.

Circuits with Resistance and Capacitance

An ***RC* circuit** is a circuit containing resistance and capacitance. As presented in Capacitance, the capacitor is an electrical component that stores electric charge, storing energy in an electric field.

[link](a) shows a simple *RC* circuit that employs a dc (direct current) voltage source ε, a resistor *R*, a capacitor *C*, and a two-position switch. The circuit allows the capacitor to be charged or discharged, depending on the position of the switch. When the switch is moved to position *A*, the capacitor charges, resulting in the circuit in part (b). When the switch is moved to position *B*, the capacitor discharges through the resistor.



(a) An *RC* circuit with a two-pole switch that can be used to charge and discharge a capacitor. (b) When the switch is moved to position *A*, the circuit reduces to a simple series connection of the voltage source, the resistor, the capacitor, and the switch. (c) When the switch is moved to position *B*, the circuit reduces to a simple series connection of the resistor, the capacitor, and the switch. The voltage source is removed from the circuit.

Charging a Capacitor

We can use Kirchhoff's loop rule to understand the charging of the capacitor. This results in the equation $\varepsilon - V_R - V_c = 0$. This equation can be used to model the charge as a function of time as the capacitor charges. Capacitance is defined as $C = q/V$, so the voltage across the capacitor is $V_C = \frac{q}{C}$. Using Ohm's law, the potential drop across the resistor is $V_R = IR$, and the current is defined as $I = dq/dt$.

Equation:

$$\begin{aligned}&\varepsilon - V_R - V_c = 0,\\&\varepsilon - IR - \tfrac{q}{C} = 0,\\&\varepsilon - R\tfrac{dq}{dt} - \tfrac{q}{C} = 0.\end{aligned}$$

This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.
Equation:

$$\begin{aligned}&\varepsilon - R\tfrac{dq}{dt} - \tfrac{q}{C} = 0,\\&\tfrac{dq}{dt} = \tfrac{\varepsilon C - q}{RC},\\&\int_0^q \frac{dq}{\varepsilon C - q} = \frac{1}{RC}\int_0^t dt.\end{aligned}$$

Let $u = \varepsilon C - q$, then $du = -dq$. The result is
Equation:

$$\begin{aligned}&-\int_0^q \frac{du}{u} = \frac{1}{RC}\int_0^t dt,\\&\ln\left(\tfrac{\varepsilon C - q}{\varepsilon C}\right) = -\tfrac{1}{RC}t,\\&\tfrac{\varepsilon C - q}{\varepsilon C} = e^{-t/RC}.\end{aligned}$$

Simplifying results in an equation for the charge on the charging capacitor as a function of time:

Note:
Equation:

$$q\left(t\right) = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right) = Q\left(1 - e^{-\frac{t}{\tau}}\right).$$

A graph of the charge on the capacitor versus time is shown in [link](a). First note that as time approaches infinity, the exponential goes to zero, so the charge approaches the maximum charge $Q = C\varepsilon$ and has units of coulombs. The units of *RC* are seconds, units of time. This quantity is known as the time constant:

Note:
Equation:

$$\tau = RC.$$

At time $t = \tau = RC$, the charge is equal to $1 - e^{-1} = 1 - 0.368 = 0.632$ of the maximum charge $Q = C\varepsilon$. Notice that the time rate change of the charge is the slope at a point of the charge versus time plot. The slope of the graph is large at time $t = 0.0$ s and approaches zero as time increases.

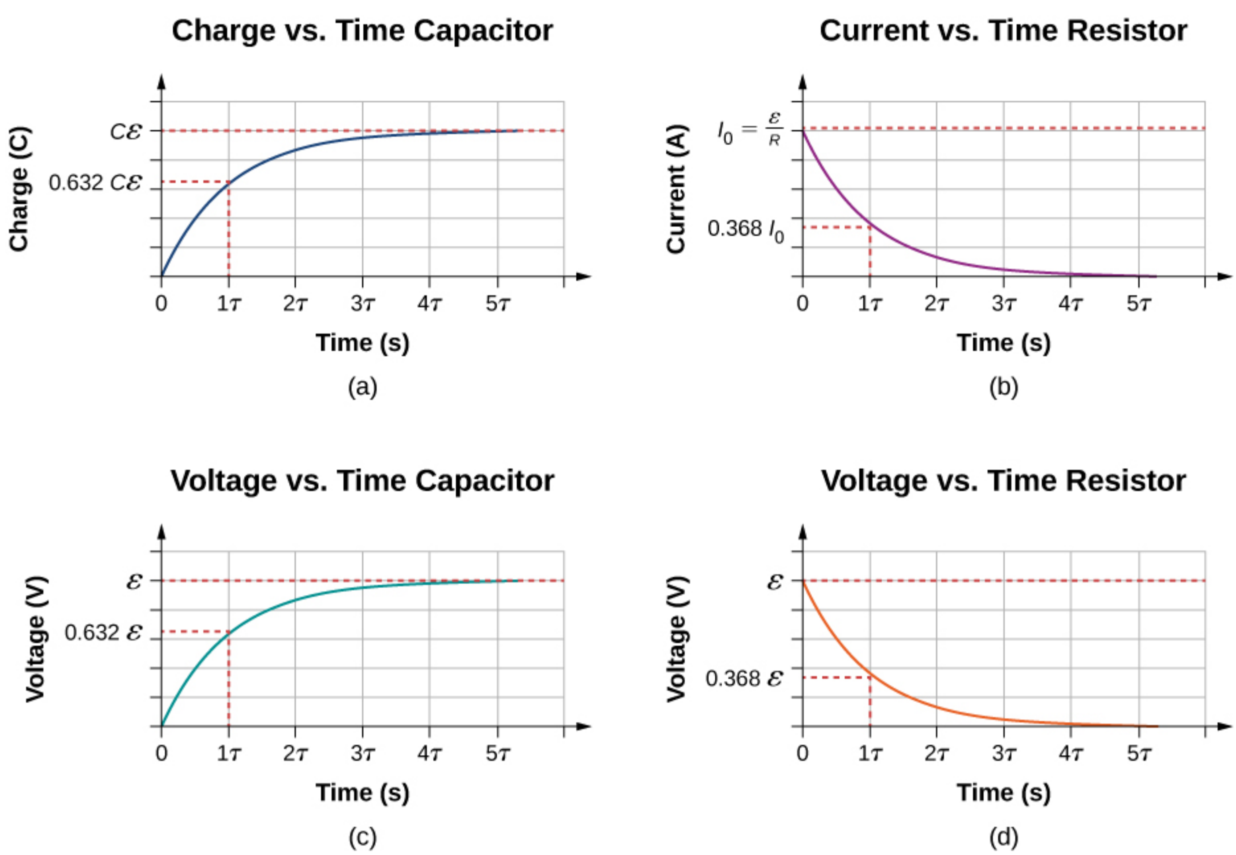
As the charge on the capacitor increases, the current through the resistor decreases, as shown in [link](b). The current through the resistor can be found by taking the time derivative of the charge.
Equation:

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}\left[C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right)\right],$$
$$I(t) = C\varepsilon\left(\frac{1}{RC}\right)e^{-\frac{t}{RC}} = \frac{\varepsilon}{R}e^{-\frac{t}{RC}} = I_o e^{-\frac{t}{RC}},$$

Note:
Equation:

$$I(t) = I_0 e^{-t/\tau}.$$

At time $t = 0.00$ s, the current through the resistor is $I_0 = \frac{\varepsilon}{R}$. As time approaches infinity, the current approaches zero. At time $t = \tau$, the current through the resistor is $I(t = \tau) = I_0 e^{-1} = 0.368 I_0$.



(a) Charge on the capacitor versus time as the capacitor charges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

[link](c) and [link](d) show the voltage differences across the capacitor and the resistor, respectively. As the charge on the capacitor increases, the current decreases, as does the voltage difference across the resistor $V_R(t) = (I_0R)e^{-t/\tau} = \varepsilon e^{-t/\tau}$. The voltage difference across the capacitor increases as $V_C(t) = \varepsilon\left(1 - e^{-t/\tau}\right)$.

Discharging a Capacitor

When the switch in [link](a) is moved to position B, the circuit reduces to the circuit in part (c), and the charged capacitor is allowed to discharge through the resistor. A graph of the charge on the capacitor as a function of time is shown in [link](a). Using Kirchhoff's loop rule to analyze the circuit as the capacitor discharges results in the equation $-V_R - V_c = 0$, which simplifies to $IR + \frac{q}{C} = 0$. Using the definition of current $\frac{dq}{dt}R = -\frac{q}{C}$ and integrating the loop equation yields an equation for the charge on the capacitor as a function of time:

Note:
Equation:

$$q(t) = Qe^{-t/\tau}.$$

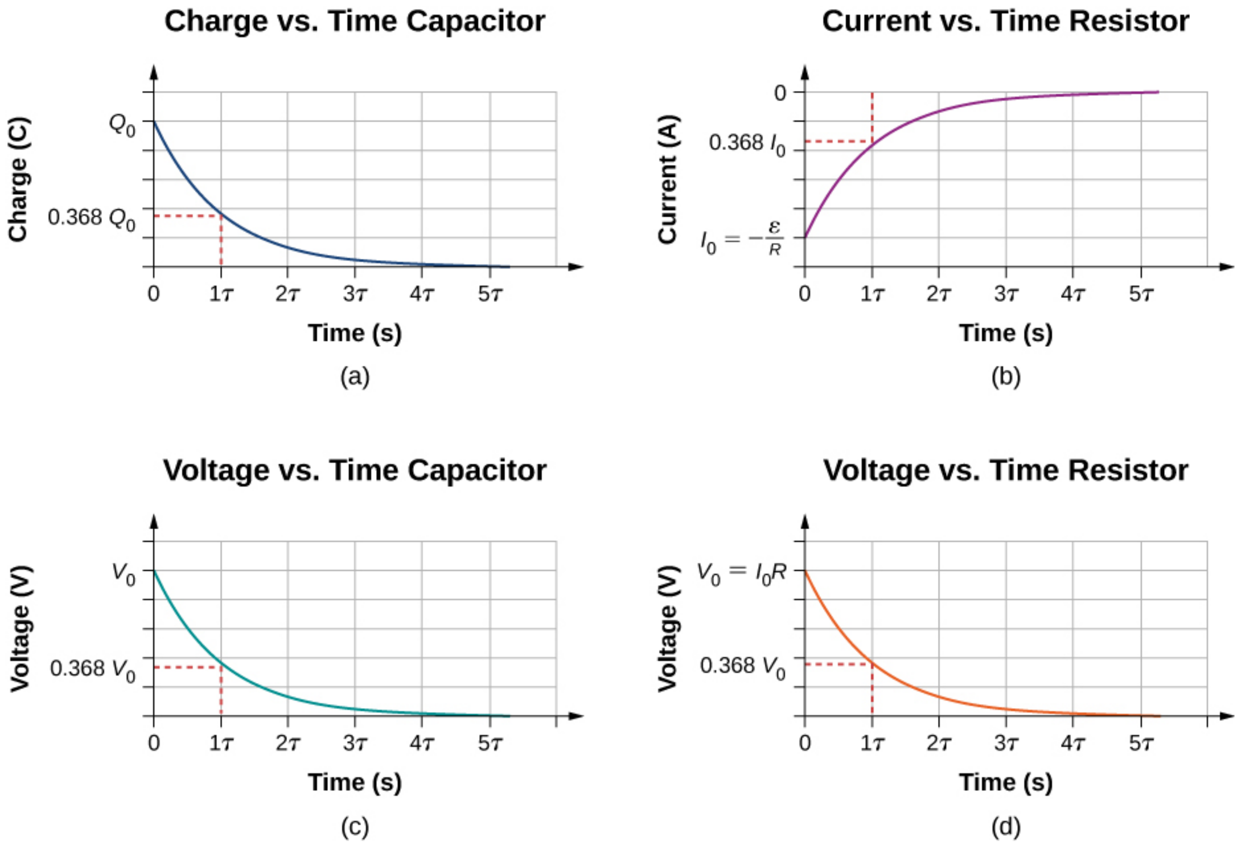
Here, Q is the initial charge on the capacitor and $\tau = RC$ is the time constant of the circuit. As shown in the graph, the charge decreases exponentially from the initial charge, approaching zero as time approaches infinity.

The current as a function of time can be found by taking the time derivative of the charge:

Note:
Equation:

$$I(t) = -\frac{Q}{RC}e^{-t/\tau}.$$

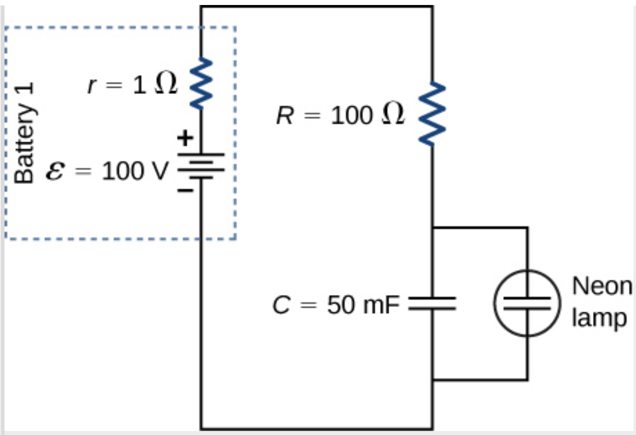
The negative sign shows that the current flows in the opposite direction of the current found when the capacitor is charging. [link](b) shows an example of a plot of charge versus time and current versus time. A plot of the voltage difference across the capacitor and the voltage difference across the resistor as a function of time are shown in parts (c) and (d) of the figure. Note that the magnitudes of the charge, current, and voltage all decrease exponentially, approaching zero as time increases.



(a) Charge on the capacitor versus time as the capacitor discharges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Now we can explain why the flash camera mentioned at the beginning of this section takes so much longer to charge than discharge: The resistance while charging is significantly greater than while discharging. The internal resistance of the battery accounts for most of the resistance while charging. As the battery ages, the increasing internal resistance makes the charging process even slower.

Example:
The Relaxation Oscillator
One application of an *RC* circuit is the relaxation oscillator, as shown below. The relaxation oscillator consists of a voltage source, a resistor, a capacitor, and a neon lamp. The neon lamp acts like an open circuit (infinite resistance) until the potential difference across the neon lamp reaches a specific voltage. At that voltage, the lamp acts like a short circuit (zero resistance), and the capacitor discharges through the neon lamp and produces light. In the relaxation oscillator shown, the voltage source charges the capacitor until the voltage across the capacitor is 80 V. When this happens, the neon in the lamp breaks down and allows the capacitor to discharge through the lamp, producing a bright flash. After the capacitor fully discharges through the neon lamp, it begins to charge again, and the process repeats. Assuming that the time it takes the capacitor to discharge is negligible, what is the time interval between flashes?



Strategy
The time period can be found from considering the equation $V_C\left(t\right) = \varepsilon\left(1 - e^{-t/\tau}\right)$, where$\tau = (R + r)C$.
Solution
The neon lamp flashes when the voltage across the capacitor reaches 80 V. The *RC* time constant is equal to $\tau = (R + r)C = (101\ \Omega)\left(50\ \times\ 10^{-3}\mathrm{F}\right) = 5.05\ \mathrm{s}$. We can solve the voltage equation for the time it takes the capacitor to reach 80 V:
Equation:

$$\begin{aligned} V_C\left(t\right) &= \varepsilon\left(1 - e^{-t/\tau}\right), \\ e^{-t/\tau} &= 1 - \frac{V_C(t)}{\varepsilon}, \\ \ln\left(e^{-t/\tau}\right) &= \ln\left(1 - \frac{V_C(t)}{\varepsilon}\right), \\ t &= -\tau\ln\left(1 - \frac{V_C(t)}{\varepsilon}\right) = -5.05\ \mathrm{s} \cdot \ln\left(1 - \frac{80\ \mathrm{V}}{100\ \mathrm{V}}\right) = 8.13\ \mathrm{s}. \end{aligned}$$

Significance
One application of the relaxation oscillator is for controlling indicator lights that flash at a frequency determined by the values for *R* and *C*. In this example, the neon lamp will flash every 8.13 seconds, a frequency of $f = \frac{1}{T} = \frac{1}{8.13\ \mathrm{s}} = 0.123\ \mathrm{Hz}$. The relaxation oscillator has many other practical uses. It is often used in electronic circuits, where the neon lamp is replaced by a transistor or a device known as a tunnel diode. The description of the transistor and tunnel diode is beyond the scope of this chapter, but you can think of them as voltage controlled switches. They are normally open switches, but when the right voltage is applied, the switch closes and conducts. The “switch” can be used to turn on another circuit, turn on a light, or run a small motor. A relaxation oscillator can be used to make the turn signals of your car blink or your cell phone to vibrate.

RC circuits have many applications. They can be used effectively as timers for applications such as intermittent windshield wipers, pace makers, and strobe lights. Some models of intermittent windshield wipers use a variable resistor to adjust the interval between sweeps of the wiper. Increasing the resistance increases the *RC* time constant, which increases the time between the operation of the wipers.

Another application is the pacemaker. The heart rate is normally controlled by electrical signals, which cause the muscles of the heart to contract and pump blood. When the heart rhythm is abnormal (the heartbeat is too high or too low), pace makers can be used to correct this abnormality. Pacemakers have sensors that detect body motion and breathing to increase the heart rate during physical activities, thus meeting the increased need for blood and oxygen, and an *RC* timing circuit can be used to control the time between voltage signals to the heart.

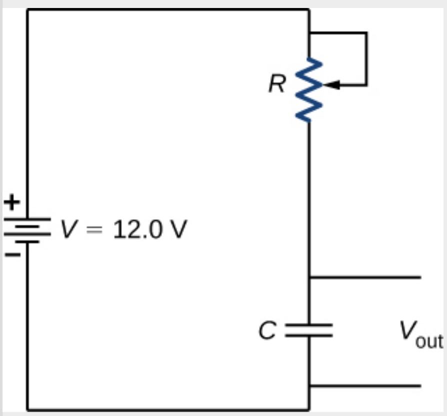
Looking ahead to the study of ac circuits (Alternating-Current Circuits), ac voltages vary as sine functions with specific frequencies. Periodic variations in voltage, or electric signals, are often recorded by scientists. These voltage signals could come from music recorded by a microphone or atmospheric data collected by radar. Occasionally, these signals can contain unwanted frequencies known as "noise." *RC* filters can be used to filter out the unwanted frequencies.

In the study of electronics, a popular device known as a 555 timer provides timed voltage pulses. The time between pulses is controlled by an *RC* circuit. These are just a few of the countless applications of *RC* circuits.

Example:

Intermittent Windshield Wipers

A relaxation oscillator is used to control a pair of windshield wipers. The relaxation oscillator consists of a 10.00-mF capacitor and a 10.00-kΩ variable resistor known as a rheostat. A knob connected to the variable resistor allows the resistance to be adjusted from $0.00\ \Omega$ to $10.00\ \text{k}\Omega$. The output of the capacitor is used to control a voltage-controlled switch. The switch is normally open, but when the output voltage reaches 10.00 V, the switch closes, energizing an electric motor and discharging the capacitor. The motor causes the windshield wipers to sweep once across the windshield and the capacitor begins to charge again. To what resistance should the rheostat be adjusted for the period of the wiper blades be 10.00 seconds?



Strategy

The resistance considers the equation $V_{\text{out}}\left(t\right)=V\left(1-e^{-t/\tau}\right)$, where $\tau=RC$. The capacitance, output voltage, and voltage of the battery are given. We need to solve this equation for the resistance.

Solution

The output voltage will be 10.00 V and the voltage of the battery is 12.00 V. The capacitance is given as 10.00 mF. Solving for the resistance yields

Equation:

$$\begin{aligned}
V_{\text{out}}\left(t\right) &= V\left(1-e^{-t/\tau}\right),\\
e^{-t/RC} &= 1-\frac{V_{\text{out}}(t)}{V},\\
\ln\left(e^{-t/RC}\right) &= \ln\left(1-\frac{V_{\text{out}}(t)}{V}\right),\\
-\frac{t}{RC} &= \ln\left(1-\frac{V_{\text{out}}(t)}{V}\right),\\
R &= \frac{-t}{C\ln\left(1-\frac{V_C(t)}{V}\right)}=\frac{-10.00\ \text{s}}{10\times10^{-3}\ \text{F}\ln\left(1-\frac{10\ \text{V}}{12\ \text{V}}\right)}=558.11\ \Omega.
\end{aligned}$$

Significance

Increasing the resistance increases the time delay between operations of the windshield wipers. When the resistance is zero, the windshield wipers run continuously. At the maximum resistance, the period of the operation of the wipers is:

Equation:

$$t=-RC\ln\left(1-\frac{V_{\text{out}}\left(t\right)}{V}\right)=-\left(10\ \times\ 10^{-3}\ \text{F}\right)\left(10\ \times\ 10^{3}\ \Omega\right)\ln\left(1-\frac{10\ \text{V}}{12\ \text{V}}\right)=179.18\ \text{s}=2.98\ \text{min}.$$

The *RC* circuit has thousands of uses and is a very important circuit to study. Not only can it be used to time circuits, it can also be used to filter out unwanted frequencies in a circuit and used in power supplies, like the one for your computer, to help turn ac voltage to dc voltage.

Summary

- An *RC* circuit is one that has both a resistor and a capacitor.
- The time constant τ for an *RC* circuit is $\tau=RC$.
- When an initially uncharged $(q=0\ \text{at}\ t=0)$ capacitor in series with a resistor is charged by a dc voltage source, the capacitor asymptotically approaches the maximum charge.
- As the charge on the capacitor increases, the current exponentially decreases from the initial current: $I_0=\varepsilon/R$.
- If a capacitor with an initial charge Q is discharged through a resistor starting at $t=0$, then its charge decreases exponentially. The current flows in the opposite direction, compared to when it charges, and the magnitude of the charge decreases with time.

Conceptual Questions

Exercise:

Problem:

A battery, switch, capacitor, and lamp are connected in series. Describe what happens to the lamp when the switch is closed.

Exercise:

Problem:

When making an ECG measurement, it is important to measure voltage variations over small time intervals. The time is limited by the *RC* constant of the circuit—it is not possible to measure time variations shorter than *RC*. How would you manipulate *R* and *C* in the circuit to allow the necessary measurements?

Solution:

The time constant can be shortened by using a smaller resistor and/or a smaller capacitor. Care should be taken when reducing the resistance because the initial current will increase as the resistance decreases.

Problems

Exercise:

Problem:

The timing device in an automobile's intermittent wiper system is based on an *RC* time constant and utilizes a $0.500\text{-}\mu\text{F}$ capacitor and a variable resistor. Over what range must *R* be made to vary to achieve time constants from 2.00 to 15.0 s?

Solution:

$4.00 \text{ to } 30.0\ \text{M}\Omega$

Exercise:

Problem:

A heart pacemaker fires 72 times a minute, each time a 25.0-nF capacitor is charged (by a battery in series with a resistor) to 0.632 of its full voltage. What is the value of the resistance?

Exercise:

Problem:

The duration of a photographic flash is related to an *RC* time constant, which is $0.100 \mu\text{s}$ for a certain camera. (a) If the resistance of the flash lamp is $0.0400\ \Omega$ during discharge, what is the size of the capacitor supplying its energy? (b) What is the time constant for charging the capacitor, if the charging resistance is $800\ \text{k}\Omega$?

Solution:

a. $2.50\ \mu\text{F}$; b. 2.00 s

Exercise:

Problem:

A 2.00- and a $7.50\text{-}\mu\text{F}$ capacitor can be connected in series or parallel, as can a 25.0- and a $100\text{-k}\Omega$ resistor. Calculate the four *RC* time constants possible from connecting the resulting capacitance and resistance in series.

Exercise:

Problem:

A $500\text{-}\Omega$ resistor, an uncharged $1.50\text{-}\mu\text{F}$ capacitor, and a 6.16-V emf are connected in series. (a) What is the initial current? (b) What is the *RC* time constant? (c) What is the current after one time constant? (d) What is the voltage on the capacitor after one time constant?

Solution:

a. 12.3 mA; b. 7.50×10^{-4}s; c. 4.53 mA; d. 3.89 V

Exercise:

Problem:

A heart defibrillator being used on a patient has an *RC* time constant of 10.0 ms due to the resistance of the patient and the capacitance of the defibrillator. (a) If the defibrillator has a capacitance of $8.00 \mu\text{F}$, what is the resistance of the path through the patient? (You may neglect the capacitance of the patient and the resistance of the defibrillator.) (b) If the initial voltage is 12.0 kV, how long does it take to decline to 6.00×10^{2} V?

Exercise:

Problem:

An ECG monitor must have an *RC* time constant less than $1.00 \times 10^{2} \mu\text{s}$ to be able to measure variations in voltage over small time intervals. (a) If the resistance of the circuit (due mostly to that of the patient's chest) is $1.00\ \text{k}\Omega$, what is the maximum capacitance of the circuit? (b) Would it be difficult in practice to limit the capacitance to less than the value found in (a)?

Solution:

a. $1.00 \times 10^{-7}\ \text{F}$; b. No, in practice it would not be difficult to limit the capacitance to less than 100 nF, since typical capacitors range from fractions of a picofarad (pF) to milifarad (mF).

Exercise:

Problem:

Using the exact exponential treatment, determine how much time is required to charge an initially uncharged 100-pF capacitor through a $75.0\text{-M}\Omega$ resistor to 90.0 of its final voltage.

Exercise:

Problem:

If you wish to take a picture of a bullet traveling at 500 m/s, then a very brief flash of light produced by an *RC* discharge through a flash tube can limit blurring. Assuming 1.00 mm of motion during one *RC* constant is acceptable, and given that the flash is driven by a $600\text{-}\mu\text{F}$ capacitor, what is the resistance in the flash tube?

Solution:

$3.33 \times 10^{-3}\ \Omega$

Glossary

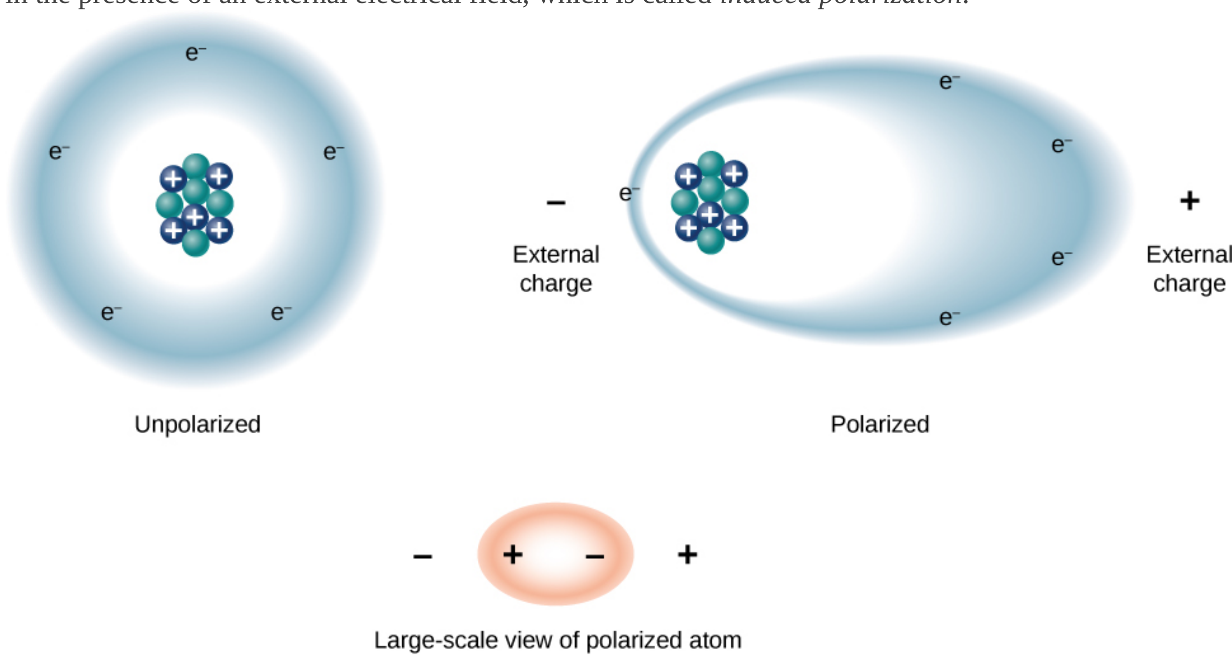
RC circuit
: circuit that contains both a resistor and a capacitor

Bonus: Molecular Model of a Dielectric

By the end of this section, you will be able to:

- Explain the polarization of a dielectric in a uniform electrical field
- Describe the effect of a polarized dielectric on the electrical field between capacitor plates
- Explain dielectric breakdown

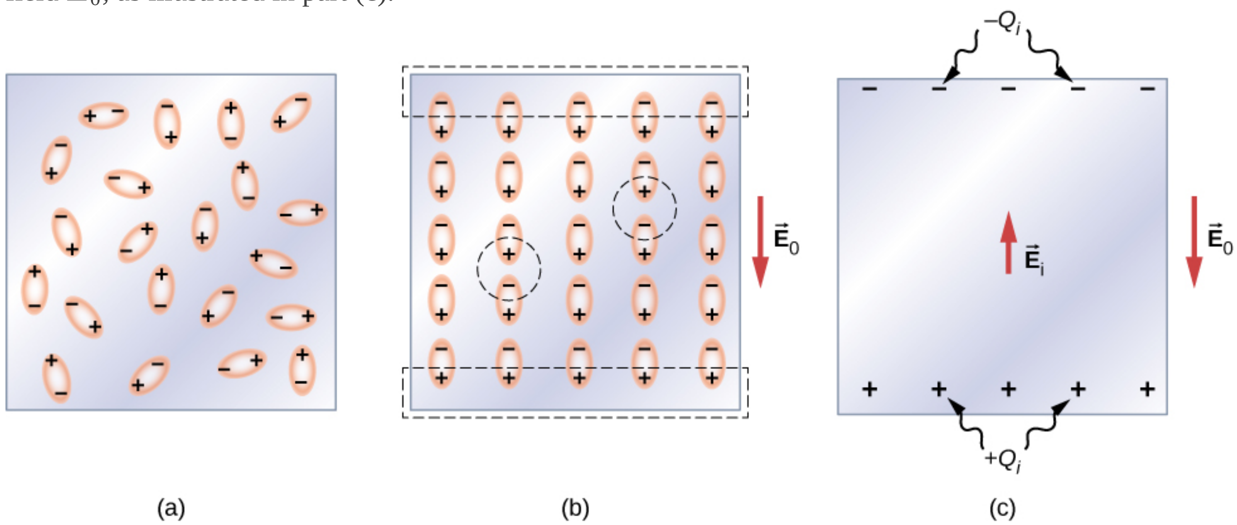
We can understand the effect of a dielectric on capacitance by looking at its behavior at the molecular level. As we have seen in earlier chapters, in general, all molecules can be classified as either *polar* or *nonpolar*. There is a net separation of positive and negative charges in an isolated polar molecule, whereas there is no charge separation in an isolated nonpolar molecule ([link]). In other words, polar molecules have permanent *electric-dipole moments* and nonpolar molecules do not. For example, a molecule of water is polar, and a molecule of oxygen is nonpolar. Nonpolar molecules can become polar in the presence of an external electrical field, which is called *induced polarization*.



The concept of polarization: In an unpolarized atom or molecule, a negatively charged electron cloud is evenly distributed around positively charged centers, whereas a polarized atom or molecule has an excess of negative charge at one side so that the other side has an excess of positive charge. However, the entire system remains electrically neutral. The charge polarization may be caused by an external electrical field. Some molecules and atoms are permanently polarized (electric dipoles) even in the absence of an external electrical field (polar molecules and atoms).

Let's first consider a dielectric composed of polar molecules. In the absence of any external electrical field, the electric dipoles are oriented randomly, as illustrated in [link](a). However, if the dielectric is placed in an external electrical field $\vec{\mathbf{E}}_0$, the polar molecules align with the external field, as shown in part (b) of the figure. Opposite charges on adjacent dipoles within the volume of dielectric neutralize each other, so there is no net charge within the dielectric (see the dashed circles in part (b)). However, this is not the case very close to the upper and lower surfaces that border the dielectric (the region enclosed by the dashed rectangles in part (b)), where the alignment does produce a net charge. Since the external electrical field merely aligns the dipoles, the dielectric as a whole is neutral, and the surface

charges induced on its opposite faces are equal and opposite. These **induced surface charges** $+Q_i$ and $-Q_i$ produce an additional electrical field $\vec{\mathbf{E}}_{\mathrm{i}}$ (an **induced electrical field**), which *opposes* the external field $\vec{\mathbf{E}}_0$, as illustrated in part (c).



A dielectric with polar molecules: (a) In the absence of an external electrical field; (b) in the presence of an external electrical field $\vec{\mathbf{E}}_0$. The dashed lines indicate the regions immediately adjacent to the capacitor plates. (c) The induced electrical field $\vec{\mathbf{E}}_{\mathrm{i}}$ inside the dielectric produced by the induced surface charge Q_i of the dielectric. Note that, in reality, the individual molecules are not perfectly aligned with an external field because of thermal fluctuations; however, the *average* alignment is along the field lines as shown.

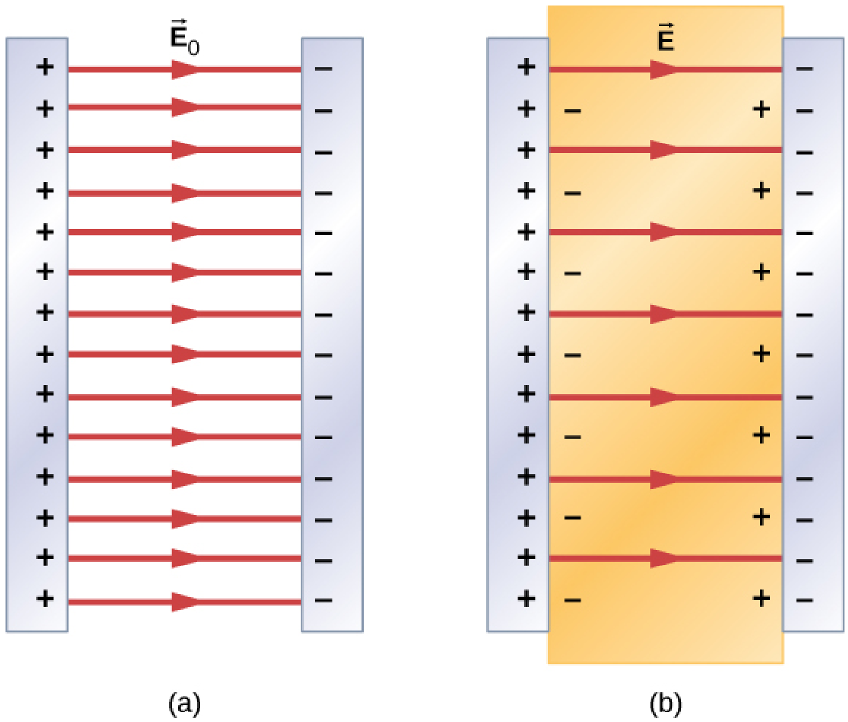
The same effect is produced when the molecules of a dielectric are nonpolar. In this case, a nonpolar molecule acquires an **induced electric-dipole moment** because the external field $\vec{\mathbf{E}}_0$ causes a separation between its positive and negative charges. The induced dipoles of the nonpolar molecules align with $\vec{\mathbf{E}}_0$ in the same way as the permanent dipoles of the polar molecules are aligned (shown in part (b)). Hence, the electrical field within the dielectric is weakened regardless of whether its molecules are polar or nonpolar.

Therefore, when the region between the parallel plates of a charged capacitor, such as that shown in [link](a), is filled with a dielectric, within the dielectric there is an electrical field $\vec{\mathbf{E}}_0$ due to the *free charge* Q_0 on the capacitor plates and an electrical field $\vec{\mathbf{E}}_{\mathrm{i}}$ due to the induced charge Q_i on the surfaces of the dielectric. Their vector sum gives the net electrical field $\vec{\mathbf{E}}$ within the dielectric between the capacitor plates (shown in part (b) of the figure):

Equation:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_{\mathrm{i}}.$$

This net field can be considered to be the field produced by an *effective charge* $Q_0 - Q_i$ on the capacitor.



Electrical field: (a) In an empty capacitor, electrical field $\vec{\mathbf{E}}_0$. (b) In a dielectric-filled capacitor, electrical field $\vec{\mathbf{E}}$.

In most dielectrics, the net electrical field $\vec{\mathbf{E}}$ is proportional to the field $\vec{\mathbf{E}}_0$ produced by the free charge. In terms of these two electrical fields, the dielectric constant κ of the material is defined as

Note:
Equation:

$$\kappa = \frac{E_0}{E}.$$

Since $\vec{\mathbf{E}}_0$ and $\vec{\mathbf{E}}_\mathrm{i}$ point in opposite directions, the magnitude E is smaller than the magnitude E_0 and therefore $\kappa > 1$. Combining [link] with [link], and rearranging the terms, yields the following expression for the induced electrical field in a dielectric:

Note:
Equation:

$$\vec{\mathbf{E}}_{\mathrm{i}} = \left(\frac{1}{\kappa} - 1\right)\vec{\mathbf{E}}_0.$$

When the magnitude of an external electrical field becomes too large, the molecules of dielectric material start to become ionized. A molecule or an atom is ionized when one or more electrons are removed from it and become free electrons, no longer bound to the molecular or atomic structure. When this happens, the material can conduct, thereby allowing charge to move through the dielectric from one capacitor plate to the other. This phenomenon is called **dielectric breakdown**. ([link] shows typical random-path patterns of electrical discharge during dielectric breakdown.) The critical value, E_{c}, of the electrical field at which the molecules of an insulator become ionized is called the **dielectric strength** of the material. The dielectric strength imposes a limit on the voltage that can be applied for a given plate separation in a capacitor. For example, the dielectric strength of air is $E_{\mathrm{c}} = 3.0\,\mathrm{MV/m}$, so for an air-filled capacitor with a plate separation of $d = 1.00\,\mathrm{mm}$, the limit on the potential difference that can be safely applied across its plates without causing dielectric breakdown is $V = E_{\mathrm{c}}d = (3.0\,\times\,10^6\,\mathrm{V/m})(1.00\,\times\,10^{-3}\,\mathrm{m}) = 3.0\,\mathrm{kV}$.

However, this limit becomes 60.0 kV when the same capacitor is filled with Teflon™, whose dielectric strength is about $60.0\,\mathrm{MV/m}$. Because of this limit imposed by the dielectric strength, the amount of charge that an air-filled capacitor can store is only $Q_0 = \kappa_{\mathrm{air}}C_0(3.0\,\mathrm{kV})$ and the charge stored on the same Teflon™-filled capacitor can be as much as

Equation:

$$Q = \kappa_{\mathrm{teflon}}C_0(60.0\,\mathrm{kV}) = \kappa_{\mathrm{teflon}}\frac{Q_0}{\kappa_{\mathrm{air}}(3.0\,\mathrm{kV})}(60.0\,\mathrm{kV}) = 20\frac{\kappa_{\mathrm{teflon}}}{\kappa_{\mathrm{air}}}Q_0 = 20\frac{2.1}{1.00059}Q_0 \cong 42\,Q_0,$$

which is about 42 times greater than a charge stored on an air-filled capacitor. Typical values of dielectric constants and dielectric strengths for various materials are given in [link]. Notice that the dielectric constant κ is exactly 1.0 for a vacuum (the empty space serves as a reference condition) and very close to 1.0 for air under normal conditions (normal pressure at room temperature). These two values are so close that, in fact, the properties of an air-filled capacitor are essentially the same as those of an empty capacitor.

Material	**Dielectric constant** κ	**Dielectric strength** $E_{\mathrm{c}}[\times 10^6\mathrm{V/m}]$
Vacuum	1	∞
Dry air (1 atm)	1.00059	3.0
Teflon™	2.1	60 to 173
Paraffin	2.3	11

Material	**Dielectric constant κ**	**Dielectric strength $E_c [\times 10^6 \text{V/m}]$**
Silicon oil	2.5	10 to 15
Polystyrene	2.56	19.7
Nylon	3.4	14
Paper	3.7	16
Fused quartz	3.78	8
Glass	4 to 6	9.8 to 13.8
Concrete	4.5	–
Bakelite	4.9	24
Diamond	5.5	2,000
Pyrex glass	5.6	14
Mica	6.0	118
Neoprene rubber	6.7	15.7 to 26.7
Water	80	–
Sulfuric acid	84 to 100	–
Titanium dioxide	86 to 173	–
Strontium titanate	310	8
Barium titanate	1,200 to 10,000	–
Calcium copper titanate	> 250,000	–

Representative Values of Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Not all substances listed in the table are good insulators, despite their high dielectric constants. Water, for example, consists of polar molecules and has a large dielectric constant of about 80. In a water molecule, electrons are more likely found around the oxygen nucleus than around the hydrogen nuclei. This makes the oxygen end of the molecule slightly negative and leaves the hydrogens end slightly positive, which makes the molecule easy to align along an external electrical field, and thus water has a large dielectric constant. However, the polar nature of water molecules also makes water a good solvent for many substances, which produces undesirable effects, because any concentration of free ions in water conducts electricity.

Example:

Electrical Field and Induced Surface Charge

Suppose that the distance between the plates of the capacitor in [link] is 2.0 mm and the area of each plate is $4.5 \times 10^{-3}\ \text{m}^2$. Determine: (a) the electrical field between the plates before and after the Teflon™ is inserted, and (b) the surface charge induced on the Teflon™ surfaces.

Strategy

In part (a), we know that the voltage across the empty capacitor is $V_0 = 40\ \text{V}$, so to find the electrical fields we use the relation $V = Ed$ and [link]. In part (b), knowing the magnitude of the electrical field, we use the expression for the magnitude of electrical field near a charged plate $E = \sigma/\varepsilon_0$, where σ is a uniform surface charge density caused by the surface charge. We use the value of free charge $Q_0 = 8.0 \times 10^{-10}\ \text{C}$ obtained in [link].

Solution

a. The electrical field E_0 between the plates of an empty capacitor is
   Equation:

$$E_0 = \frac{V_0}{d} = \frac{40\ \text{V}}{2.0 \times 10^{-3}\text{m}} = 2.0 \times 10^4\ \text{V/m}.$$

The electrical field *E* with the Teflon™ in place is
   Equation:

$$E = \frac{1}{\kappa}E_0 = \frac{1}{2.1}2.0 \times 10^4\ \text{V/m} = 9.5 \times 10^3\ \text{V/m}.$$

b. The effective charge on the capacitor is the difference between the free charge Q_0 and the induced charge Q_i. The electrical field in the Teflon™ is caused by this effective charge. Thus
   Equation:

$$E = \frac{1}{\varepsilon_0}\sigma = \frac{1}{\varepsilon_0}\frac{Q_0 - Q_\text{i}}{A}.$$

We invert this equation to obtain Q_i, which yields
   Equation:

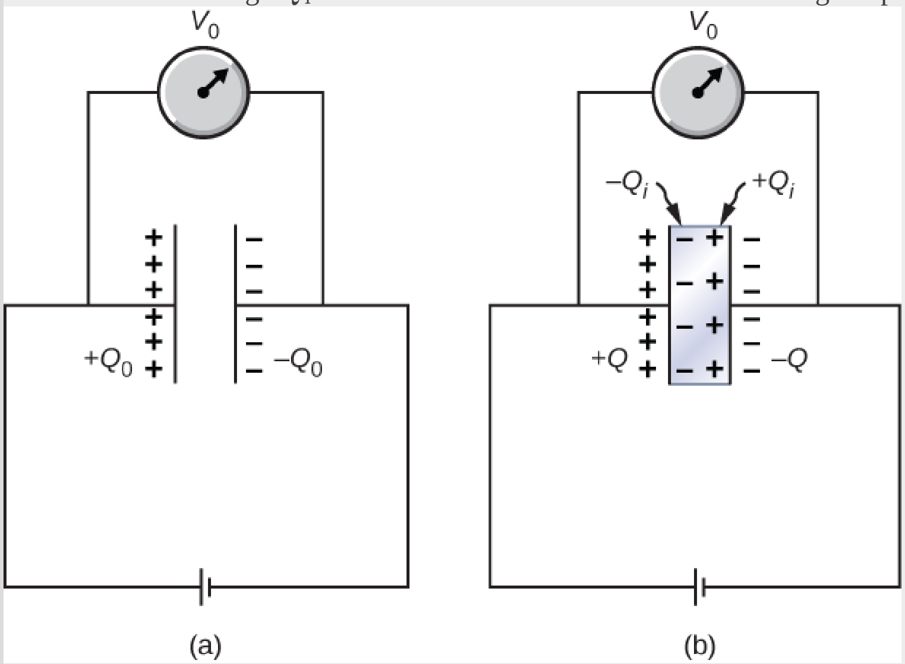
$$\begin{aligned} Q_\text{i} &= Q_0 - \varepsilon_0 AE \\ &= 8.0 \times 10^{-10}\text{C} - \left(8.85 \times 10^{-12}\ \frac{\text{C}^2}{\text{N·m}^2}\right)\left(4.5 \times 10^{-3}\ \text{m}^2\right)\left(9.5 \times 10^3\ \frac{\text{V}}{\text{m}}\right) \\ &= 4.2 \times 10^{-10}\text{C} = 0.42\,\text{nC}. \end{aligned}$$

Example:

Inserting a Dielectric into a Capacitor Connected to a Battery

When a battery of voltage V_0 is connected across an empty capacitor of capacitance C_0, the charge on its plates is Q_0, and the electrical field between its plates is E_0. A dielectric of dielectric constant κ is inserted between the plates *while the battery remains in place*, as shown in [link]. (a) Find the capacitance *C*, the voltage *V* across the capacitor, and the electrical field *E* between the plates after the

dielectric is inserted. (b) Obtain an expression for the free charge Q on the plates of the filled capacitor and the induced charge Q_{i} on the dielectric surface in terms of the original plate charge Q_0.



A dielectric is inserted into the charged capacitor while the capacitor remains connected to the battery.

Strategy

We identify the known values: V_0, C_0,E_0, κ, and Q_0. Our task is to express the unknown values in terms of these known values.

Solution

(a) The capacitance of the filled capacitor is $C=\kappa C_0$. Since the battery is always connected to the capacitor plates, the potential difference between them does not change; hence, $V=V_0$. Because of that, the electrical field in the filled capacitor is the same as the field in the empty capacitor, so we can obtain directly that

Equation:

$$E=\frac{V}{d}=\frac{V_0}{d}=E_0.$$

(b) For the filled capacitor, the free charge on the plates is

Equation:

$$Q=CV=(\kappa C_0)V_0=\kappa(C_0V_0)=\kappa Q_0.$$

The electrical field E in the filled capacitor is due to the effective charge $Q-Q_{\mathrm{i}}$ ([link](b)). Since $E=E_0$, we have

Equation:

$$\frac{Q-Q_{\mathrm{i}}}{\varepsilon_0 A}=\frac{Q_0}{\varepsilon_0 A}.$$

Solving this equation for Q_{i}, we obtain for the induced charge

Equation:

$$Q_{\mathrm{i}} = Q - Q_0 = \kappa Q_0 - Q_0 = (\kappa - 1)Q_0.$$

Significance

Notice that for materials with dielectric constants larger than 2 (see [link]), the induced charge on the surface of dielectric is larger than the charge on the plates of a vacuum capacitor. The opposite is true for gasses like air whose dielectric constant is smaller than 2.

Note:

Exercise:

Problem:

Check Your Understanding Continuing with [link], show that when the battery is connected across the plates the energy stored in dielectric-filled capacitor is $U = \kappa U_0$ (larger than the energy U_0 of an empty capacitor kept at the same voltage). Compare this result with the result $U = U_0/\kappa$ found previously for an isolated, charged capacitor.

Note:

Exercise:

Problem:

Check Your Understanding Repeat the calculations of [link] for the case in which the battery remains connected while the dielectric is placed in the capacitor.

Solution:

a. $C_0 = 20\,\mathrm{pF}$, $C = 42\,\mathrm{pF}$; b. $Q_0 = 0.8\,\mathrm{nC}$, $Q = 1.7\,\mathrm{nC}$; c. $V_0 = V = 40\,\mathrm{V}$; d. $U_0 = 16\,\mathrm{nJ}$, $U = 34\,\mathrm{nJ}$

Summary

- When a dielectric is inserted between the plates of a capacitor, equal and opposite surface charge is induced on the two faces of the dielectric. The induced surface charge produces an induced electrical field that opposes the field of the free charge on the capacitor plates.
- The dielectric constant of a material is the ratio of the electrical field in vacuum to the net electrical field in the material. A capacitor filled with dielectric has a larger capacitance than an empty capacitor.
- The dielectric strength of an insulator represents a critical value of electrical field at which the molecules in an insulating material start to become ionized. When this happens, the material can conduct and dielectric breakdown is observed.

Key Equations

Capacitance	$C=\frac{Q}{V}$
Capacitance of a parallel-plate capacitor	$C=\varepsilon_0\frac{A}{d}$
Capacitance of a vacuum spherical capacitor	$C=4\pi\varepsilon_0\frac{R_1R_2}{R_2-R_1}$
Capacitance of a vacuum cylindrical capacitor	$C=\frac{2\pi\varepsilon_0 l}{\ln(R_2/R_1)}$
Capacitance of a series combination	$\frac{1}{C_\text{S}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}+\cdots$
Capacitance of a parallel combination	$C_\text{P}=C_1+C_2+C_3+\cdots$
Energy density	$u_E=\frac{1}{2}\varepsilon_0 E^2$
Energy stored in a capacitor	$U_C=\frac{1}{2}V^2C=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}QV$
Capacitance of a capacitor with dielectric	$C=\kappa C_0$
Energy stored in an isolated capacitor with dielectric	$U=\frac{1}{\kappa}U_0$
Dielectric constant	$\kappa=\frac{E_0}{E}$
Induced electrical field in a dielectric	$\vec{\mathbf{E}}_\text{i}=\left(\frac{1}{\kappa}-1\right)\vec{\mathbf{E}}_0$

Conceptual Questions

Exercise:

Problem: Distinguish between dielectric strength and dielectric constant.

Solution:

Dielectric strength is a critical value of an electrical field above which an insulator starts to conduct; a dielectric constant is the ratio of the electrical field in vacuum to the net electrical field in a material.

Exercise:

Problem: Water is a good solvent because it has a high dielectric constant. Explain.

Exercise:

Problem:

Water has a high dielectric constant. Explain why it is then not used as a dielectric material in capacitors.

Solution:

Water is a good solvent.

Exercise:

Problem:

Elaborate on why molecules in a dielectric material experience net forces on them in a non-uniform electrical field but not in a uniform field.

Exercise:

Problem:

Explain why the dielectric constant of a substance containing permanent molecular electric dipoles decreases with increasing temperature.

Solution:

When energy of thermal motion is large (high temperature), an electrical field must be large too in order to keep electric dipoles aligned with it.

Exercise:

Problem:

Give a reason why a dielectric material increases capacitance compared with what it would be with air between the plates of a capacitor. How does a dielectric material also allow a greater voltage to be applied to a capacitor? (The dielectric thus increases C and permits a greater V.)

Exercise:

Problem:

Elaborate on the way in which the polar character of water molecules helps to explain water's relatively large dielectric constant.

Solution:

answers may vary

Exercise:

Problem:

Sparks will occur between the plates of an air-filled capacitor at a lower voltage when the air is humid than when it is dry. Discuss why, considering the polar character of water molecules.

Problems

Exercise:

Problem:

Two flat plates containing equal and opposite charges are separated by material 4.0 mm thick with a dielectric constant of 5.0. If the electrical field in the dielectric is 1.5 MV/m, what are (a) the charge density on the capacitor plates, and (b) the induced charge density on the surfaces of the dielectric?

Exercise:

Problem:

For a Teflon™-filled, parallel-plate capacitor, the area of the plate is $50.0\ \mathrm{cm}^2$ and the spacing between the plates is 0.50 mm. If the capacitor is connected to a 200-V battery, find (a) the free charge on the capacitor plates, (b) the electrical field in the dielectric, and (c) the induced charge on the dielectric surfaces.

Solution:

a. 37 nC; b. 0.4 MV/m; c. 19 nC

Exercise:

Problem:

Find the capacitance of a parallel-plate capacitor having plates with a surface area of $5.00\ m^2$ and separated by 0.100 mm of Teflon™.

Exercise:

Problem:

(a) What is the capacitance of a parallel-plate capacitor with plates of area $1.50\ \mathrm{m}^2$ that are separated by 0.0200 mm of neoprene rubber? (b) What charge does it hold when 9.00 V is applied to it?

Solution:

a. 4.4 μF; b. $4.0\ \times\ 10^{-5}$ C

Exercise:

Problem:

Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electrical field is $E = 3.20\ \times\ 10^5\ \mathrm{V/m}$. When the space is filled with dielectric, the electrical field is $E = 2.50\ \times\ 10^5\ \mathrm{V/m}$. (a) What is the surface charge density on each surface of the dielectric? (b) What is the dielectric constant?

Exercise:

Problem:

The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of $1.60\ \times\ 10^7\ \mathrm{V/m}$. The capacitor has to have a capacitance of 1.25 nF and must be able to withstand a maximum potential difference 5.5 kV. What is the minimum area the plates of the capacitor may have?

Solution:

$0.0135\ \mathrm{m}^2$

Exercise:

Problem:

When a 360-nF air capacitor is connected to a power supply, the energy stored in the capacitor is $18.5\ \mu\mathrm{J}$. While the capacitor is connected to the power supply, a slab of dielectric is inserted that completely fills the space between the plates. This increases the stored energy by $23.2\ \mu\mathrm{J}$. (a) What is the potential difference between the capacitor plates? (b) What is the dielectric constant of the slab?

Exercise:

Problem:

A parallel-plate capacitor has square plates that are 8.00 cm on each side and 3.80 mm apart. The space between the plates is completely filled with two square slabs of dielectric, each 8.00 cm on a side and 1.90 mm thick. One slab is Pyrex glass and the other slab is polystyrene. If the potential difference between the plates is 86.0 V, find how much electrical energy can be stored in this capacitor.

Solution:

$0.185\ \mu\mathrm{J}$

Additional Problems

Exercise:

Problem:

A capacitor is made from two flat parallel plates placed 0.40 mm apart. When a charge of $0.020\ \mu\mathrm{C}$ is placed on the plates the potential difference between them is 250 V. (a) What is the capacitance of the plates? (b) What is the area of each plate? (c) What is the charge on the plates when the potential difference between them is 500 V? (d) What maximum potential difference can be applied between the plates so that the magnitude of electrical fields between the plates does not exceed 3.0 MV/m?

Exercise:

Problem:

An air-filled (empty) parallel-plate capacitor is made from two square plates that are 25 cm on each side and 1.0 mm apart. The capacitor is connected to a 50-V battery and fully charged. It is then disconnected from the battery and its plates are pulled apart to a separation of 2.00 mm. (a) What is the capacitance of this new capacitor? (b) What is the charge on each plate? (c) What is the electrical field between the plates?

Solution:

a. 0.277 nF; b. 27.7 nC; c. 50 kV/m

Exercise:

Problem:

Suppose that the capacitance of a variable capacitor can be manually changed from 100 to 800 pF by turning a dial connected to one set of plates by a shaft, from 0° to 180°. With the dial set at 180° (corresponding to $C = 800\,\text{pF}$), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to 0°. (a) What is the charge on the capacitor? (b) What is the voltage across the capacitor when the dial is set to 0°?

Exercise:

Problem:

Earth can be considered as a spherical capacitor with two plates, where the negative plate is the surface of Earth and the positive plate is the bottom of the ionosphere, which is located at an altitude of approximately 70 km. The potential difference between Earth's surface and the ionosphere is about 350,000 V. (a) Calculate the capacitance of this system. (b) Find the total charge on this capacitor. (c) Find the energy stored in this system.

Solution:

a. 0.065 F; b. 23,000 C; c. 4.0 GJ

Exercise:

Problem:

A 4.00-μF capacitor and a 6.00-μF capacitor are connected in parallel across a 600-V supply line. (a) Find the charge on each capacitor and voltage across each. (b) The charged capacitors are disconnected from the line and from each other. They are then reconnected to each other with terminals of unlike sign together. Find the final charge on each capacitor and the voltage across each.

Exercise:

Problem:

Three capacitors having capacitances of 8.40, 8.40, and 4.20 μF, respectively, are connected in series across a 36.0-V potential difference. (a) What is the charge on the $4.20\text{-}\mu\text{F}$ capacitor? (b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination?

Solution:

a. $75.6\ \mu\text{C}$; b. 10.8 V

Exercise:

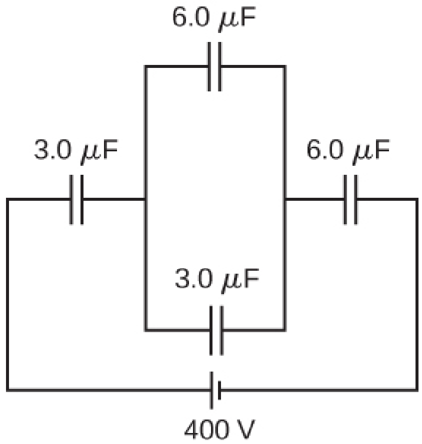
Problem:

A parallel-plate capacitor with capacitance $5.0\ \mu\text{F}$ is charged with a 12.0-V battery, after which the battery is disconnected. Determine the minimum work required to increase the separation between the plates by a factor of 3.

Exercise:

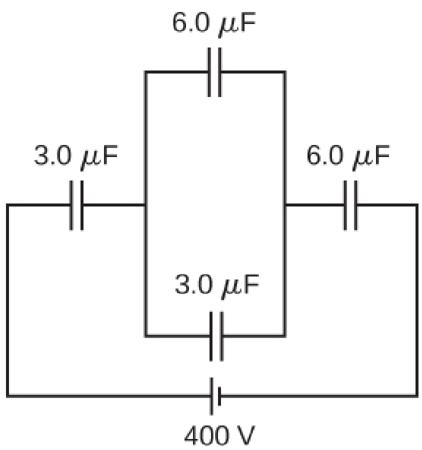
Problem:

(a) How much energy is stored in the electrical fields in the capacitors (in total) shown below? (b) Is this energy equal to the work done by the 400-V source in charging the capacitors?



Solution:

a. 0.13 J; b. no, because of resistive heating in connecting wires that is always present, but the circuit schematic does not indicate resistors



Exercise:

Problem:

Three capacitors having capacitances 8.4, 8.4, and 4.2 μF are connected in series across a 36.0-V potential difference. (a) What is the total energy stored in all three capacitors? (b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the total energy now stored in the capacitors?

Exercise:

Problem:

(a) An $8.00\text{-}\mu\text{F}$ capacitor is connected in parallel to another capacitor, producing a total capacitance of $5.00\ \mu\text{F}$. What is the capacitance of the second capacitor? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Solution:

a. $-3.00\ \mu\text{F}$; b. You cannot have a negative C_2 capacitance. c. The assumption that they were hooked up in parallel, rather than in series, is incorrect. A parallel connection always produces a greater capacitance, while here a smaller capacitance was assumed. This could only happen if the capacitors are connected in series.

Exercise:

Problem:

(a) On a particular day, it takes $9.60\ \times\ 10^3\ \text{J}$ of electrical energy to start a truck's engine. Calculate the capacitance of a capacitor that could store that amount of energy at 12.0 V. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Exercise:

Problem:

(a) A certain parallel-plate capacitor has plates of area $4.00\ \text{m}^2$, separated by 0.0100 mm of nylon, and stores 0.170 C of charge. What is the applied voltage? (b) What is unreasonable about this result? (c) Which assumptions are responsible or inconsistent?

Solution:

a. 14.2 kV; b. The voltage is unreasonably large, more than 100 times the breakdown voltage of nylon. c. The assumed charge is unreasonably large and cannot be stored in a capacitor of these dimensions.

Exercise:

Problem:

A prankster applies 450 V to an $80.0\text{-}\mu\text{F}$ capacitor and then tosses it to an unsuspecting victim. The victim's finger is burned by the discharge of the capacitor through 0.200 g of flesh. Estimate, what is the temperature increase of the flesh? Is it reasonable to assume that no thermodynamic phase change happened?

Challenge Problems

Exercise:

Problem:

A spherical capacitor is formed from two concentric spherical conducting spheres separated by vacuum. The inner sphere has radius 12.5 cm and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor. (a) What is the capacitance of the capacitor? (b) What is the magnitude of the electrical field at $r = 12.6\ \text{cm}$, just outside the inner sphere? (c) What is the magnitude of the electrical field at $r = 14.7\ \text{cm}$, just inside the outer sphere? (d) For a parallel-plate capacitor the electrical field is uniform in the region between the plates, except near the edges of the plates. Is this also true for a spherical capacitor?

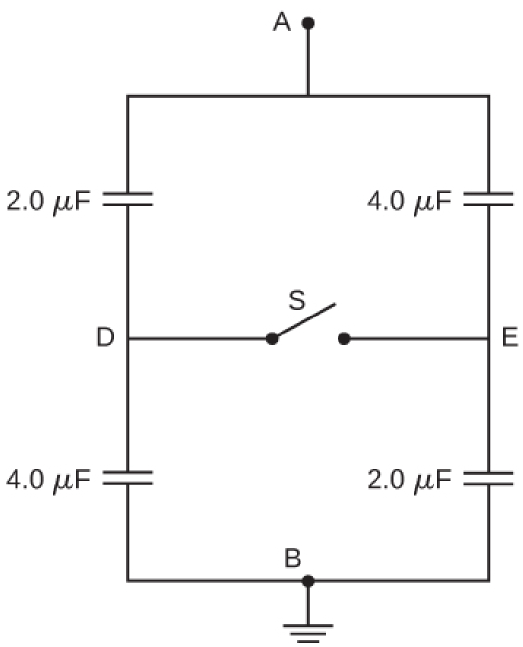
Solution:

a. 89.6 pF; b. 6.09 kV/m; c. 4.47 kV/m; d. no

Exercise:

Problem:

The network of capacitors shown below are all uncharged when a 300-V potential is applied between points A and B with the switch S open. (a) What is the potential difference $V_E - V_D$? (b) What is the potential at point E after the switch is closed? (c) How much charge flows through the switch after it is closed?



Exercise:

Problem:

Electronic flash units for cameras contain a capacitor for storing the energy used to produce the flash. In one such unit the flash lasts for 1/675 fraction of a second with an average light power output of 270 kW. (a) If the conversion of electrical energy to light is 95% efficient (because the rest of the energy goes to thermal energy), how much energy must be stored in the capacitor for one flash? (b) The capacitor has a potential difference between its plates of 125 V when the stored energy equals the value stored in part (a). What is the capacitance?

Solution:

a. 421 J; b. 53.9 mF

Exercise:

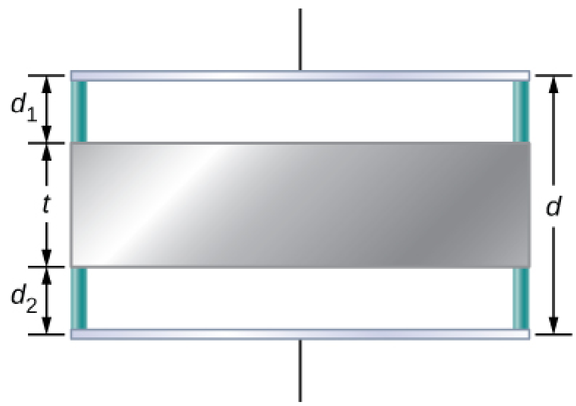
Problem:

A spherical capacitor is formed from two concentric spherical conducting shells separated by a vacuum. The inner sphere has radius 12.5 cm and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor. (a) What is the energy density at $r = 12.6\ \text{cm}$, just outside the inner sphere? (b) What is the energy density at $r = 14.7\ \text{cm}$, just inside the outer sphere? (c) For the parallel-plate capacitor the energy density is uniform in the region between the plates, except near the edges of the plates. Is this also true for the spherical capacitor?

Exercise:

Problem:

A metal plate of thickness *t* is held in place between two capacitor plates by plastic pegs, as shown below. The effect of the pegs on the capacitance is negligible. The area of each capacitor plate and the area of the top and bottom surfaces of the inserted plate are all *A*. What is the capacitance of this system?



Solution:

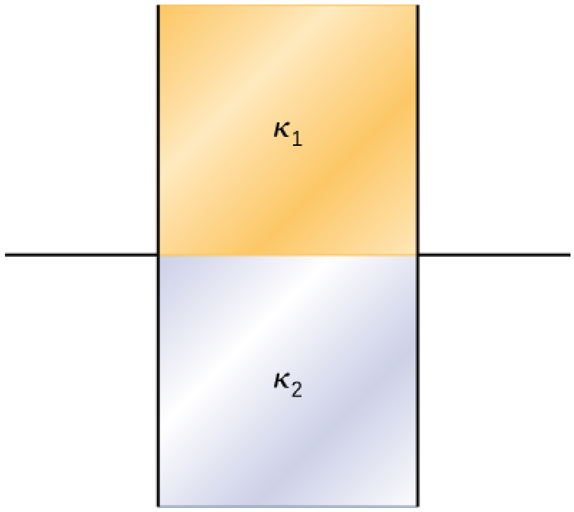
$C = \varepsilon_0 A/(d_1 + d_2)$

Exercise:

Problem:

A parallel-plate capacitor is filled with two dielectrics, as shown below. When the plate area is *A* and separation between plates is *d*, show that the capacitance is given by

$C = \varepsilon_0 \frac{A}{d}\ \frac{\kappa_1 + \kappa_2}{2}$.

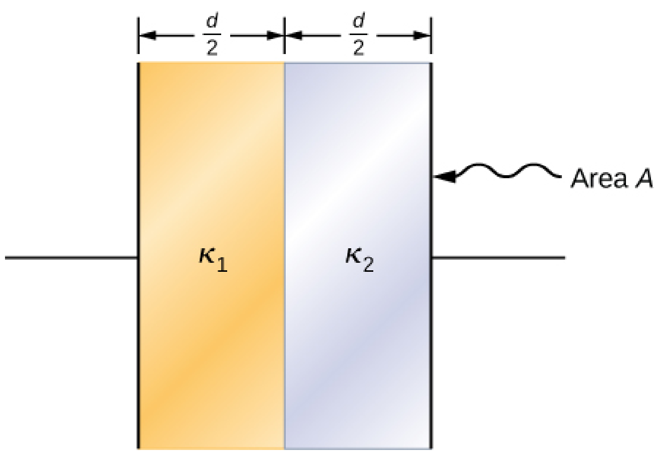


Exercise:

Problem:

A parallel-plate capacitor is filled with two dielectrics, as shown below. Show that the capacitance is given by

$C = 2\varepsilon_0 \frac{A}{d} \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$.



Solution:

proof

Exercise:

Problem:

A capacitor has parallel plates of area $12\,\text{cm}^2$ separated by 2.0 mm. The space between the plates is filled with polystyrene. (a) Find the maximum permissible voltage across the capacitor to avoid dielectric breakdown. (b) When the voltage equals the value found in part (a), find the surface charge density on the surface of the dielectric.

Glossary

dielectric breakdown
: phenomenon that occurs when an insulator becomes a conductor in a strong electrical field

dielectric strength
: critical electrical field strength above which molecules in insulator begin to break down and the insulator starts to conduct

induced electric-dipole moment
: dipole moment that a nonpolar molecule may acquire when it is placed in an electrical field

induced electrical field
: electrical field in the dielectric due to the presence of induced charges

induced surface charges
: charges that occur on a dielectric surface due to its polarization

Introduction
class="introduction"

An industrial electromagnet is capable of lifting thousands of pounds of metallic waste. (credit: modification of work by “BedfordAl”/Flickr)

For the past few chapters, we have been studying electrostatic forces and fields, which are caused by electric charges at rest. These electric fields can move other free charges, such as producing a current in a circuit; however, the electrostatic forces and fields themselves come from other static charges. In this chapter, we see that when an electric charge moves, it generates other forces and fields. These additional forces and fields are what we commonly call magnetism.

Before we examine the origins of magnetism, we first describe what it is and how magnetic fields behave. Once we are more familiar with magnetic effects, we can explain how they arise from the behavior of atoms and molecules, and how magnetism is related to electricity. The connection between electricity and magnetism is fascinating from a theoretical point of view, but it is also immensely practical, as shown by an industrial electromagnet that can lift thousands of pounds of metal.

Magnetic Fields and Lines
By the end of this section, you will be able to:

- Define the magnetic field based on a moving charge experiencing a force
- Apply the right-hand rule to determine the direction of a magnetic force based on the motion of a charge in a magnetic field
- Sketch magnetic field lines to understand which way the magnetic field points and how strong it is in a region of space

We have outlined the properties of magnets, described how they behave, and listed some of the applications of magnetic properties. Even though there are no such things as isolated magnetic charges, we can still define the attraction and repulsion of magnets as based on a field. In this section, we define the magnetic field, determine its direction based on the right-hand rule, and discuss how to draw magnetic field lines.

Defining the Magnetic Field

A magnetic field is defined by the force that a charged particle experiences moving in this field, after we account for the gravitational and any additional electric forces possible on the charge. The magnitude of this force is proportional to the amount of charge *q*, the speed of the charged particle *v*, and the magnitude of the applied magnetic field. The direction of this force is perpendicular to both the direction of the moving charged particle and the direction of the applied magnetic field. Based on these observations, we define the magnetic field strength *B* based on the **magnetic force** $\vec{\mathbf{F}}$ on a charge *q* moving at velocity $\vec{\mathbf{v}}$ as the cross product of the velocity and magnetic field, that is,

Note:
Equation:

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}.$$

In fact, this is how we define the magnetic field $\vec{\mathbf{B}}$—in terms of the force on a charged particle moving in a magnetic field. The magnitude of the force is determined from the definition of the cross product as it relates to the magnitudes of each of the vectors. In other words, the magnitude of the force satisfies

Note:
Equation:

$$F = qvB\sin\theta$$

where θ is the angle between the velocity and the magnetic field.

The SI unit for magnetic field strength B is called the **tesla** (T) after the eccentric but brilliant inventor Nikola Tesla (1856–1943), where

Equation:

$$1\,\mathrm{T} = \frac{1\,\mathrm{N}}{\mathrm{A}\cdot\mathrm{m}}.$$

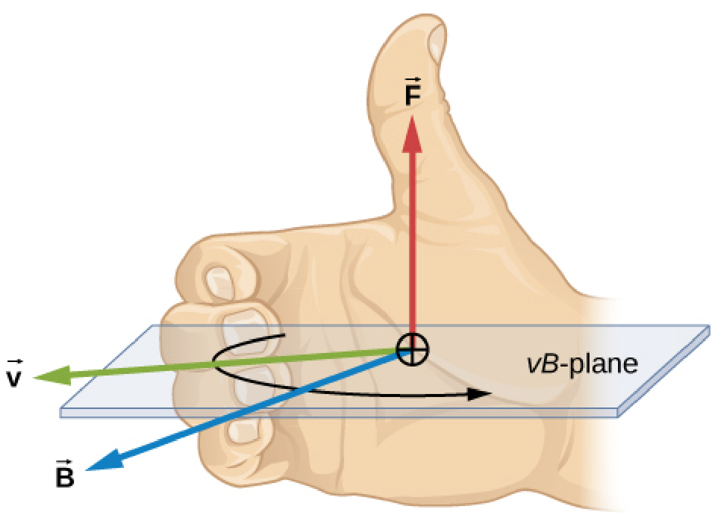
A smaller unit, called the **gauss** (G), where $1\,\mathrm{G} = 10^{-4}\mathrm{T}$, is sometimes used. The strongest permanent magnets have fields near 2 T; superconducting electromagnets may attain 10 T or more. Earth's magnetic field on its surface is only about $5\ \times\ 10^{-5}\mathrm{T}$, or 0.5 G.

Note:

Problem-Solving Strategy: Direction of the Magnetic Field by the Right-Hand Rule

The direction of the magnetic force $\vec{\mathbf{F}}$ is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, as determined by the **right-hand rule-1** (or RHR-1), which is illustrated in [link].

1. Orient your right hand so that your fingers curl in the plane defined by the velocity and magnetic field vectors.
2. Using your right hand, sweep from the velocity toward the magnetic field with your fingers through the smallest angle possible.
3. The magnetic force is directed where your thumb is pointing.
4. If the charge was negative, reverse the direction found by these steps.



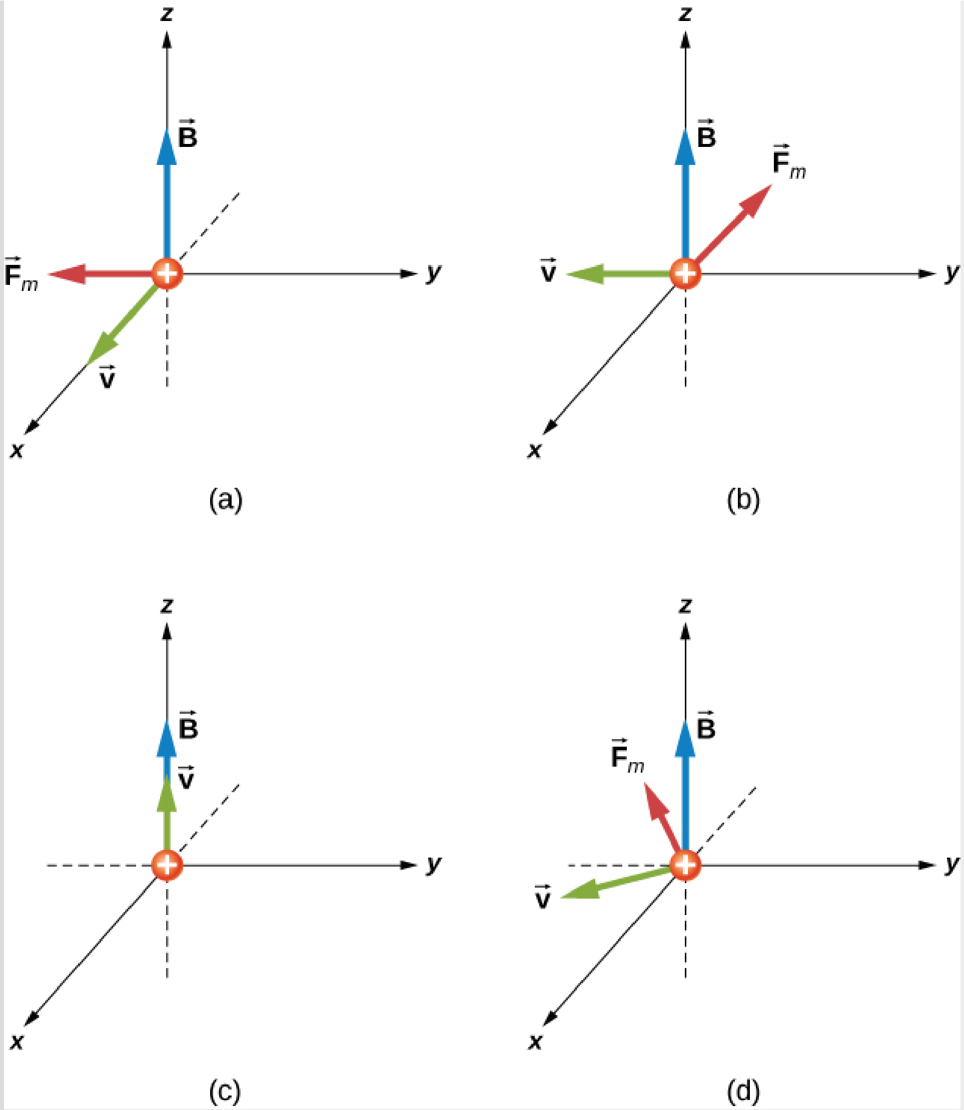
Magnetic fields exert forces on moving charges. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ and

follows the right-hand rule-1 (RHR-1) as shown. The magnitude of the force is proportional to q, v, B, and the sine of the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.

Note:
Visit this website for additional practice with the direction of magnetic fields.

There is no magnetic force on static charges. However, there is a magnetic force on charges moving at an angle to a magnetic field. When charges are stationary, their electric fields do not affect magnets. However, when charges move, they produce magnetic fields that exert forces on other magnets. When there is relative motion, a connection between electric and magnetic forces emerges—each affects the other.

Example:
An Alpha-Particle Moving in a Magnetic Field
An alpha-particle $\left(q = 3.2 \times 10^{-19}\text{C}\right)$ moves through a uniform magnetic field whose magnitude is 1.5 T. The field is directly parallel to the positive *z*-axis of the rectangular coordinate system of [link]. What is the magnetic force on the alpha-particle when it is moving (a) in the positive *x*-direction with a speed of $5.0 \times 10^4\text{m/s}$? (b) in the negative *y*-direction with a speed of $5.0 \times 10^4\text{m/s}$? (c) in the positive *z*-direction with a speed of $5.0 \times 10^4\text{m/s}$? (d) with a velocity $\vec{\mathbf{v}} = \left(2.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}} + 1.0\widehat{\mathbf{k}}\right) \times 10^4\text{m/s}$?



The magnetic forces on an alpha-particle moving in a uniform magnetic field. The field is the same in each drawing, but the velocity is different.

Strategy
We are given the charge, its velocity, and the magnetic field strength and direction. We can thus use the equation $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ or $F = qvB\sin\theta$ to calculate the force. The direction of the force is determined by RHR-1.
Solution

a. First, to determine the direction, start with your fingers pointing in the positive *x*-direction. Sweep your fingers upward in the direction of magnetic field. Your thumb should point in the negative *y*-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:
Equation:

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \left(3.2 \times 10^{-19}\mathrm{C}\right)\left(5.0 \times 10^{4}\mathrm{m/s}\,\hat{\mathbf{i}}\right) \times \left(1.5\,\mathrm{T}\,\widehat{\mathbf{k}}\right) = -2.4 \times 10^{-14}\mathrm{N}\,\hat{\mathbf{j}}.$$

b. First, to determine the directionality, start with your fingers pointing in the negative *y*-direction. Sweep your fingers upward in the direction of magnetic field as in the previous problem. Your thumb should be open in the negative *x*-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:
Equation:

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \left(3.2 \times 10^{-19}\mathrm{C}\right)\left(-5.0 \times 10^{4}\mathrm{m/s}\,\hat{\mathbf{j}}\right) \times \left(1.5\,\mathrm{T}\,\widehat{\mathbf{k}}\right) = -2.4 \times 10^{-14}\mathrm{N}\,\hat{\mathbf{i}}.$$

An alternative approach is to use [link] to find the magnitude of the force. This applies for both parts (a) and (b). Since the velocity is perpendicular to the magnetic field, the angle between them is 90 degrees. Therefore, the magnitude of the force is:
Equation:

$$F = qvB\sin\theta = \left(3.2 \times 10^{-19}\mathrm{C}\right)\left(5.0 \times 10^{4}\mathrm{m/s}\right)(1.5\ \mathrm{T})\sin(90^\circ) = 2.4 \times 10^{-14}\mathrm{N}.$$

c. Since the velocity and magnetic field are parallel to each other, there is no orientation of your hand that will result in a force direction. Therefore, the force on this moving charge is zero. This is confirmed by the cross product. When you cross two vectors pointing in the same direction, the result is equal to zero.

d. First, to determine the direction, your fingers could point in any orientation; however, you must sweep your fingers upward in the direction of the magnetic field. As you rotate your hand, notice that the thumb can point in any *x*- or *y*-direction possible, but not in the *z*-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:
Equation:

$$\begin{aligned}\vec{\mathbf{F}} &= q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \left(3.2 \times 10^{-19}\mathrm{C}\right)\left(\left(2.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}} + 1.0\widehat{\mathbf{k}}\right) \times 10^{4}\mathrm{m/s}\right) \times \left(1.5\,\mathrm{T}\,\widehat{\mathbf{k}}\right)\\ &= \left(-14.4\hat{\mathbf{i}} - 9.6\hat{\mathbf{j}}\right) \times 10^{-15}\mathrm{N}.\end{aligned}$$

This solution can be rewritten in terms of a magnitude and angle in the *xy*-plane:
Equation:

$$\begin{aligned}\left|\vec{\mathbf{F}}\right| &= \sqrt{F_x^2 + F_y^2} = \sqrt{(-14.4)^2 + (-9.6)^2} \times 10^{-15}\mathrm{N} = 1.7 \times 10^{-14}\mathrm{N}\\ \theta &= \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-9.6 \times 10^{-15}\mathrm{N}}{-14.4 \times 10^{-15}\mathrm{N}}\right) = 34^\circ.\end{aligned}$$

The magnitude of the force can also be calculated using [link]. The velocity in this question, however, has three components. The *z*-component of the velocity can be neglected, because it is parallel to the magnetic field and therefore generates no force. The magnitude of the velocity

is calculated from the *x*- and *y*-components. The angle between the velocity in the *xy*-plane and the magnetic field in the *z*-plane is 90 degrees. Therefore, the force is calculated to be:

Equation:

$$\begin{array}{ccc}|\vec{\mathbf{v}}| & = & \sqrt{(2)^2+(-3)^2}\ \times\ 10^4\frac{\text{m}}{\text{s}}=3.6\ \times\ 10^4\frac{\text{m}}{\text{s}}\\ F & = & qvB\sin\theta=(3.2\ \times\ 10^{-19}\text{C})(3.6\ \times\ 10^4\text{m/s})(1.5\text{ T})\sin(90^\circ)=1.7\ \times\ 10^{-14}\text{N}.\end{array}$$

This is the same magnitude of force calculated by unit vectors.

Significance

The cross product in this formula results in a third vector that must be perpendicular to the other two. Other physical quantities, such as angular momentum, also have three vectors that are related by the cross product. Note that typical force values in magnetic force problems are much larger than the gravitational force. Therefore, for an isolated charge, the magnetic force is the dominant force governing the charge's motion.

Note:

Exercise:

Problem:

Check Your Understanding Repeat the previous problem with the magnetic field in the *x*-direction rather than in the *z*-direction. Check your answers with RHR-1.

Solution:

a. 0 N; b. $2.4\ \times\ 10^{-14}\widehat{\mathbf{k}}\text{N}$; c. $2.4\ \times\ 10^{-14}\hat{\mathbf{j}}\text{ N}$; d. $(7.2\hat{\mathbf{j}}+2.2\widehat{\mathbf{k}})\ \times\ 10^{-15}\text{N}$

Representing Magnetic Fields

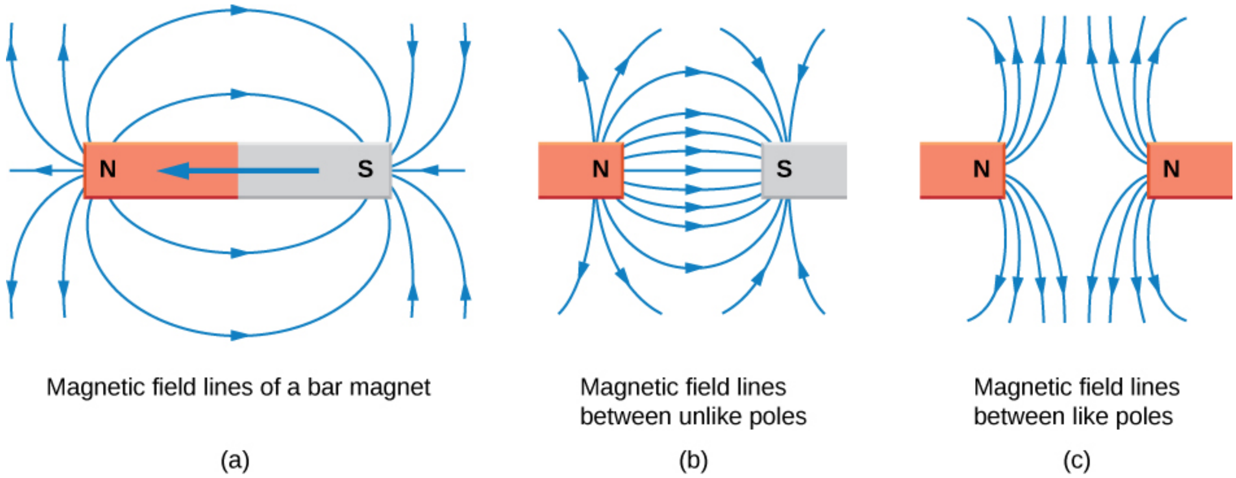
The representation of magnetic fields by **magnetic field lines** is very useful in visualizing the strength and direction of the magnetic field. As shown in [link], each of these lines forms a closed loop, even if not shown by the constraints of the space available for the figure. The field lines emerge from the north pole (N), loop around to the south pole (S), and continue through the bar magnet back to the north pole.

Magnetic field lines have several hard-and-fast rules:

1. The direction of the magnetic field is tangent to the field line at any point in space. A small compass will point in the direction of the field line.
2. The strength of the field is proportional to the closeness of the lines. It is exactly proportional to the number of lines per unit area perpendicular to the lines (called the areal density).
3. Magnetic field lines can never cross, meaning that the field is unique at any point in space.

4. Magnetic field lines are continuous, forming closed loops without a beginning or end. They are directed from the north pole to the south pole.

The last property is related to the fact that the north and south poles cannot be separated. It is a distinct difference from electric field lines, which generally begin on positive charges and end on negative charges or at infinity. If isolated magnetic charges (referred to as magnetic monopoles) existed, then magnetic field lines would begin and end on them.



Magnetic field lines are defined to have the direction in which a small compass points when placed at a location in the field. The strength of the field is proportional to the closeness (or density) of the lines. If the interior of the magnet could be probed, the field lines would be found to form continuous, closed loops. To fit in a reasonable space, some of these drawings may not show the closing of the loops; however, if enough space were provided, the loops would be closed.

Summary

- Charges moving across a magnetic field experience a force determined by $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. The force is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.
- The direction of the force on a moving charge is given by the right hand rule 1 (RHR-1): Sweep your fingers in a velocity, magnetic field plane. Start by pointing them in the direction of velocity and sweep towards the magnetic field. Your thumb points in the direction of the magnetic force for positive charges.
- Magnetic fields can be pictorially represented by magnetic field lines, which have the following properties:

    1. The field is tangent to the magnetic field line.
    2. Field strength is proportional to the line density.
    3. Field lines cannot cross.
    4. Field lines form continuous, closed loops.

- Magnetic poles always occur in pairs of north and south—it is not possible to isolate north and south poles.

Conceptual Questions

Exercise:

Problem:

Discuss the similarities and differences between the electrical force on a charge and the magnetic force on a charge.

Solution:

Both are field dependent. Electrical force is dependent on charge, whereas magnetic force is dependent on current or rate of charge flow.

Exercise:

Problem:

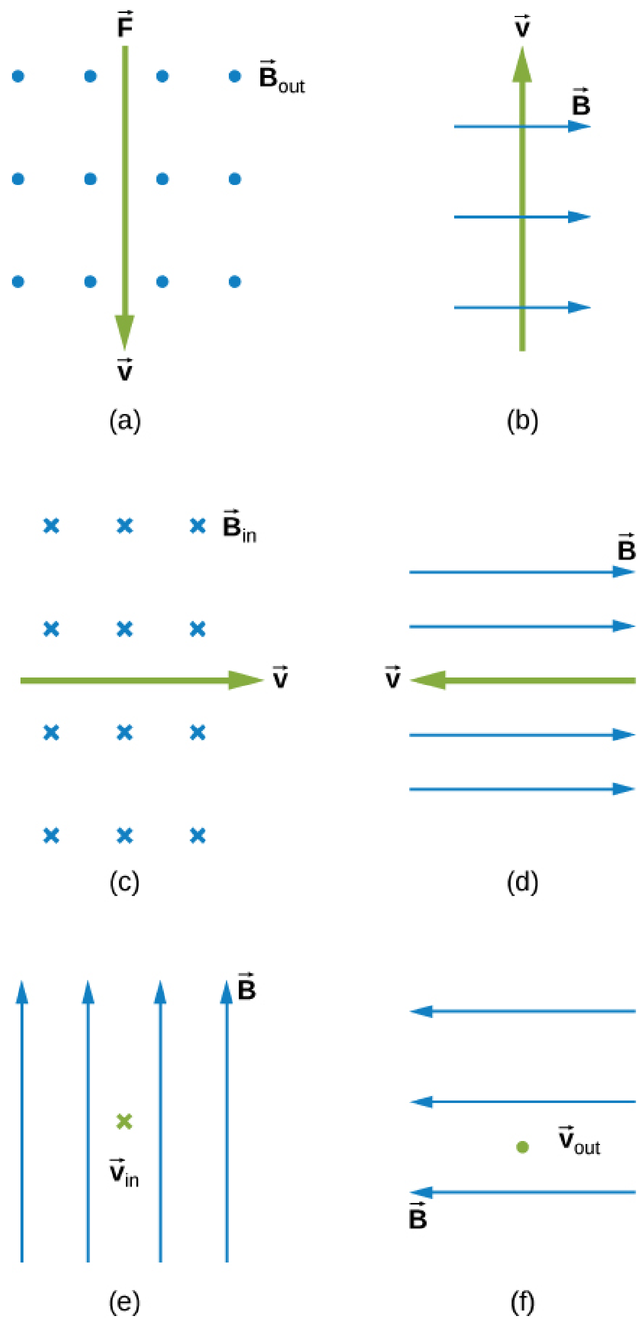
(a) Is it possible for the magnetic force on a charge moving in a magnetic field to be zero? (b) Is it possible for the electric force on a charge moving in an electric field to be zero? (c) Is it possible for the resultant of the electric and magnetic forces on a charge moving simultaneously through both fields to be zero?

Problems

Exercise:

Problem:

What is the direction of the magnetic force on a positive charge that moves as shown in each of the six cases?



Solution:

a. left; b. into the page; c. up the page; d. no force; e. right; f. down

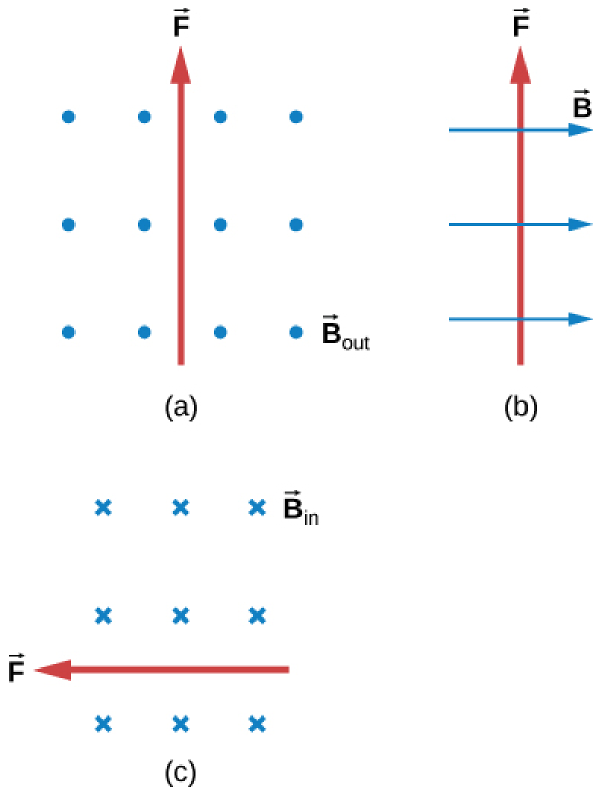
Exercise:

Problem: Repeat previous exercise for a negative charge.

Exercise:

Problem:

What is the direction of the velocity of a negative charge that experiences the magnetic force shown in each of the three cases, assuming it moves perpendicular to *B*?



Solution:

a. right; b. into the page; c. down

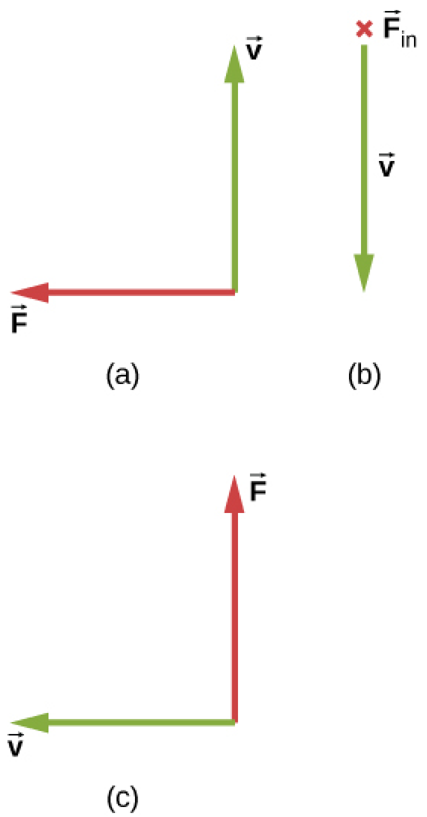
Exercise:

Problem: Repeat previous exercise for a positive charge.

Exercise:

Problem:

What is the direction of the magnetic field that produces the magnetic force on a positive charge as shown in each of the three cases, assuming $\vec{B}$ is perpendicular to $\vec{v}$?



Solution:

a. into the page; b. left; c. out of the page

Exercise:

Problem: Repeat previous exercise for a negative charge.

Exercise:

Problem:

(a) Aircraft sometimes acquire small static charges. Suppose a supersonic jet has a 0.500-μC charge and flies due west at a speed of 660. m/s over Earth's south magnetic pole, where the $8.00 \times 10^{-5} - \text{T}$ magnetic field points straight down into the ground. What are the direction and the magnitude of the magnetic force on the plane? (b) Discuss whether the value obtained in part (a) implies this is a significant or negligible effect.

Solution:

a. 2.64×10^{-8} N; north b. The force is very small, so this implies that the effect of static charges on airplanes is negligible.

Exercise:

Problem:

(a) A cosmic ray proton moving toward Earth at $5.00 \times 10^7 \mathrm{m/s}$ experiences a magnetic force of $1.70 \times 10^{-16}\ \mathrm{N}$. What is the strength of the magnetic field if there is a 45º angle between it and the proton’s velocity? (b) Is the value obtained in part a. consistent with the known strength of Earth’s magnetic field on its surface? Discuss.

Exercise:

Problem:

An electron moving at $4.00 \times 10^3 \mathrm{m/s}$ in a 1.25-T magnetic field experiences a magnetic force of $1.40 \times 10^{-16}\ \mathrm{N}$. What angle does the velocity of the electron make with the magnetic field? There are two answers.

Solution:

$10.1^\circ; 169.9^\circ$

Exercise:

Problem:

(a) A physicist performing a sensitive measurement wants to limit the magnetic force on a moving charge in her equipment to less than $1.00 \times 10^{-12}\ \mathrm{N}$. What is the greatest the charge can be if it moves at a maximum speed of 30.0 m/s in Earth’s field? (b) Discuss whether it would be difficult to limit the charge to less than the value found in (a) by comparing it with typical static electricity and noting that static is often absent.

Glossary

gauss
: G, unit of the magnetic field strength; $1\ \mathrm{G} = 10^{-4}\mathrm{T}$

magnetic field lines
: continuous curves that show the direction of a magnetic field; these lines point in the same direction as a compass points, toward the magnetic south pole of a bar magnet

magnetic force
: force applied to a charged particle moving through a magnetic field

right-hand rule-1
: using your right hand to determine the direction of either the magnetic force, velocity of a charged particle, or magnetic field

tesla
: SI unit for magnetic field: 1 T = 1 N/A-m

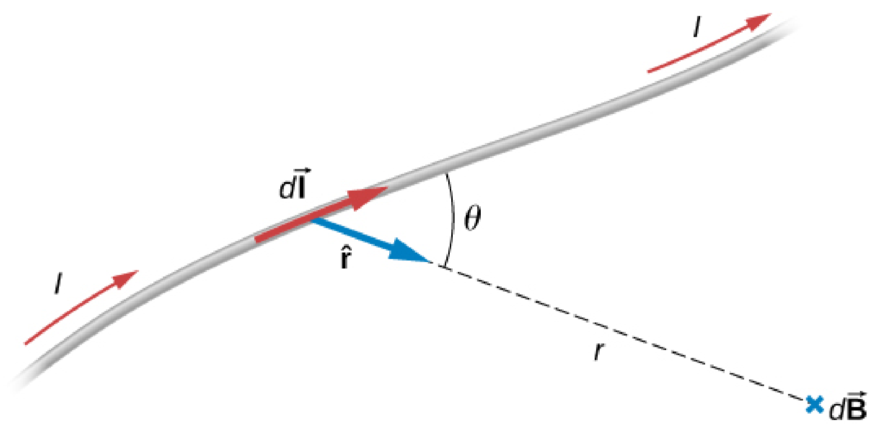
The Biot-Savart Law
By the end of this section, you will be able to:

- Explain how to derive a magnetic field from an arbitrary current in a line segment
- Calculate magnetic field from the Biot-Savart law in specific geometries, such as a current in a line and a current in a circular arc

We have seen that mass produces a gravitational field and also interacts with that field. Charge produces an electric field and also interacts with that field. Since moving charge (that is, current) interacts with a magnetic field, we might expect that it also creates that field—and it does.

The equation used to calculate the magnetic field produced by a current is known as the Biot-Savart law. It is an empirical law named in honor of two scientists who investigated the interaction between a straight, current-carrying wire and a permanent magnet. This law enables us to calculate the magnitude and direction of the magnetic field produced by a current in a wire. The **Biot-Savart law** states that at any point *P* ([link]), the magnetic field $d\vec{\mathbf{B}}$ due to an element $d\vec{\mathbf{l}}$ of a current-carrying wire is given by
Equation:

$$d\vec{\mathbf{B}}=\frac{\mu_0}{4\pi}\frac{Id\vec{\mathbf{l}}\times\hat{\mathbf{r}}}{r^2}.$$



A current element $Id\vec{\mathbf{l}}$ produces a magnetic field at point *P* given by the Biot-Savart law.

The constant μ_0 is known as the **permeability of free space** and is exactly

Note:

Equation:

$$\mu_0 = 4\pi \times 10^{-7} \mathrm{T} \cdot \mathrm{m/A}$$

in the SI system. The infinitesimal wire segment $d\vec{\mathbf{l}}$ is in the same direction as the current I (assumed positive), r is the distance from $d\vec{\mathbf{l}}$ to P and $\hat{\mathbf{r}}$ is a unit vector that points from $d\vec{\mathbf{l}}$ to P, as shown in the figure.

The direction of $d\vec{\mathbf{B}}$ is determined by applying the right-hand rule to the vector product $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The magnitude of $d\vec{\mathbf{B}}$ is

Note:
Equation:

$$dB = \frac{\mu_0}{4\pi} \frac{I\,dl\,\sin\theta}{r^2}$$

where θ is the angle between $d\vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$. Notice that if $\theta = 0$, then $d\vec{\mathbf{B}} = \vec{\mathbf{0}}$. The field produced by a current element $Id\vec{\mathbf{l}}$ has no component parallel to $d\vec{\mathbf{l}}$.

The magnetic field due to a finite length of current-carrying wire is found by integrating [link] along the wire, giving us the usual form of the Biot-Savart law.

Note:
Biot-Savart law
The magnetic field $\vec{\mathbf{B}}$ due to an element $d\vec{\mathbf{l}}$ of a current-carrying wire is given by
Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I\,d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}.$$

Since this is a vector integral, contributions from different current elements may not point in the same direction. Consequently, the integral is often difficult to evaluate, even for fairly simple geometries. The following strategy may be helpful.

Note:

Problem-Solving Strategy: Solving Biot-Savart Problems

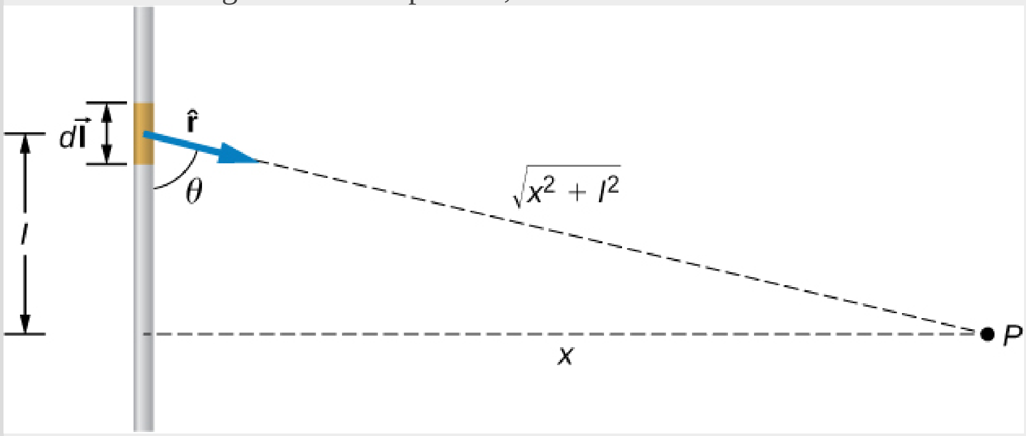
To solve Biot-Savart law problems, the following steps are helpful:

1. Identify that the Biot-Savart law is the chosen method to solve the given problem. If there is symmetry in the problem comparing $\vec{\mathbf{B}}$ and $d\vec{\mathbf{l}}$, Ampère's law may be the preferred method to solve the question.
2. Draw the current element length $d\vec{\mathbf{l}}$ and the unit vector $\hat{\mathbf{r}}$, noting that $d\vec{\mathbf{l}}$ points in the direction of the current and $\hat{\mathbf{r}}$ points from the current element toward the point where the field is desired.
3. Calculate the cross product $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field according to the Biot-Savart law.
4. Use [link] and substitute all given quantities into the expression to solve for the magnetic field. Note all variables that remain constant over the entire length of the wire may be factored out of the integration.
5. Use the right-hand rule to verify the direction of the magnetic field produced from the current or to write down the direction of the magnetic field if only the magnitude was solved for in the previous part.

Example:

Calculating Magnetic Fields of Short Current Segments

A short wire of length 1.0 cm carries a current of 2.0 A in the vertical direction ([link]). The rest of the wire is shielded so it does not add to the magnetic field produced by the wire. Calculate the magnetic field at point *P*, which is 1 meter from the wire in the *x*-direction.



A small line segment carries a current *I* in the vertical direction. What is the magnetic field at a distance *x* from the segment?

Strategy

We can determine the magnetic field at point *P* using the Biot-Savart law. Since the current segment is much smaller than the distance *x*, we can drop the integral from the expression. The integration is converted back into a summation, but only for small *dl*, which we now write as Δl. Another way to think about it is that each of the radius values is nearly the same, no matter where the current element is on the line segment, if Δl is small compared to *x*. The angle θ is calculated using a tangent function. Using the numbers given, we can calculate the magnetic field at *P*.

Solution

The angle between $\Delta \vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$ is calculated from trigonometry, knowing the distances *l* and *x* from the problem:

Equation:

$$\theta = \tan^{-1}\left(\frac{1\text{ m}}{0.01\text{ m}}\right) = 89.4^{\circ}.$$

The magnetic field at point *P* is calculated by the Biot-Savart law:

Equation:

$$B = \frac{\mu_0}{4\pi}\frac{I\Delta l\sin\theta}{r^2} = (1\times 10^{-7}\text{T}\cdot\text{m/A})\left(\frac{2\text{ A}(0.01\text{ m})\sin(89.4^{\circ})}{(1\text{ m})^2}\right) = 2.0\times 10^{-9}\text{T}.$$

From the right-hand rule and the Biot-Savart law, the field is directed into the page.

Significance

This approximation is only good if the length of the line segment is very small compared to the distance from the current element to the point. If not, the integral form of the Biot-Savart law must be used over the entire line segment to calculate the magnetic field.

Note:

Exercise:

Problem:

Check Your Understanding Using [link], at what distance would *P* have to be to measure a magnetic field half of the given answer?

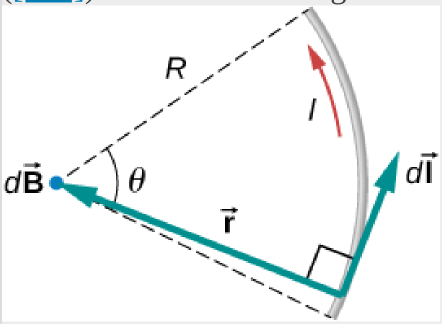
Solution:

1.41 meters

Example:

Calculating Magnetic Field of a Circular Arc of Wire

A wire carries a current I in a circular arc with radius R swept through an arbitrary angle θ ([link]). Calculate the magnetic field at the center of this arc at point P.



A wire segment carrying a current I. The path $d\vec{\mathbf{l}}$ and radial direction $\hat{\mathbf{r}}$ are indicated.

Strategy

We can determine the magnetic field at point P using the Biot-Savart law. The radial and path length directions are always at a right angle, so the cross product turns into multiplication. We also know that the distance along the path dl is related to the radius times the angle θ (in radians). Then we can pull all constants out of the integration and solve for the magnetic field.

Solution

The Biot-Savart law starts with the following equation:

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}.$$

As we integrate along the arc, all the contributions to the magnetic field are in the same direction (out of the page), so we can work with the magnitude of the field. The cross product turns into multiplication because the path dl and the radial direction are perpendicular. We can also substitute the arc length formula, $dl = rd\theta$:

Equation:

$$B = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{Ir\, d\theta}{r^2}.$$

The current and radius can be pulled out of the integral because they are the same regardless of where we are on the path. This leaves only the integral over the angle,

Equation:

$$B = \frac{\mu_0 I}{4\pi r} \int_{\text{wire}} d\theta.$$

The angle varies on the wire from 0 to θ; hence, the result is

Equation:

$$B = \frac{\mu_0 I\theta}{4\pi r}.$$

Significance

The direction of the magnetic field at point *P* is determined by the right-hand rule, as shown in the previous chapter. If there are other wires in the diagram along with the arc, and you are asked to find the net magnetic field, find each contribution from a wire or arc and add the results by superposition of vectors. Make sure to pay attention to the direction of each contribution. Also note that in a symmetric situation, like a straight or circular wire, contributions from opposite sides of point *P* cancel each other.

Note:

Exercise:

Problem:

Check Your Understanding The wire loop forms a full circle of radius *R* and current *I*. What is the magnitude of the magnetic field at the center?

Solution:

$\frac{\mu_0 I}{2R}$

Summary

- The magnetic field created by a current-carrying wire is found by the Biot-Savart law.
- The current element $Id\vec{\mathbf{l}}$ produces a magnetic field a distance *r* away.

Conceptual Questions

Exercise:

Problem:

For calculating magnetic fields, what are the advantages and disadvantages of the Biot-Savart law?

Solution:

Biot-Savart law's advantage is that it works with any magnetic field produced by a current loop. The disadvantage is that it can take a long time.

Exercise:

Problem:

Describe the magnetic field due to the current in two wires connected to the two terminals of a source of emf and twisted tightly around each other.

Exercise:

Problem: How can you decide if a wire is infinite?

Solution:

If you were to go to the start of a line segment and calculate the angle θ to be approximately 0°, the wire can be considered infinite. This judgment is based also on the precision you need in the result.

Exercise:

Problem:

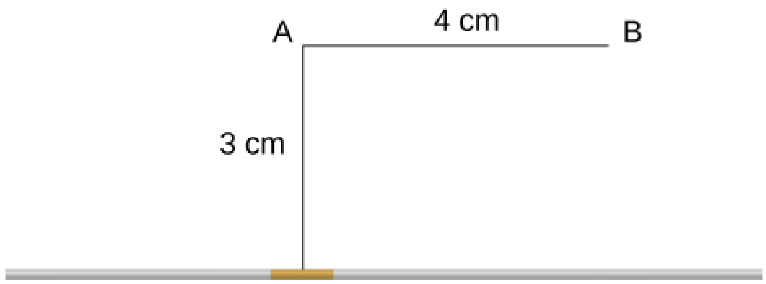
Identical currents are carried in two circular loops; however, one loop has twice the diameter as the other loop. Compare the magnetic fields created by the loops at the center of each loop.

Problems

Exercise:

Problem:

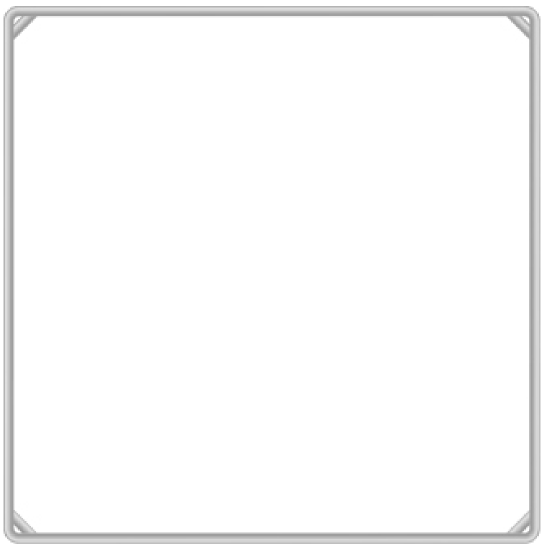
A 10-A current flows through the wire shown. What is the magnitude of the magnetic field due to a 0.5-mm segment of wire as measured at (a) point A and (b) point B?



Exercise:

Problem:

Ten amps flow through a square loop where each side is 20 cm in length. At each corner of the loop is a 0.01-cm segment that connects the longer wires as shown. Calculate the magnitude of the magnetic field at the center of the loop.

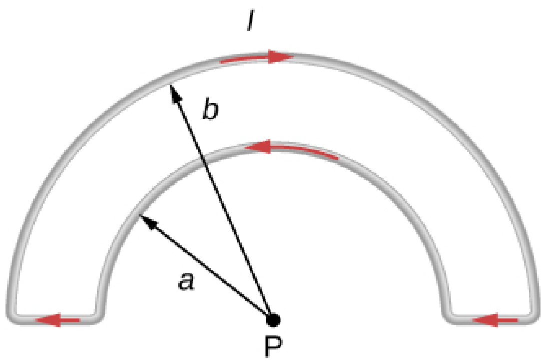


Solution:

$5.66 \times 10^{-5} \text{T}$

Exercise:

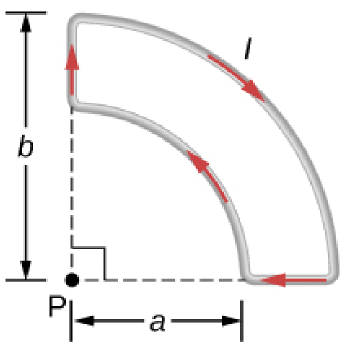
Problem: What is the magnetic field at P due to the current *I* in the wire shown?



Exercise:

Problem:

The accompanying figure shows a current loop consisting of two concentric circular arcs and two perpendicular radial lines. Determine the magnetic field at point P.



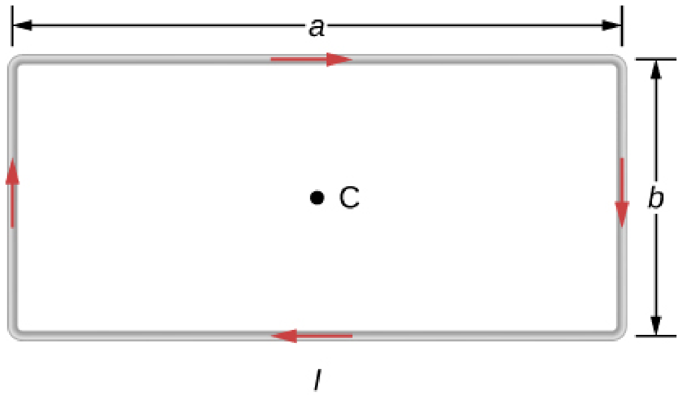
Solution:

$B = \frac{\mu_o I}{8} \ \frac{1}{a} - \frac{1}{b}$ out of the page

Exercise:

Problem:

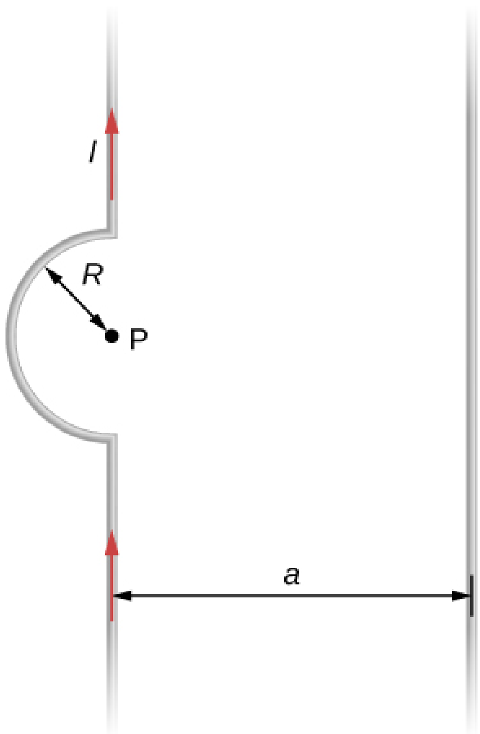
Find the magnetic field at the center C of the rectangular loop of wire shown in the accompanying figure.



Exercise:

Problem:

Two long wires, one of which has a semicircular bend of radius *R*, are positioned as shown in the accompanying figure. If both wires carry a current *I*, how far apart must their parallel sections be so that the net magnetic field at P is zero? Does the current in the straight wire flow up or down?



Solution:

$a = \frac{2R}{\pi}$; the current in the wire to the right must flow up the page.

Glossary

Biot-Savart law
: an equation giving the magnetic field at a point produced by a current-carrying wire

permeability of free space
: μ_0, measure of the ability of a material, in this case free space, to support a magnetic field

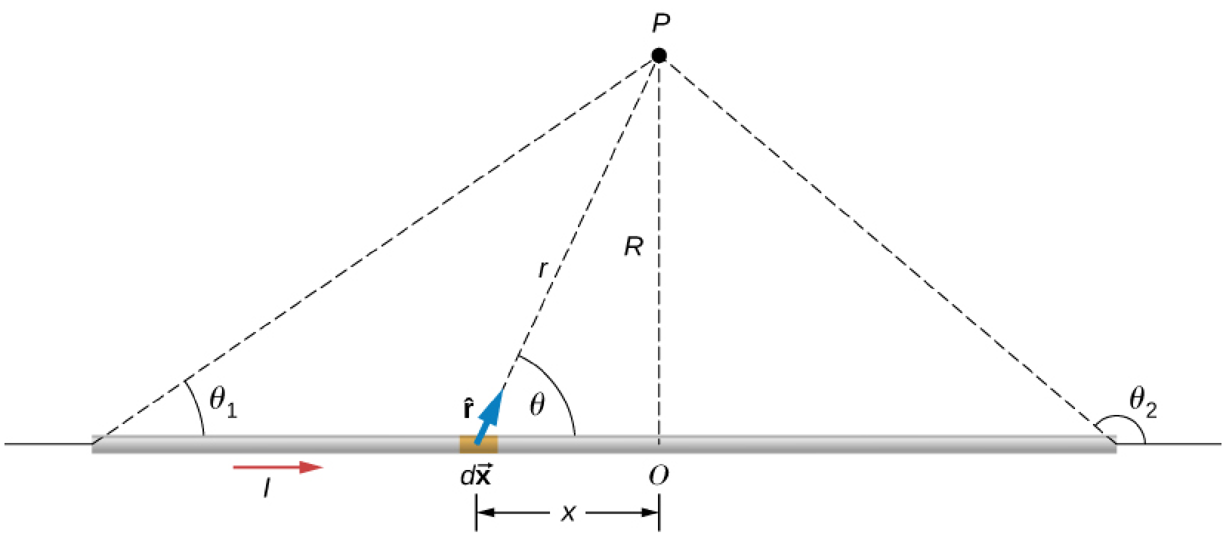
Magnetic Field Due to a Thin Straight Wire

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a thin, straight wire.
- Determine the dependence of the magnetic field from a thin, straight wire based on the distance from it and the current flowing in the wire.
- Sketch the magnetic field created from a thin, straight wire by using the second right-hand rule.

How much current is needed to produce a significant magnetic field, perhaps as strong as Earth's field? Surveyors will tell you that overhead electric power lines create magnetic fields that interfere with their compass readings. Indeed, when Oersted discovered in 1820 that a current in a wire affected a compass needle, he was not dealing with extremely large currents. How does the shape of wires carrying current affect the shape of the magnetic field created? We noted in Chapter 28 that a current loop created a magnetic field similar to that of a bar magnet, but what about a straight wire? We can use the Biot-Savart law to answer all of these questions, including determining the magnetic field of a long straight wire.

[link] shows a section of an infinitely long, straight wire that carries a current *I*. What is the magnetic field at a point *P*, located a distance *R* from the wire?



A section of a thin, straight current-carrying wire. The independent

variable θ has the limits θ_1 and θ_2.

Let’s begin by considering the magnetic field due to the current element $I\,d\vec{\mathbf{x}}$ located at the position *x*. Using the right-hand rule 1 from the previous chapter, $d\vec{\mathbf{x}}\ \times\ \hat{\mathbf{r}}$ points out of the page for any element along the wire. At point *P*, therefore, the magnetic fields due to all current elements have the same direction. This means that we can calculate the net field there by evaluating the scalar sum of the contributions of the elements. With $\left|d\vec{\mathbf{x}}\ \times\ \hat{\mathbf{r}}\right| = (dx)(1)\sin\theta$, we have from the Biot-Savart law

Equation:

$$B = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{I\sin\theta\,dx}{r^2}.$$

The wire is symmetrical about point *O*, so we can set the limits of the integration from zero to infinity and double the answer, rather than integrate from negative infinity to positive infinity. Based on the picture and geometry, we can write expressions for *r* and $\sin\theta$ in terms of *x* and *R*, namely:

Equation:

$$\begin{array}{rcl} r & = & \sqrt{x^2 + R^2} \\ \sin\theta & = & \frac{R}{\sqrt{x^2+R^2}}. \end{array}$$

Substituting these expressions into [link], the magnetic field integration becomes

Equation:

$$B = \frac{\mu_o I}{2\pi}\int_0^{\infty} \frac{R\,dx}{\left(x^2 + R^2\right)^{3/2}}.$$

Evaluating the integral yields

Equation:

$$B = \frac{\mu_o I}{2\pi R}\left[\frac{x}{(x^2+R^2)^{1/2}}\right]_0^{\infty}.$$

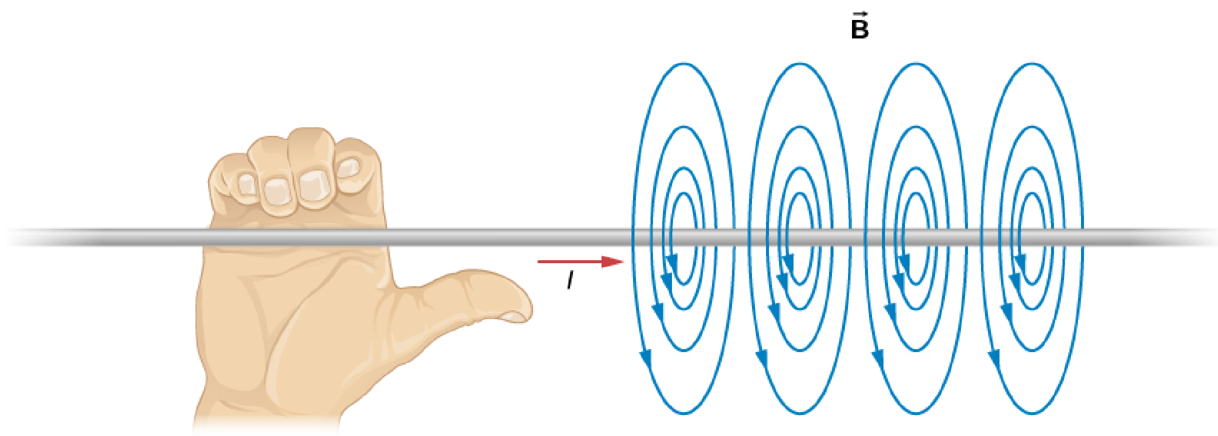
Substituting the limits gives us the solution

Note:

Equation:

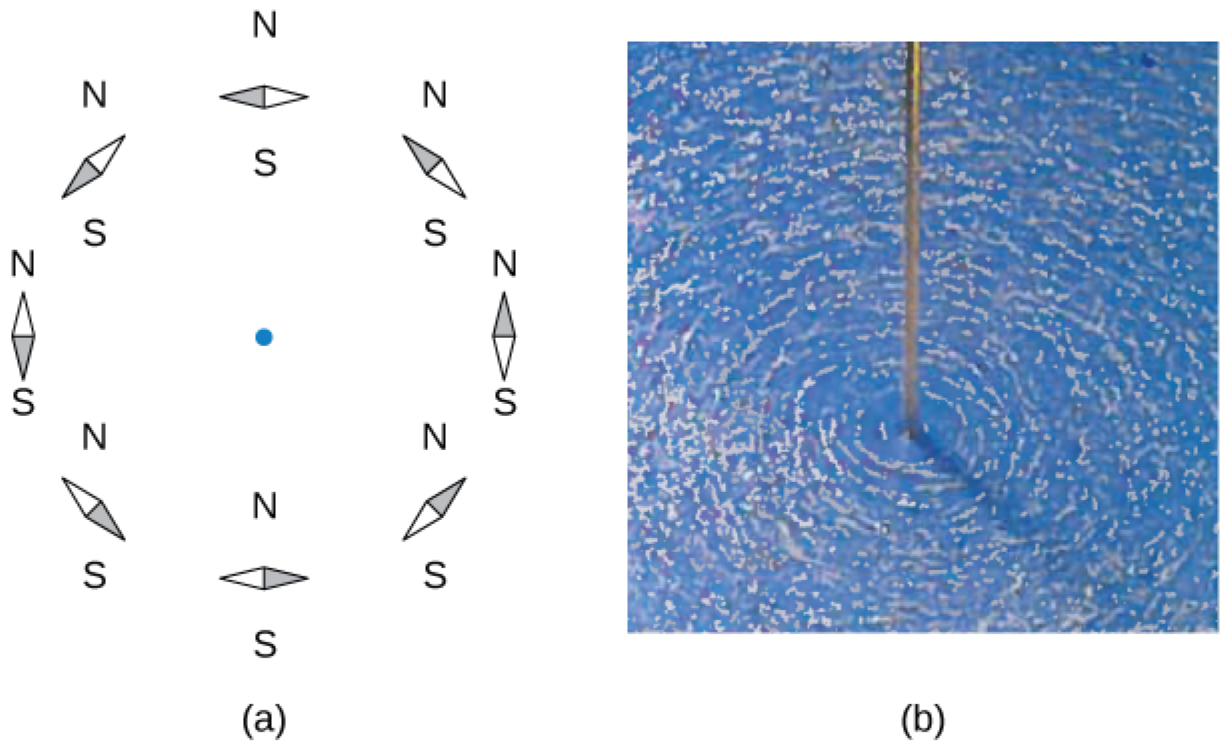
$$B = \frac{\mu_o I}{2\pi R}.$$

The magnetic field lines of the infinite wire are circular and centered at the wire ([link]), and they are identical in every plane perpendicular to the wire. Since the field decreases with distance from the wire, the spacing of the field lines must increase correspondingly with distance. The direction of this magnetic field may be found with a second form of the right-hand rule (illustrated in [link]). If you hold the wire with your right hand so that your thumb points along the current, then your fingers wrap around the wire in the same sense as $\vec{\mathbf{B}}$.



Some magnetic field lines of an infinite wire. The direction of $\vec{\mathbf{B}}$ can be found with a form of the right-hand rule.

The direction of the field lines can be observed experimentally by placing several small compass needles on a circle near the wire, as illustrated in [link]. When there is no current in the wire, the needles align with Earth’s magnetic field. However, when a large current is sent through the wire, the compass needles all point tangent to the circle. Iron filings sprinkled on a horizontal surface also delineate the field lines, as shown in [link].

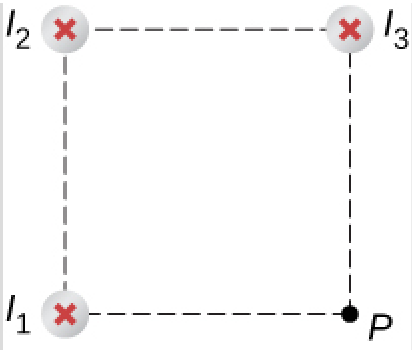


The shape of the magnetic field lines of a long wire can be seen using (a) small compass needles and (b) iron filings.

Example:

Calculating Magnetic Field Due to Three Wires

Three wires sit at the corners of a square, all carrying currents of 2 amps into the page as shown in [link]. Calculate the magnitude of the magnetic field at the other corner of the square, point *P*, if the length of each side of the square is 1 cm.



Three wires have current flowing into the page. The magnetic field is determined at the fourth corner of the square.

Strategy

The magnetic field due to each wire at the desired point is calculated. The diagonal distance is calculated using the Pythagorean theorem. Next, the direction of each magnetic field's contribution is determined by drawing a circle centered at the point of the wire and out toward the desired point. The direction of the magnetic field contribution from that wire is tangential to the curve. Lastly, working with these vectors, the resultant is calculated.

Solution

Wires 1 and 3 both have the same magnitude of magnetic field contribution at point *P*:

Equation:

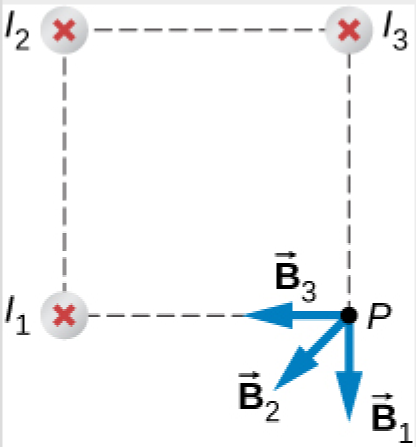
$$B_1 = B_3 = \frac{\mu_o I}{2\pi R} = \frac{(4\pi \times 10^{-7}\mathrm{T \cdot m/A})(2\ \mathrm{A})}{2\pi(0.01\ \mathrm{m})} = 4 \times 10^{-5}\mathrm{T}.$$

Wire 2 has a longer distance and a magnetic field contribution at point *P* of:

Equation:

$$B_2 = \frac{\mu_o I}{2\pi R} = \frac{(4\pi \times 10^{-7}\mathrm{T\cdot m/A})(2\,\mathrm{A})}{2\pi(0.01414\,\mathrm{m})} = 3 \times 10^{-5}\mathrm{T}.$$

The vectors for each of these magnetic field contributions are shown.



The magnetic field in the *x*-direction has contributions from wire 3 and the *x*-component of wire 2:

Equation:

$$B_{\mathrm{net}\,x} = -4 \times 10^{-5}\mathrm{T} - 2.83 \times 10^{-5}\mathrm{T}\cos(45^\circ) = -6 \times 10^{-5}\mathrm{T}.$$

The *y*-component is similarly the contributions from wire 1 and the *y*-component of wire 2:

Equation:

$$B_{\mathrm{net}\,y} = -4 \times 10^{-5}\mathrm{T} - 2.83 \times 10^{-5}\mathrm{T}\sin(45^\circ) = -6 \times 10^{-5}\mathrm{T}.$$

Therefore, the net magnetic field is the resultant of these two components:

Equation:

$$B_{\mathrm{net}} = \sqrt{B^2_{\mathrm{net}\,x} + B^2_{\mathrm{net}\,y}}$$
$$B_{\mathrm{net}} = \sqrt{(-6 \times 10^{-5}\mathrm{T})^2 + (-6 \times 10^{-5}\mathrm{T})^2}$$
$$B_{\mathrm{net}} = 8 \times 10^{-5}\mathrm{T}.$$

Significance
The geometry in this problem results in the magnetic field contributions in the *x*- and *y*-directions having the same magnitude. This is not necessarily the case if the currents were different values or if the wires were located in different positions. Regardless of the numerical results, working on the components of the vectors will yield the resulting magnetic field at the point in need.

Note:
Exercise:

Problem:

Check Your Understanding Using [link], keeping the currents the same in wires 1 and 3, what should the current be in wire 2 to counteract the magnetic fields from wires 1 and 3 so that there is no net magnetic field at point P?

Solution:

4 amps flowing out of the page

Summary

- The strength of the magnetic field created by current in a long straight wire is given by $B = \frac{\mu_0 I}{2\pi R}$ (long straight wire) where I is the current, R is the shortest distance to the wire, and the constant $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/s}$ is the permeability of free space.
- The direction of the magnetic field created by a long straight wire is given by right-hand rule 2 (RHR-2): Point the thumb of the right hand in the direction of current, and the fingers curl in the direction of the magnetic field loops created by it.

Conceptual Questions

Exercise:

Problem:

How would you orient two long, straight, current-carrying wires so that there is no net magnetic force between them? (*Hint*: What orientation would lead to one wire not experiencing a magnetic field from the other?)

Solution:

You would make sure the currents flow perpendicular to one another.

Problems

Exercise:

Problem:

A typical current in a lightning bolt is 10^4 A. Estimate the magnetic field 1 m from the bolt.

Exercise:

Problem:

The magnitude of the magnetic field 50 cm from a long, thin, straight wire is $8.0\ \mu\text{T}$. What is the current through the long wire?

Solution:

20 A

Exercise:

Problem:

A transmission line strung 7.0 m above the ground carries a current of 500 A. What is the magnetic field on the ground directly below the wire? Compare your answer with the magnetic field of Earth.

Exercise:

Problem:

A long, straight, horizontal wire carries a left-to-right current of 20 A. If the wire is placed in a uniform magnetic field of magnitude $4.0 \times 10^{-5}\mathrm{T}$ that is directed vertically downward, what is the resultant magnitude of the magnetic field 20 cm above the wire? 20 cm below the wire?

Solution:

Both answers have the magnitude of magnetic field of $4.5 \times 10^{-5}\mathrm{T}$.

Exercise:

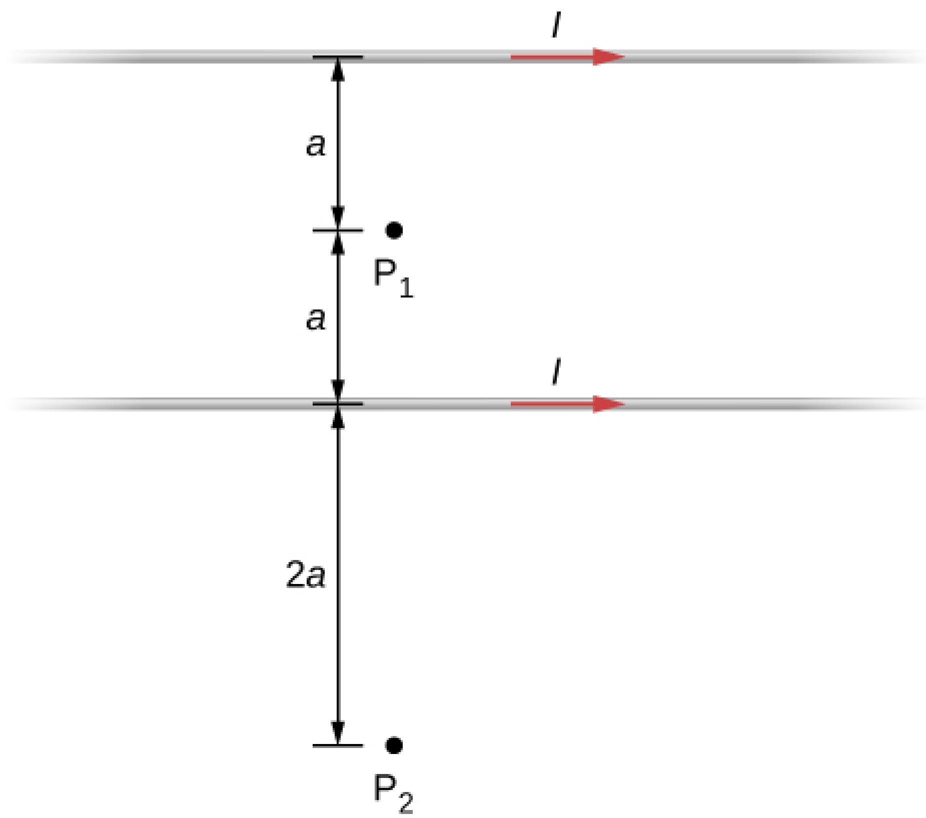
Problem:

The two long, parallel wires shown in the accompanying figure carry currents in the same direction. If $I_1 = 10\ \mathrm{A}$ and $I_2 = 20\ \mathrm{A}$, what is the magnetic field at point P?

Exercise:

Problem:

The accompanying figure shows two long, straight, horizontal wires that are parallel and a distance 2*a* apart. If both wires carry current *I* in the same direction, (a) what is the magnetic field at P_1? (b) P_2?



Solution:

At P1, the net magnetic field is zero. At P2, $B = \frac{3\mu_o I}{8\pi a}$ into the page.

Exercise:

Problem:

Repeat the calculations of the preceding problem with the direction of the current in the lower wire reversed.

Exercise:

Problem:

Consider the area between the wires of the preceding problem. At what distance from the top wire is the net magnetic field a minimum? Assume that the currents are equal and flow in opposite directions.

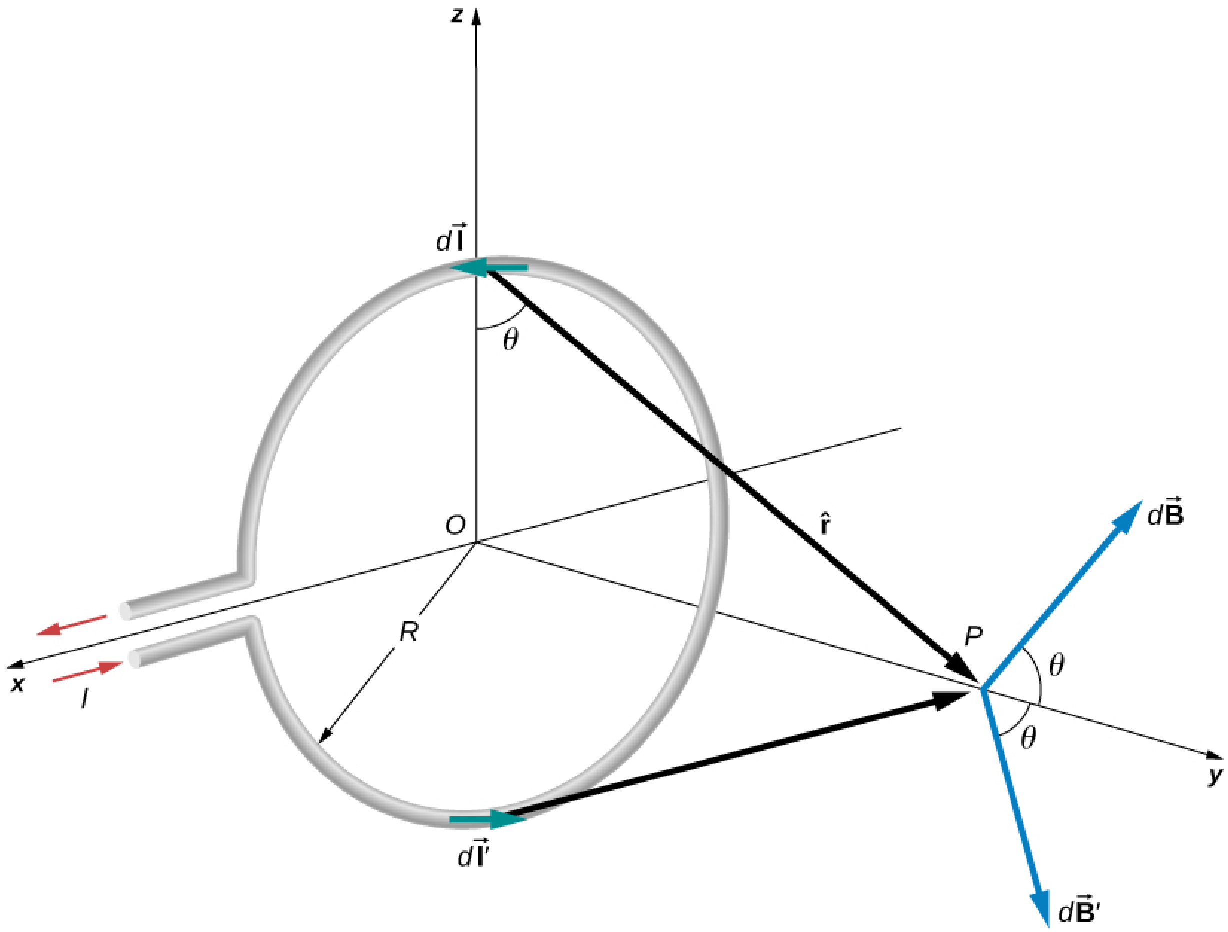
Solution:

The magnetic field is at a minimum at distance *a* from the top wire, or half-way between the wires.

Magnetic Field of a Current Loop
By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a current in a loop of wire at a point along a line perpendicular to the plane of the loop.
- Determine the magnetic field of an arc of current.

The circular loop of [link] has a radius R, carries a current I, and lies in the xz-plane. What is the magnetic field due to the current at an arbitrary point P along the axis of the loop?



Determining the magnetic field at point P along the axis of a current-carrying loop of wire.

We can use the Biot-Savart law to find the magnetic field due to a current. We first consider arbitrary segments on opposite sides of the loop to qualitatively show by the vector results that the net magnetic field direction is along the central axis from the loop. From there, we can use the Biot-Savart law to derive the expression for magnetic field.

Let P be a distance y from the center of the loop. From the right-hand rule, the magnetic field $d\vec{\mathbf{B}}$ at P, produced by the current element $I\,d\vec{\mathbf{l}}$, is directed at an angle θ above the y-axis as shown. Since $d\vec{\mathbf{l}}$ is parallel along the x-axis and $\hat{\mathbf{r}}$ is in the yz-plane, the two vectors are perpendicular, so we have

Equation:

$$dB = \frac{\mu_0}{4\pi}\,\frac{I\,dl\,\sin\pi/2}{r^2} = \frac{\mu_0}{4\pi}\,\frac{I\,dl}{y^2+R^2}$$

where we have used $r^2 = y^2 + R^2$.

Now consider the magnetic field $d\vec{\mathbf{B}}'$ due to the current element $I\,d\vec{\mathbf{l}}'$, which is directly opposite $I\,d\vec{\mathbf{l}}$ on the loop. The magnitude of $d\vec{\mathbf{B}}'$ is also given by [link], but it is directed at an angle θ below the y-axis. The components of $d\vec{\mathbf{B}}$ and $d\vec{\mathbf{B}}'$ perpendicular to the y-axis therefore cancel, and in calculating the net magnetic field, only the components along the y-axis need to be considered. The components perpendicular to the axis of the loop sum to zero in pairs. Hence at point P:

Equation:

$$\vec{\mathbf{B}} = \hat{\mathbf{j}}\int\limits_{\text{loop}} dB\,\cos\theta = \hat{\mathbf{j}}\frac{\mu_0 I}{4\pi}\int\limits_{\text{loop}}\frac{\cos\theta\,dl}{y^2+R^2}.$$

For all elements $d\vec{\mathbf{l}}$ on the wire, y, R, and $\cos\theta$ are constant and are related by

Equation:

$$\cos\theta = \frac{R}{\sqrt{y^2 + R^2}}.$$

Now from [link], the magnetic field at P is

Equation:

$$\vec{\mathbf{B}} = \hat{\mathbf{j}} \frac{\mu_0 I R}{4\pi (y^2 + R^2)^{3/2}} \int_{\text{loop}} dl = \frac{\mu_0 I R^2}{2 (y^2 + R^2)^{3/2}} \hat{\mathbf{j}}$$

where we have used $\int_{\text{loop}} dl = 2\pi R$. As discussed in the previous chapter, the closed current loop is a magnetic dipole of moment $\vec{\mu} = IA\widehat{\mathbf{n}}$. For this example, $A = \pi R^2$ and $\widehat{\mathbf{n}} = \hat{\mathbf{j}}$, so the magnetic field at P can also be written as

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0 \mu \hat{\mathbf{j}}}{2\pi (y^2 + R^2)^{3/2}}.$$

By setting $y = 0$ in [link], we obtain the magnetic field at the center of the loop:

Note:

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R} \hat{\mathbf{j}}.$$

This equation becomes $B = \mu_0 nI/(2R)$ for a flat coil of *n* loops per length. It can also be expressed as

Equation:

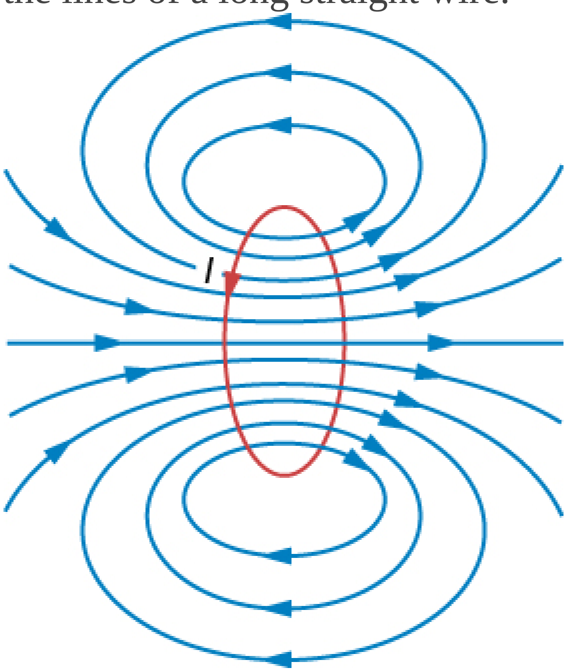
$$\vec{\mathbf{B}} = \frac{\mu_0 \vec{\mu}}{2\pi R^3}.$$

If we consider $y \gg R$ in [link], the expression reduces to an expression known as the magnetic field from a dipole:

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0 \vec{\mu}}{2\pi y^3}.$$

The calculation of the magnetic field due to the circular current loop at points off-axis requires rather complex mathematics, so we'll just look at the results. The magnetic field lines are shaped as shown in [link]. Notice that one field line follows the axis of the loop. This is the field line we just found. Also, very close to the wire, the field lines are almost circular, like the lines of a long straight wire.

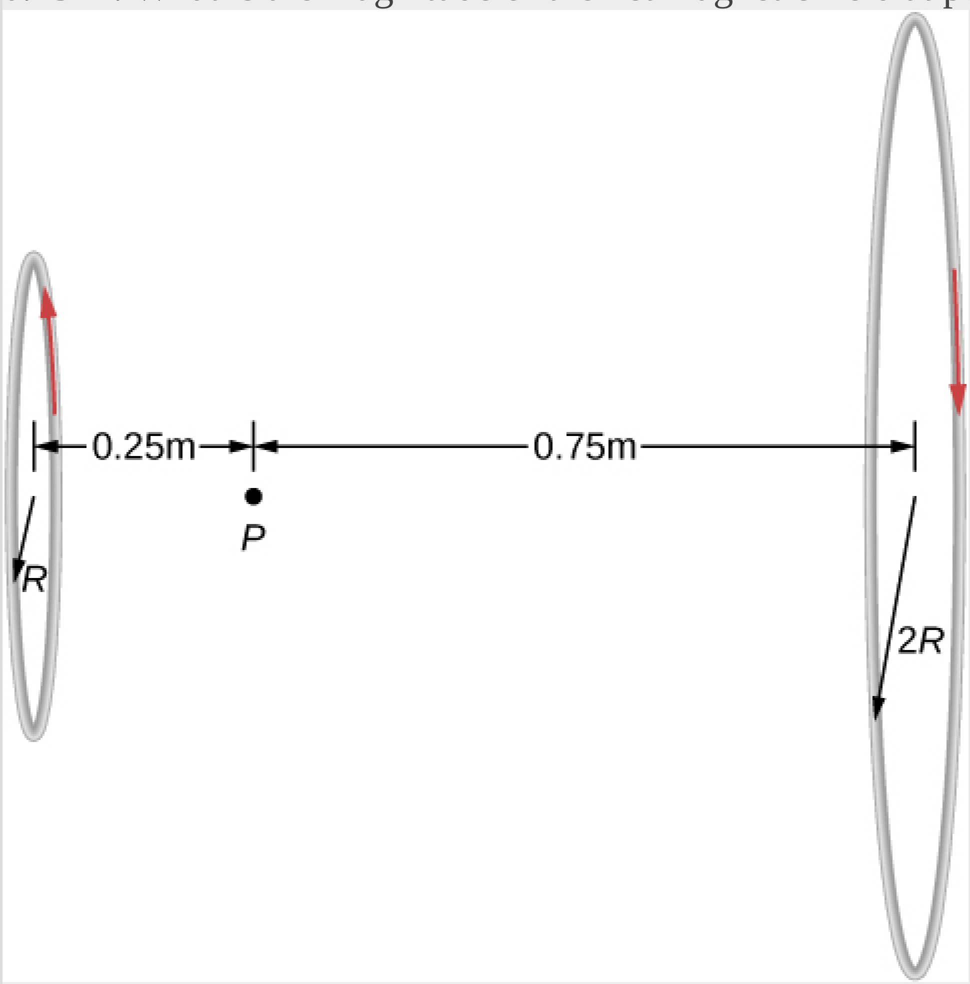


Sketch of the magnetic field lines of a circular current loop.

Example:

Magnetic Field between Two Loops

Two loops of wire carry the same current of 10 mA, but flow in opposite directions as seen in [link]. One loop is measured to have a radius of $R = 50\ \text{cm}$ while the other loop has a radius of $2R = 100\ \text{cm}$. The distance from the first loop to the point where the magnetic field is measured is 0.25 m, and the distance from that point to the second loop is 0.75 m. What is the magnitude of the net magnetic field at point P?



Two loops of different radii have the same current but flowing in opposite directions. The magnetic field at point *P* is measured to be zero.

Strategy

The magnetic field at point *P* has been determined in [link]. Since the currents are flowing in opposite directions, the net magnetic field is the difference between the two fields generated by the coils. Using the given quantities in the problem, the net magnetic field is then calculated.

Solution

Solving for the net magnetic field using [link] and the given quantities in the problem yields

Equation:

$$\begin{aligned}
B &= \frac{\mu_0 I {R_1}^2}{2({y_1}^2+{R_1}^2)^{3/2}} - \frac{\mu_0 I {R_2}^2}{2({y_2}^2+{R_2}^2)^{3/2}} \\
B &= \frac{(4\pi \times 10^{-7}\,\mathrm{T\cdot m/A})(0.010\,\mathrm{A})(0.5\,\mathrm{m})^2}{2((0.25\,\mathrm{m})^2+(0.5\,\mathrm{m})^2)^{3/2}} - \frac{(4\pi \times 10^{-7}\,\mathrm{T\cdot m/A})(0.010\,\mathrm{A})(1.0\,\mathrm{m})^2}{2((0.75\,\mathrm{m})^2+(1.0\,\mathrm{m})^2)^{3/2}} \\
B &= 5.77 \times 10^{-9}\,\mathrm{T}\ \text{to the right.}
\end{aligned}$$

Significance

Helmholtz coils typically have loops with equal radii with current flowing in the same direction to have a strong uniform field at the midpoint between the loops. A similar application of the magnetic field distribution created by Helmholtz coils is found in a magnetic bottle that can temporarily trap charged particles. See Magnetic Forces and Fields for a discussion on this.

Note:

Exercise:

Problem:

Check Your Understanding Using [link], at what distance would you have to move the first coil to have zero measurable magnetic field at point *P*?

Solution:

0.608 meters

Summary

- The magnetic field strength at the center of a circular loop is given by $B = \frac{\mu_0 I}{2R} \text{(at center of loop)}$, where *R* is the radius of the loop. RHR-2 gives the direction of the field about the loop.

Conceptual Questions

Exercise:

Problem: Is the magnetic field of a current loop uniform?

Exercise:

Problem:

What happens to the length of a suspended spring when a current passes through it?

Solution:

The spring reduces in length since each coil with have a north pole-produced magnetic field next to a south pole of the next coil.

Exercise:

Problem:

Two concentric circular wires with different diameters carry currents in the same direction. Describe the force on the inner wire.

Problems

Exercise:

Problem:

When the current through a circular loop is 6.0 A, the magnetic field at its center is $2.0 \times 10^{-4}\mathrm{T}$. What is the radius of the loop?

Solution:

0.019 m

Exercise:

Problem:

How many turns must be wound on a flat, circular coil of radius 20 cm in order to produce a magnetic field of magnitude $4.0 \times 10^{-5}\mathrm{T}$ at the center of the coil when the current through it is 0.85 A?

Exercise:

Problem:

A flat, circular loop has 20 turns. The radius of the loop is 10.0 cm and the current through the wire is 0.50 A. Determine the magnitude of the magnetic field at the center of the loop.

Solution:

N $\times$ $6.28 \times 10^{-5}\mathrm{T}$

Exercise:

Problem:

A circular loop of radius *R* carries a current *I*. At what distance along the axis of the loop is the magnetic field one-half its value at the center of the loop?

Exercise:

Problem:

Two flat, circular coils, each with a radius *R* and wound with *N* turns, are mounted along the same axis so that they are parallel a distance *d* apart. What is the magnetic field at the midpoint of the common axis if a current *I* flows in the same direction through each coil?

Solution:

$$B=\frac{\mu_o I R^2}{\left(\left(\frac{d}{2}\right)^2+R^2\right)^{3/2}}$$

Exercise:

Problem:

For the coils in the preceding problem, what is the magnetic field at the center of either coil?

Ampère's Law

By the end of this section, you will be able to:

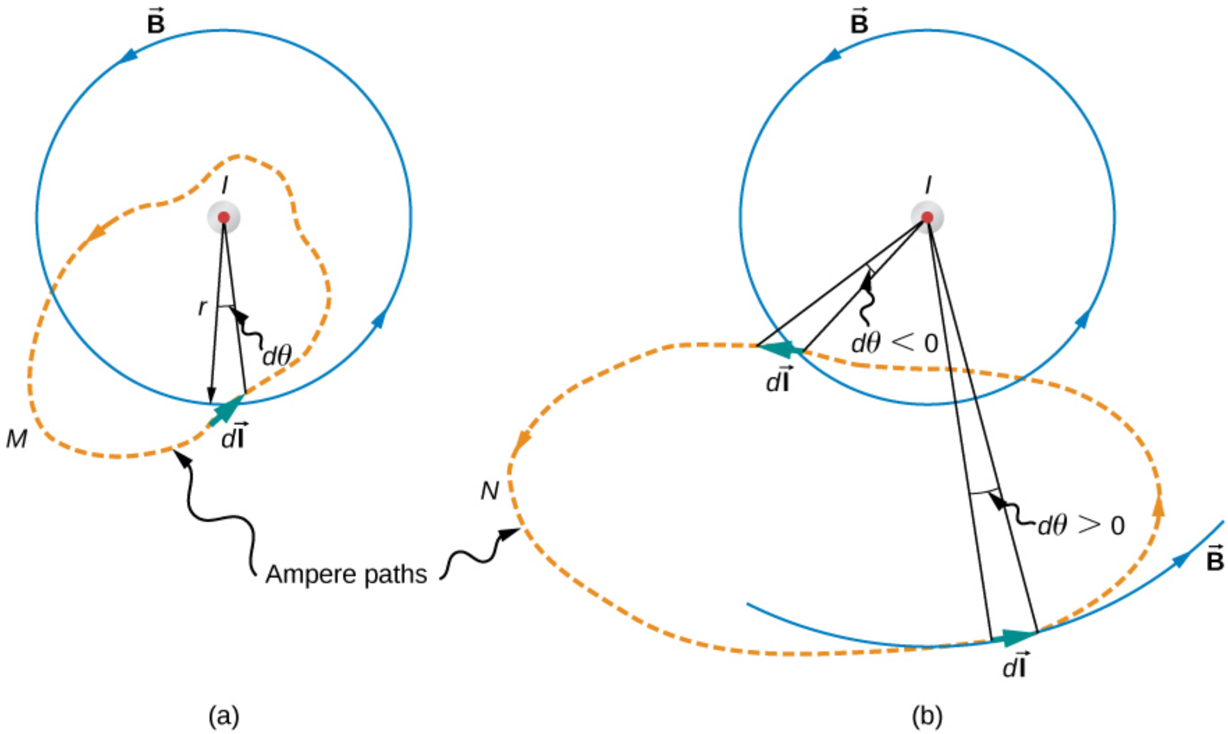
- Explain how Ampère's law relates the magnetic field produced by a current to the value of the current
- Calculate the magnetic field from a long straight wire, either thin or thick, by Ampère's law

A fundamental property of a static magnetic field is that, unlike an electrostatic field, it is not conservative. A conservative field is one that does the same amount of work on a particle moving between two different points regardless of the path chosen. Magnetic fields do not have such a property. Instead, there is a relationship between the magnetic field and its source, electric current. It is expressed in terms of the line integral of $\vec{\mathbf{B}}$ and is known as **Ampère's law**. This law can also be derived directly from the Biot-Savart law. We now consider that derivation for the special case of an infinite, straight wire.

[link] shows an arbitrary plane perpendicular to an infinite, straight wire whose current *I* is directed out of the page. The magnetic field lines are circles directed counterclockwise and centered on the wire. To begin, let's consider $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ over the closed paths *M* and *N*. Notice that one path (*M*) encloses the wire, whereas the other (*N*) does not. Since the field lines are circular, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ is the product of *B* and the projection of *dl* onto the circle passing through $d\vec{\mathbf{l}}$. If the radius of this particular circle is *r*, the projection is $rd\theta$, and

Equation:

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Br\, d\theta.$$



The current I of a long, straight wire is directed out of the page. The integral $\oint d\theta$ equals 2π and 0, respectively, for paths M and N.

With $\overrightarrow{\mathbf{B}}$ given by [link],

Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint \left(\frac{\mu_0 I}{2\pi r}\right) r\, d\theta = \frac{\mu_0 I}{2\pi} \oint d\theta.$$

For path M, which circulates around the wire, $\oint_M d\theta = 2\pi$ and

Equation:

$$\oint_M \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I.$$

Path N, on the other hand, circulates through both positive (counterclockwise) and negative (clockwise) $d\theta$ (see [link]), and since it is closed, $\oint_N d\theta = 0$. Thus for path N,

Equation:

$$\oint_N \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0.$$

The extension of this result to the general case is Ampère's law.

Note:
Ampère's law
Over an arbitrary closed path,

Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I$$

where I is the total current passing through any open surface S whose perimeter is the path of integration. Only currents inside the path of integration need be considered.

To determine whether a specific current I is positive or negative, curl the fingers of your right hand in the direction of the path of integration, as shown in [link]. If I passes through S in the same direction as your extended thumb, I is positive; if I passes through S in the direction opposite to your extended thumb, it is negative.

Note:
Problem-Solving Strategy: Ampère's Law

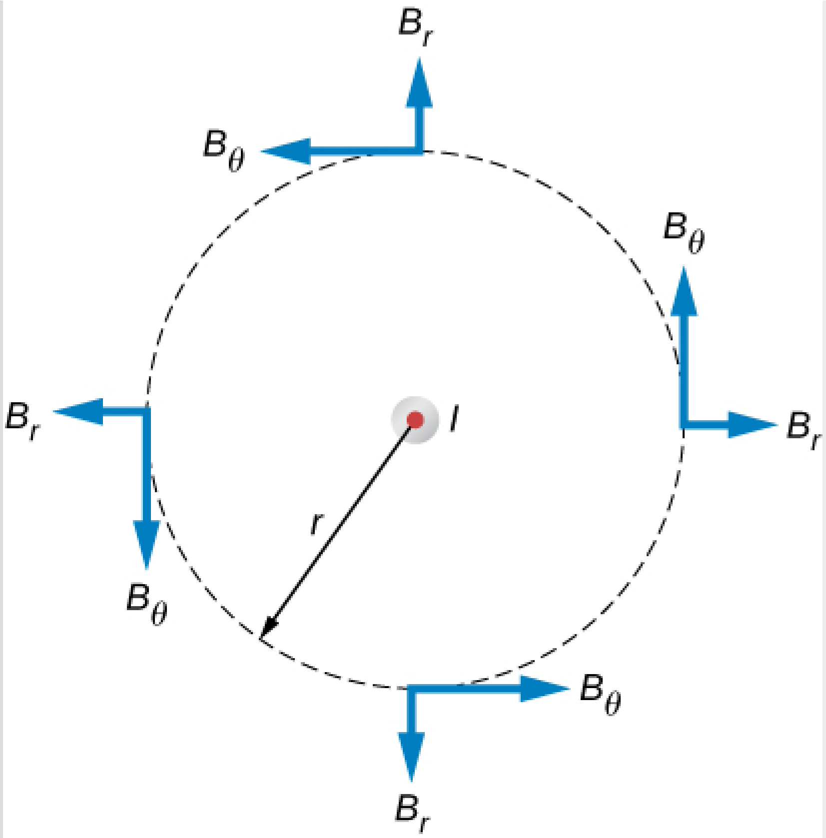
To calculate the magnetic field created from current in wire(s), use the following steps:

1. Identify the symmetry of the current in the wire(s). If there is no symmetry, use the Biot-Savart law to determine the magnetic field.
2. Determine the direction of the magnetic field created by the wire(s) by right-hand rule 2.
3. Chose a path loop where the magnetic field is either constant or zero.
4. Calculate the current inside the loop.
5. Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ around the closed loop.
6. Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

Example:

Using Ampère's Law to Calculate the Magnetic Field Due to a Wire

Use Ampère's law to calculate the magnetic field due to a steady current *I* in an infinitely long, thin, straight wire as shown in [link].



The possible components of the magnetic field B due to a current I, which is directed out of the page. The radial component is zero because the angle between the magnetic field and the path is at a right angle.

Strategy

Consider an arbitrary plane perpendicular to the wire, with the current directed out of the page. The possible magnetic field components in this plane, B_r and B_θ, are shown at arbitrary points on a circle of radius r centered on the wire. Since the field is cylindrically symmetric, neither B_r nor B_θ varies with the position on this circle. Also from symmetry, the radial lines, if they exist, must be directed either all inward or all outward from the wire. This means, however, that there must be a net magnetic flux across an arbitrary cylinder concentric with the wire. The radial component

of the magnetic field must be zero because $\vec{\mathbf{B}}_r \cdot d\vec{\mathbf{l}} = 0$. Therefore, we can apply Ampère’s law to the circular path as shown.

Solution

Over this path $\vec{\mathbf{B}}$ is constant and parallel to $d\vec{\mathbf{l}}$, so

Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B_\theta \oint dl = B_\theta(2\pi r).$$

Thus Ampère’s law reduces to

Equation:

$$B_\theta(2\pi r) = \mu_0 I.$$

Finally, since B_θ is the only component of $\vec{\mathbf{B}}$, we can drop the subscript and write

Equation:

$$B = \frac{\mu_0 I}{2\pi r}.$$

This agrees with the Biot-Savart calculation above.

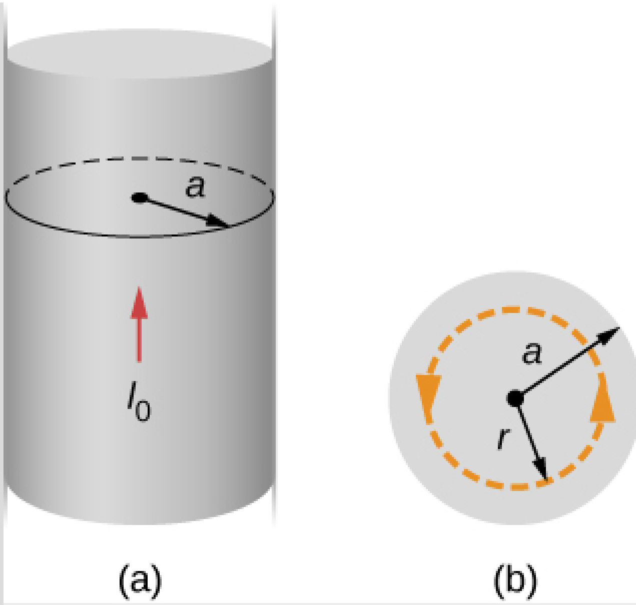
Significance

Ampère’s law works well if you have a path to integrate over which $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ has results that are easy to simplify. For the infinite wire, this works easily with a path that is circular around the wire so that the magnetic field factors out of the integration. If the path dependence looks complicated, you can always go back to the Biot-Savart law and use that to find the magnetic field.

Example:

Calculating the Magnetic Field of a Thick Wire with Ampère’s Law

The radius of the long, straight wire of [link] is a, and the wire carries a current I_0 that is distributed uniformly over its cross-section. Find the magnetic field both inside and outside the wire.



(a) A model of a current-carrying wire of radius a and current I_0. (b) A cross-section of the same wire showing the radius a and the Ampère’s loop of radius r.

Strategy
This problem has the same geometry as [link], but the enclosed current changes as we move the integration path from outside the wire to inside the wire, where it doesn’t capture the entire current enclosed (see [link]).
Solution
For any circular path of radius r that is centered on the wire,
Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint Bdl = B\oint dl = B(2\pi r).$$

From Ampère’s law, this equals the total current passing through any surface bounded by the path of integration.
Consider first a circular path that is inside the wire $(r \le a)$ such as that shown in part (a) of [link]. We need the current I passing through the area enclosed by the path. It’s equal to the current density J times the area

enclosed. Since the current is uniform, the current density inside the path equals the current density in the whole wire, which is $I_0/\pi a^2$. Therefore the current *I* passing through the area enclosed by the path is
Equation:

$$I = \frac{\pi r^2}{\pi a^2} I_0 = \frac{r^2}{a^2} I_0.$$

We can consider this ratio because the current density *J* is constant over the area of the wire. Therefore, the current density of a part of the wire is equal to the current density in the whole area. Using Ampère’s law, we obtain
Equation:

$$B(2\pi r) = \mu_0 \left(\frac{r^2}{a^2} \right) I_0,$$

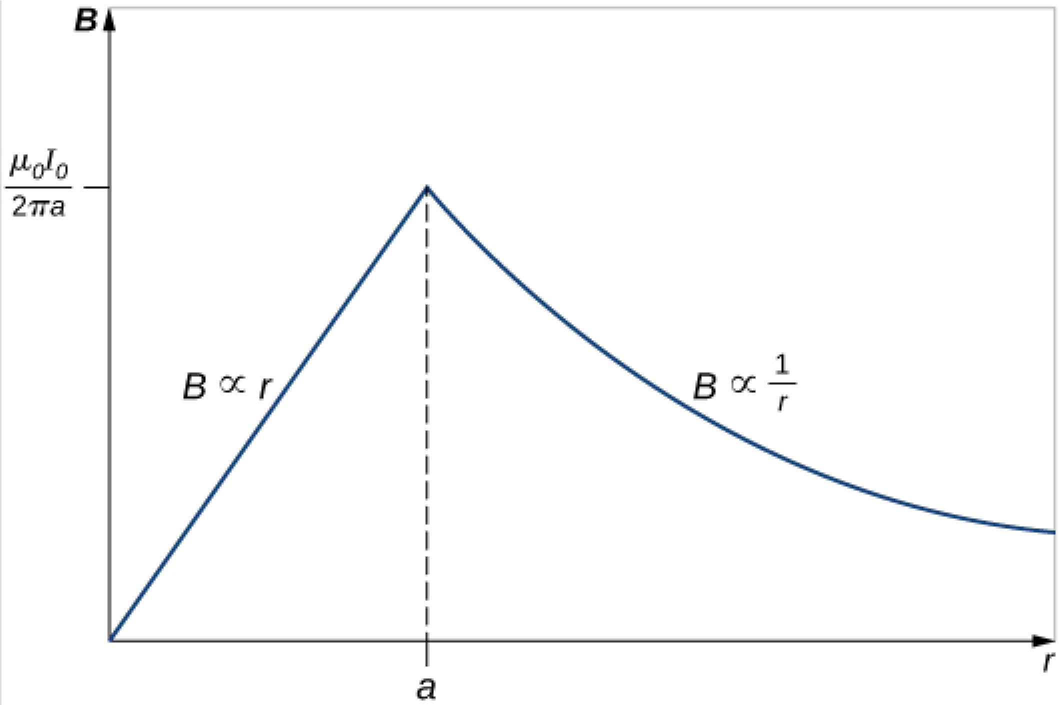
and the magnetic field inside the wire is
Equation:

$$B = \frac{\mu_0 I_0}{2\pi} \frac{r}{a^2} \ (r \le a).$$

Outside the wire, the situation is identical to that of the infinite thin wire of the previous example; that is,
Equation:

$$B = \frac{\mu_0 I_0}{2\pi r} \ (r \ge a).$$

The variation of *B* with *r* is shown in [link].



Variation of the magnetic field produced by a current I_0 in a long, straight wire of radius a.

Significance

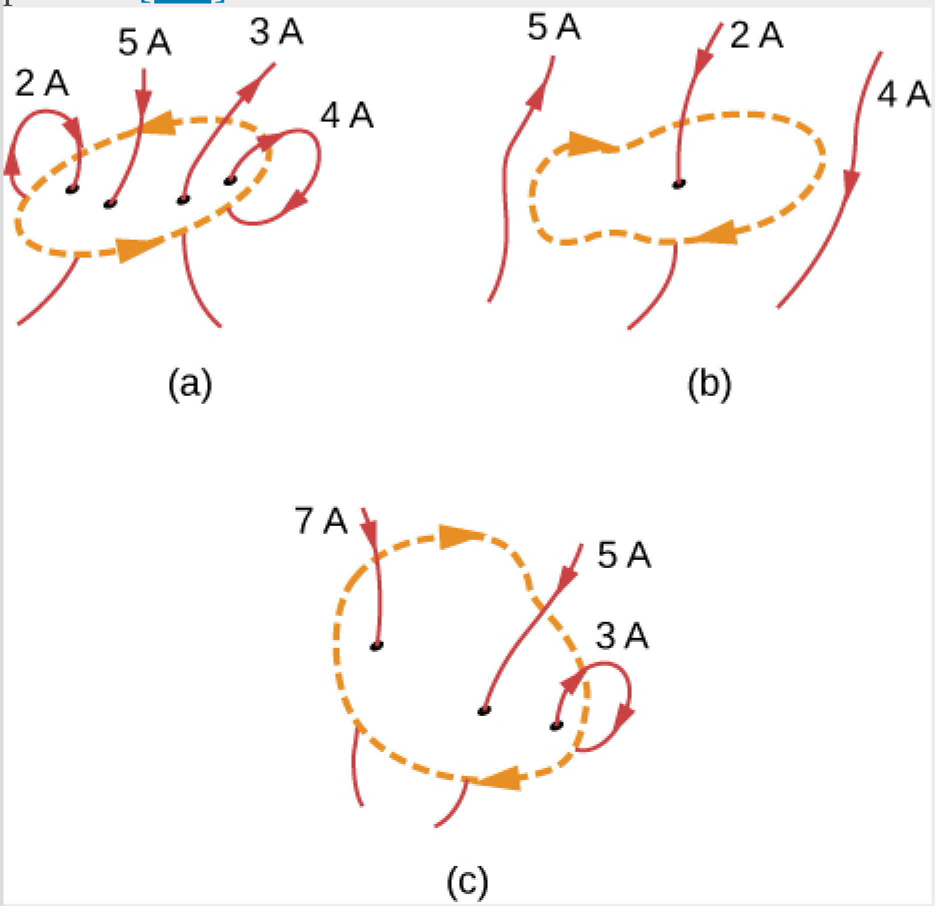
The results show that as the radial distance increases inside the thick wire, the magnetic field increases from zero to a familiar value of the magnetic field of a thin wire. Outside the wire, the field drops off regardless of whether it was a thick or thin wire.

This result is similar to how Gauss’s law for electrical charges behaves inside a uniform charge distribution, except that Gauss’s law for electrical charges has a uniform volume distribution of charge, whereas Ampère’s law here has a uniform area of current distribution. Also, the drop-off outside the thick wire is similar to how an electric field drops off outside of a linear charge distribution, since the two cases have the same geometry and neither case depends on the configuration of charges or currents once the loop is outside the distribution.

Example:

Using Ampère's Law with Arbitrary Paths

Use Ampère's law to evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the current configurations and paths in [link].



Current configurations and paths for [link].

Strategy

Ampère's law states that $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I$ where I is the total current passing through the enclosed loop. The quickest way to evaluate the integral is to calculate $\mu_0 I$ by finding the net current through the loop. Positive currents flow with your right-hand thumb if your fingers wrap around in the direction of the loop. This will tell us the sign of the answer.

Solution

(a) The current going downward through the loop equals the current going out of the loop, so the net current is zero. Thus, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0.$

(b) The only current to consider in this problem is 2A because it is the only current inside the loop. The right-hand rule shows us the current going downward through the loop is in the positive direction. Therefore, the answer is $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(2\text{ A}) = 2.51 \times 10^{-6}\text{T} \cdot \text{m}.$

(c) The right-hand rule shows us the current going downward through the loop is in the positive direction. There are $7\text{A} + 5\text{A} = 12\text{A}$ of current going downward and –3 A going upward. Therefore, the total current is 9 A and $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(9\text{ A}) = 5.65 \times 10^{-6}\text{T} \cdot \text{m}.$

Significance

If the currents all wrapped around so that the same current went into the loop and out of the loop, the net current would be zero and no magnetic field would be present. This is why wires are very close to each other in an electrical cord. The currents flowing toward a device and away from a device in a wire equal zero total current flow through an Ampère loop around these wires. Therefore, no stray magnetic fields can be present from cords carrying current.

Note:

Exercise:

Problem:

Check Your Understanding Consider using Ampère’s law to calculate the magnetic fields of a finite straight wire and of a circular loop of wire. Why is it not useful for these calculations?

Solution:

In these cases the integrals around the Ampèrian loop are very difficult because there is no symmetry, so this method would not be useful.

Summary

- The magnetic field created by current following any path is the sum (or integral) of the fields due to segments along the path (magnitude and direction as for a straight wire), resulting in a general relationship between current and field known as Ampère’s law.
- Ampère’s law can be used to determine the magnetic field from a thin wire or thick wire by a geometrically convenient path of integration. The results are consistent with the Biot-Savart law.

Conceptual Questions

Exercise:

Problem:

Is Ampère’s law valid for all closed paths? Why isn’t it normally useful for calculating a magnetic field?

Solution:

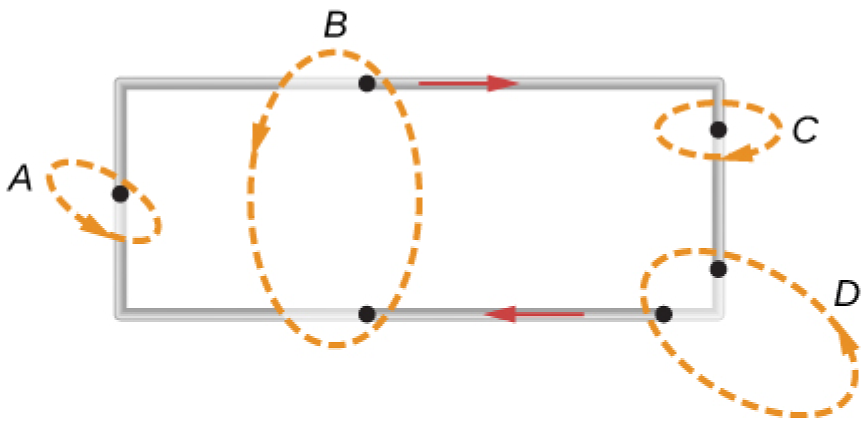
Ampère’s law is valid for all closed paths, but it is not useful for calculating fields when the magnetic field produced lacks symmetry that can be exploited by a suitable choice of path.

Problems

Exercise:

Problem:

A current I flows around the rectangular loop shown in the accompanying figure. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the paths A, B, C, and D.



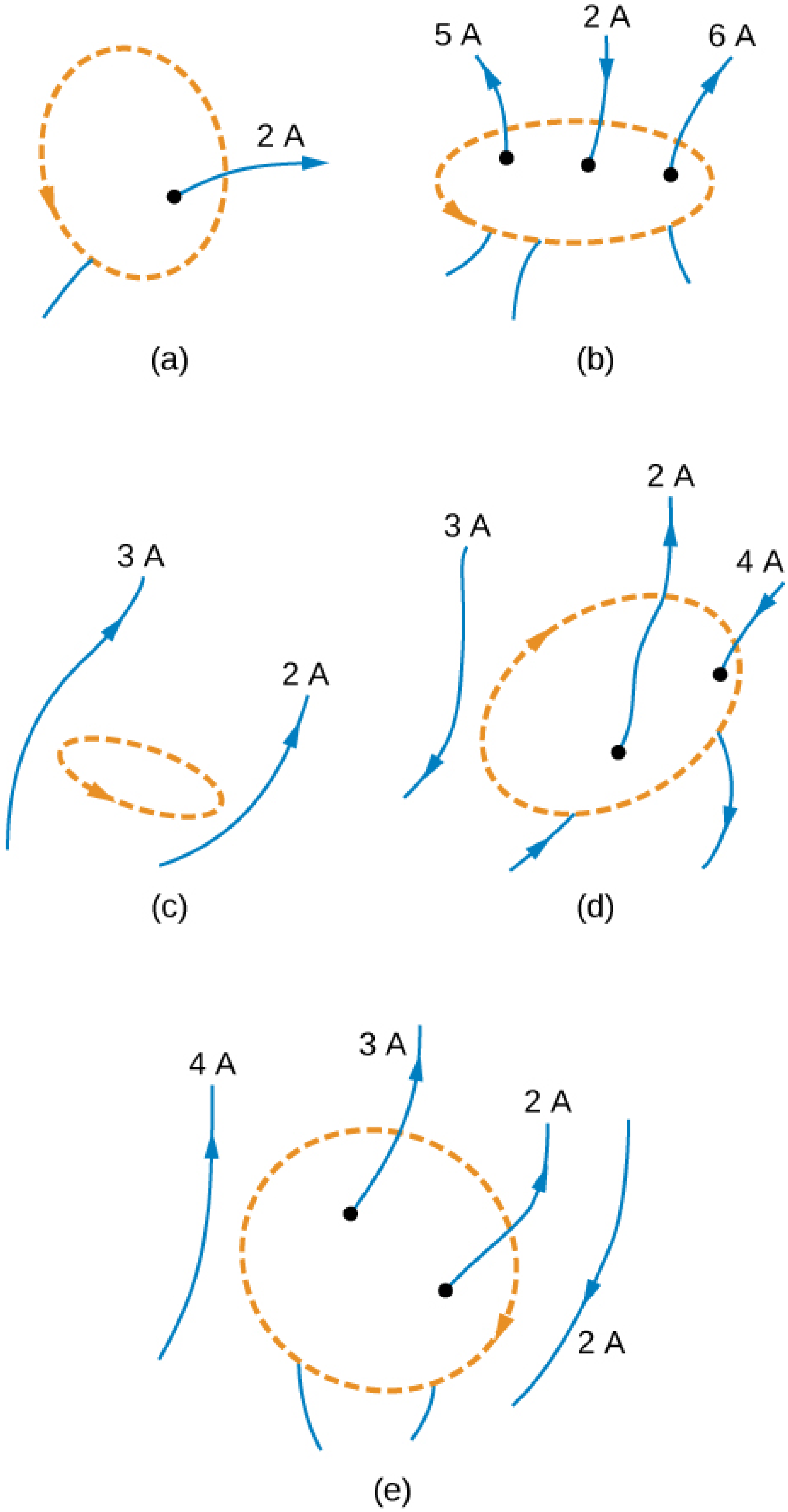
Solution:

a. $\mu_0 I$; b. 0; c. $\mu_0 I$; d. 0

Exercise:

Problem:

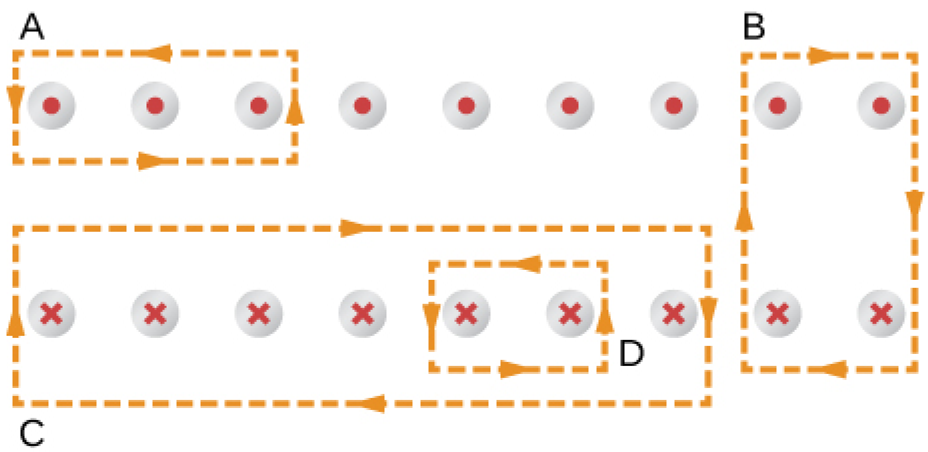
Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for each of the cases shown in the accompanying figure.


2 A
(a)
5 A
2 A
6 A
(b)
3 A
2 A
(c)
3 A
2 A
4 A
(d)
4 A
3 A
2 A
2 A
(e)

Exercise:

Problem:

The coil whose lengthwise cross section is shown in the accompanying figure carries a current *I* and has *N* evenly spaced turns distributed along the length l. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the paths indicated.



Solution:

a. $3\mu_0 I$; b. 0; c. $7\mu_0 I$; d. $-2\mu_0 I$

Exercise:

Problem:

A superconducting wire of diameter 0.25 cm carries a current of 1000 A. What is the magnetic field just outside the wire?

Exercise:

Problem:

A long, straight wire of radius *R* carries a current *I* that is distributed uniformly over the cross-section of the wire. At what distance from the axis of the wire is the magnitude of the magnetic field a maximum?

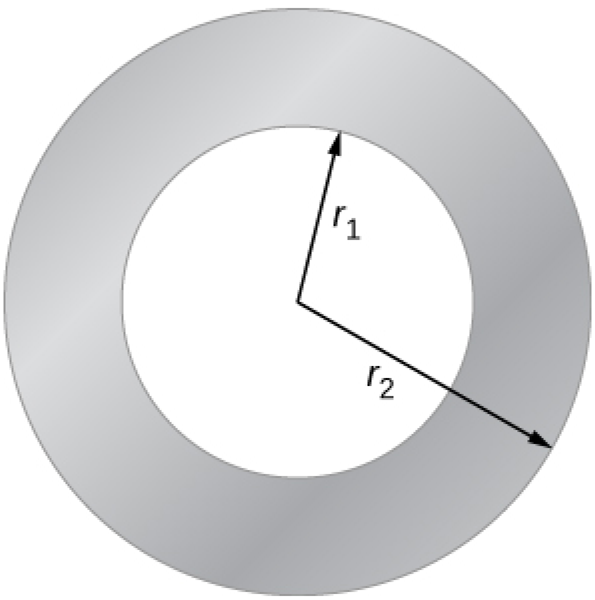
Solution:

at the radius *R*

Exercise:

Problem:

The accompanying figure shows a cross-section of a long, hollow, cylindrical conductor of inner radius $r_1 = 3.0 \text{ cm}$ and outer radius $r_2 = 5.0 \text{ cm}$. A 50-A current distributed uniformly over the cross-section flows into the page. Calculate the magnetic field at $r = 2.0 \text{ cm}, \; r = 4.0 \text{ cm}, \text{ and } r = 6.0 \text{ cm}.$

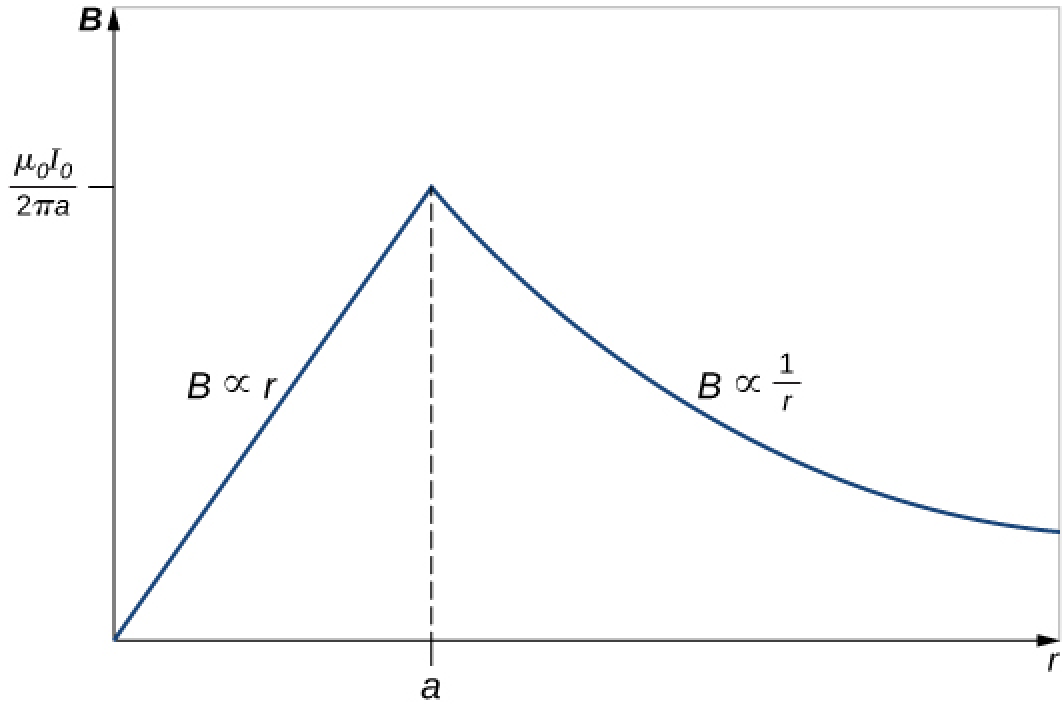


Exercise:

Problem:

A long, solid, cylindrical conductor of radius 3.0 cm carries a current of 50 A distributed uniformly over its cross-section. Plot the magnetic field as a function of the radial distance *r* from the center of the conductor.

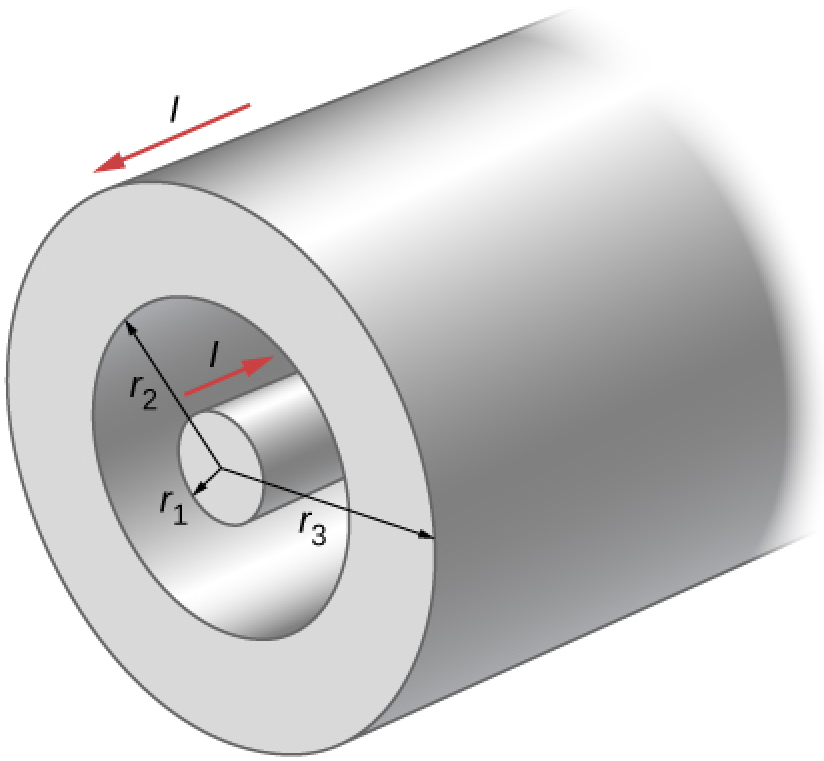
Solution:



Exercise:

Problem:

A portion of a long, cylindrical coaxial cable is shown in the accompanying figure. A current I flows down the center conductor, and this current is returned in the outer conductor. Determine the magnetic field in the regions (a) $r \le r_1$, (b) $r_2 \ge r \ge r_1$, (c) $r_3 \ge r \ge r_2$, and (d) $r \ge r_3$. Assume that the current is distributed uniformly over the cross sections of the two parts of the cable.



Glossary

Ampère’s law
: physical law that states that the line integral of the magnetic field around an electric current is proportional to the current

Solenoids and Toroids
By the end of this section, you will be able to:

- Establish a relationship for how the magnetic field of a solenoid varies with distance and current by using both the Biot-Savart law and Ampère’s law
- Establish a relationship for how the magnetic field of a toroid varies with distance and current by using Ampère’s law

Two of the most common and useful electromagnetic devices are called solenoids and toroids. In one form or another, they are part of numerous instruments, both large and small. In this section, we examine the magnetic field typical of these devices.

Solenoids

A long wire wound in the form of a helical coil is known as a **solenoid**. Solenoids are commonly used in experimental research requiring magnetic fields. A solenoid is generally easy to wind, and near its center, its magnetic field is quite uniform and directly proportional to the current in the wire.

[link] shows a solenoid consisting of N turns of wire tightly wound over a length L. A current I is flowing along the wire of the solenoid. The number of turns per unit length is N/L; therefore, the number of turns in an infinitesimal length dy are $(N/L)dy$ turns. This produces a current
Equation:

$$dI = \frac{NI}{L}dy.$$

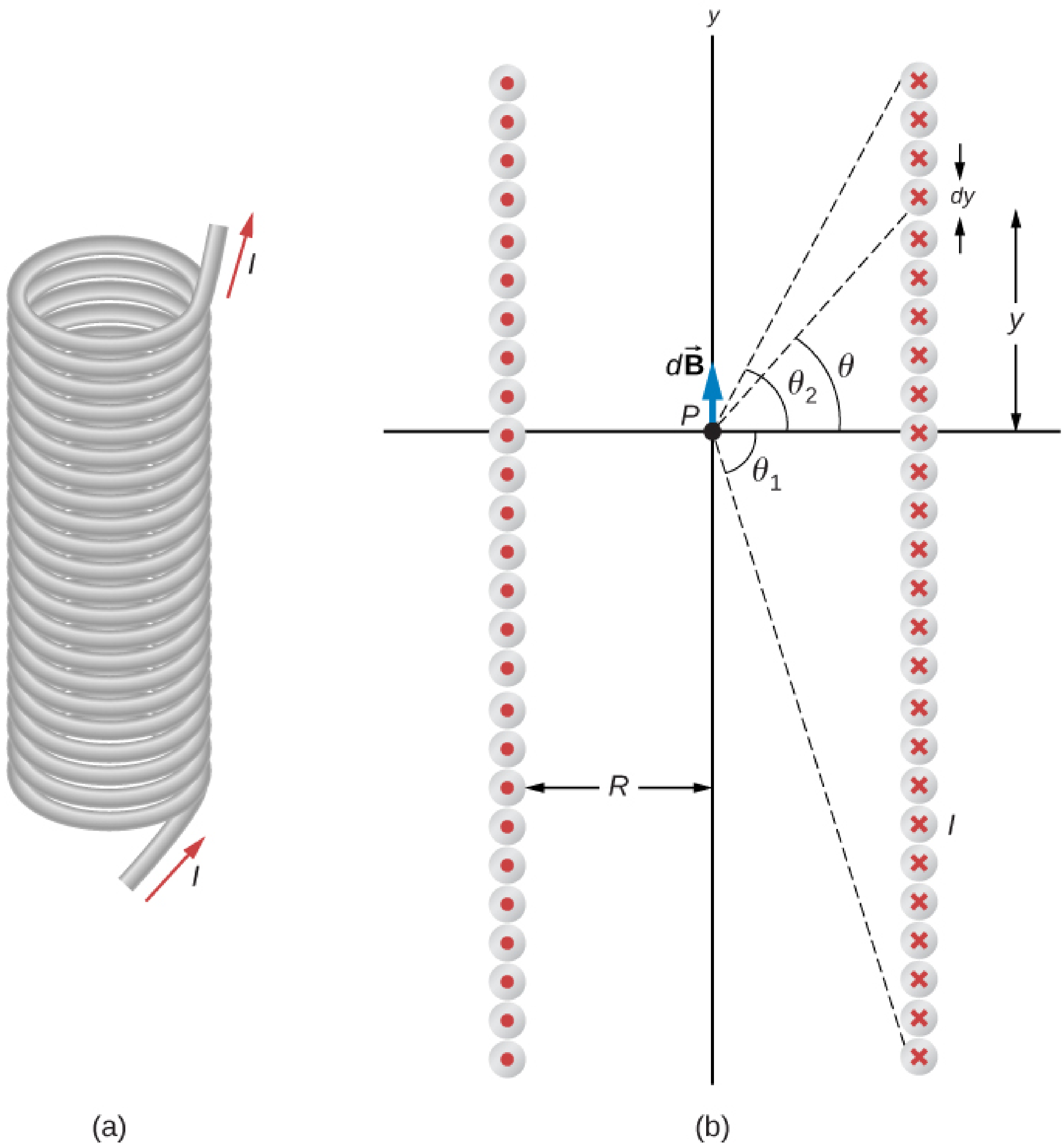
We first calculate the magnetic field at the point P of [link]. This point is on the central axis of the solenoid. We are basically cutting the solenoid into thin slices that are dy thick and treating each as a current loop. Thus, dI is the current through each slice. The magnetic field $d\vec{\mathbf{B}}$ due to the current dI in dy can be found with the help of [link] and [link]:
Equation:

$$d\vec{\mathbf{B}} = \frac{\mu_0 R^2 dI}{2(y^2 + R^2)^{3/2}} \hat{\mathbf{j}} = \left(\frac{\mu_0 I R^2 N}{2L} \hat{\mathbf{j}} \right) \frac{dy}{(y^2 + R^2)^{3/2}}$$

where we used [link] to replace dI. The resultant field at P is found by integrating $d\vec{\mathbf{B}}$ along the entire length of the solenoid. It's easiest to evaluate this integral by changing the independent variable from y to θ. From inspection of [link], we have:

Equation:

$$\sin\theta = \frac{y}{\sqrt{y^2 + R^2}}.$$



(a) A solenoid is a long wire wound in the shape of a helix. (b) The magnetic field at the point *P* on the axis of the solenoid is the net field due to all of the current loops.

Taking the differential of both sides of this equation, we obtain

Equation:

$$\begin{aligned}\cos\theta\, d\theta &= \left[-\frac{y^2}{(y^2+R^2)^{3/2}} + \frac{1}{\sqrt{y^2+R^2}}\right] dy \\ &= \frac{R^2 dy}{(y^2+R^2)^{3/2}}.\end{aligned}$$

When this is substituted into the equation for $d\vec{\mathbf{B}}$, we have

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0 I N}{2L}\hat{\mathbf{j}} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I N}{2L}(\sin\theta_2 - \sin\theta_1)\hat{\mathbf{j}},$$

which is the magnetic field along the central axis of a finite solenoid.

Of special interest is the infinitely long solenoid, for which $L \to \infty$. From a practical point of view, the infinite solenoid is one whose length is much larger than its radius $(L \gg R)$. In this case, $\theta_1 = \frac{-\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. Then from [link], the magnetic field along the central axis of an infinite solenoid is

Equation:

$$\vec{\mathbf{B}} = \frac{\mu_0 I N}{2L}\hat{\mathbf{j}}\left[\sin(\pi/2) - \sin(-\pi/2)\right] = \frac{\mu_0 I N}{L}\hat{\mathbf{j}}$$

or

Equation:

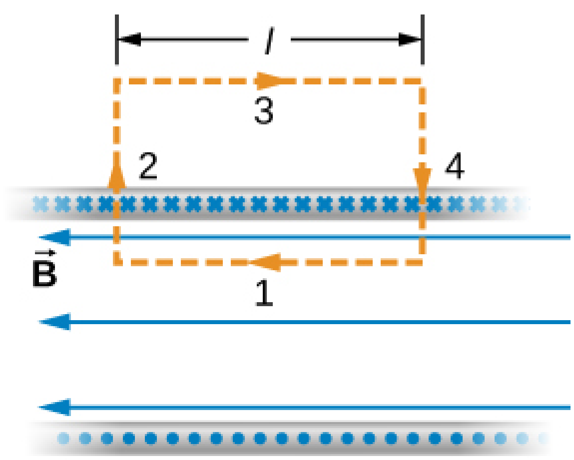
$$\vec{\mathbf{B}} = \mu_0 n I \hat{\mathbf{j}},$$

where *n* is the number of turns per unit length. You can find the direction of $\vec{\mathbf{B}}$ with a right-hand rule: Curl your fingers in the direction of the current,

and your thumb points along the magnetic field in the interior of the solenoid.

We now use these properties, along with Ampère’s law, to calculate the magnitude of the magnetic field at any location inside the infinite solenoid. Consider the closed path of [link]. Along segment 1, $\vec{\mathbf{B}}$ is uniform and parallel to the path. Along segments 2 and 4, $\vec{\mathbf{B}}$ is perpendicular to part of the path and vanishes over the rest of it. Therefore, segments 2 and 4 do not contribute to the line integral in Ampère’s law. Along segment 3, $\vec{\mathbf{B}} = 0$ because the magnetic field is zero outside the solenoid. If you consider an Ampère’s law loop outside of the solenoid, the current flows in opposite directions on different segments of wire. Therefore, there is no enclosed current and no magnetic field according to Ampère’s law. Thus, there is no contribution to the line integral from segment 3. As a result, we find
Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \int_1 \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Bl.$$



The path of integration used in Ampère’s law to evaluate the magnetic field of an infinite solenoid.

The solenoid has *n* turns per unit length, so the current that passes through the surface enclosed by the path is nlI. Therefore, from Ampère's law,
Equation:

$$Bl = \mu_0 nlI$$

and

Note:
Equation:

$$B = \mu_0 nI$$

within the solenoid. This agrees with what we found earlier for *B* on the central axis of the solenoid. Here, however, the location of segment 1 is arbitrary, so we have found that this equation gives the magnetic field everywhere inside the infinite solenoid.

Outside the solenoid, one can draw an Ampère's law loop around the entire solenoid. This would enclose current flowing in both directions. Therefore, the net current inside the loop is zero. According to Ampère's law, if the net current is zero, the magnetic field must be zero. Therefore, for locations outside of the solenoid's radius, the magnetic field is zero.

When a patient undergoes a magnetic resonance imaging (MRI) scan, the person lies down on a table that is moved into the center of a large solenoid that can generate very large magnetic fields. The solenoid is capable of these high fields from high currents flowing through superconducting wires. The large magnetic field is used to change the spin of protons in the patient's body. The time it takes for the spins to align or relax (return to

original orientation) is a signature of different tissues that can be analyzed to see if the structures of the tissues is normal ([link]).

In an MRI machine, a large magnetic field is generated by the cylindrical solenoid surrounding the patient. (credit: Liz West)

Example:

Magnetic Field Inside a Solenoid

A solenoid has 300 turns wound around a cylinder of diameter 1.20 cm and length 14.0 cm. If the current through the coils is 0.410 A, what is the magnitude of the magnetic field inside and near the middle of the solenoid?

Strategy

We are given the number of turns and the length of the solenoid so we can find the number of turns per unit length. Therefore, the magnetic field inside and near the middle of the solenoid is given by [link]. Outside the solenoid, the magnetic field is zero.

Solution
The number of turns per unit length is
Equation:

$$n = \frac{300\,\text{turns}}{0.140\,\text{m}} = 2.14 \times 10^3 \text{turns/m}.$$

The magnetic field produced inside the solenoid is
Equation:

$$\begin{aligned} B &= \mu_0 n I = (4\pi \times 10^{-7} \text{T} \cdot \text{m/A})(2.14 \times 10^3 \text{turns/m})(0.410\,\text{A}) \\ B &= 1.10 \times 10^{-3} \text{T}. \end{aligned}$$

Significance
This solution is valid only if the length of the solenoid is reasonably large compared with its diameter. This example is a case where this is valid.

Note:
Exercise:

Problem:

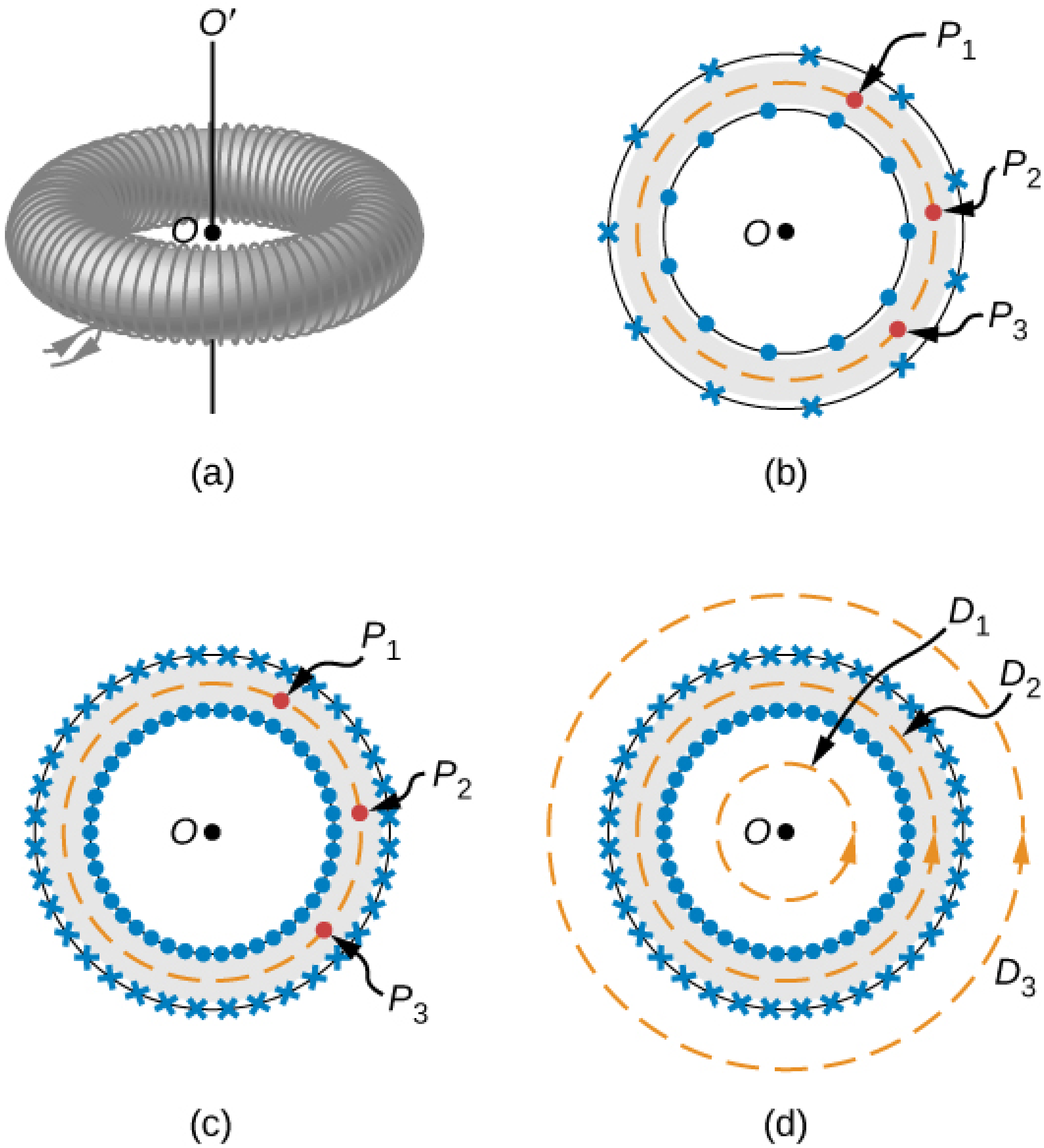
Check Your Understanding What is the ratio of the magnetic field produced from using a finite formula over the infinite approximation for an angle θ of (a) 85°? (b) 89°? The solenoid has 1000 turns in 50 cm with a current of 1.0 A flowing through the coils

Solution:

a. 1.00382; b. 1.00015

Toroids

A toroid is a donut-shaped coil closely wound with one continuous wire, as illustrated in part (a) of [link]. If the toroid has N windings and the current in the wire is I, what is the magnetic field both inside and outside the toroid?



(a) A toroid is a coil wound into a donut-shaped object. (b) A loosely wound toroid does not have cylindrical symmetry. (c) In a tightly wound toroid, cylindrical symmetry is a very

good approximation. (d) Several paths of integration for Ampère’s law.

We begin by assuming cylindrical symmetry around the axis *OO*’. Actually, this assumption is not precisely correct, for as part (b) of [link] shows, the view of the toroidal coil varies from point to point (for example, $P_1,\ P_2$, and P_3) on a circular path centered around *OO*’. However, if the toroid is tightly wound, all points on the circle become essentially equivalent [part (c) of [link]], and cylindrical symmetry is an accurate approximation.

With this symmetry, the magnetic field must be tangent to and constant in magnitude along any circular path centered on *OO*’. This allows us to write for each of the paths D_1, D_2, and D_3 shown in part (d) of [link],

Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r).$$

Ampère’s law relates this integral to the net current passing through any surface bounded by the path of integration. For a path that is external to the toroid, either no current passes through the enclosing surface (path D_1), or the current passing through the surface in one direction is exactly balanced by the current passing through it in the opposite direction $(\text{path } D_3)$. In either case, there is no net current passing through the surface, so

Equation:

$$\oint B(2\pi r) = 0$$

and

Equation:

$$B = 0 \quad (\text{outside the toroid}).$$

The turns of a toroid form a helix, rather than circular loops. As a result, there is a small field external to the coil; however, the derivation above holds if the coils were circular.

For a circular path within the toroid (path D_2), the current in the wire cuts the surface *N* times, resulting in a net current *NI* through the surface. We now find with Ampère's law,

Equation:

$$B(2\pi r) = \mu_0 NI$$

and

Note:
Equation:

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(within the toroid).}$$

The magnetic field is directed in the counterclockwise direction for the windings shown. When the current in the coils is reversed, the direction of the magnetic field also reverses.

The magnetic field inside a toroid is not uniform, as it varies inversely with the distance *r* from the axis *OO*'. However, if the central radius *R* (the radius midway between the inner and outer radii of the toroid) is much larger than the cross-sectional diameter of the coils *r*, the variation is fairly small, and the magnitude of the magnetic field may be calculated by [link] where $r = R$.

Summary

- The magnetic field strength inside a solenoid is
  Equation:

  $$B=\mu_0 nI \quad \text{(inside a solenoid)}$$

  where *n* is the number of loops per unit length of the solenoid. The field inside is very uniform in magnitude and direction.
- The magnetic field strength inside a toroid is
  Equation:

  $$B=\frac{\mu_o NI}{2\pi r} \quad \text{(within the toroid)}$$

  where *N* is the number of windings. The field inside a toroid is not uniform and varies with the distance as $1/r$.

Conceptual Questions

Exercise:

Problem:

Is the magnetic field inside a toroid completely uniform? Almost uniform?

Exercise:

Problem:

Explain why $\vec{\mathbf{B}}=0$ inside a long, hollow copper pipe that is carrying an electric current parallel to the axis. Is $\vec{\mathbf{B}}=0$ outside the pipe?

Solution:

If there is no current inside the loop, there is no magnetic field (see Ampère’s law). Outside the pipe, there may be an enclosed current

through the copper pipe, so the magnetic field may not be zero outside the pipe.

Problems

Exercise:

Problem:

A solenoid is wound with 2000 turns per meter. When the current is 5.2 A, what is the magnetic field within the solenoid?

Solution:

$B = 1.3 \times 10^{-2} \mathrm{T}$

Exercise:

Problem:

A solenoid has 12 turns per centimeter. What current will produce a magnetic field of $2.0 \times 10^{-2} \mathrm{T}$ within the solenoid?

Exercise:

Problem:

If a current is 2.0 A, how many turns per centimeter must be wound on a solenoid in order to produce a magnetic field of $2.0 \times 10^{-3} \mathrm{T}$ within it?

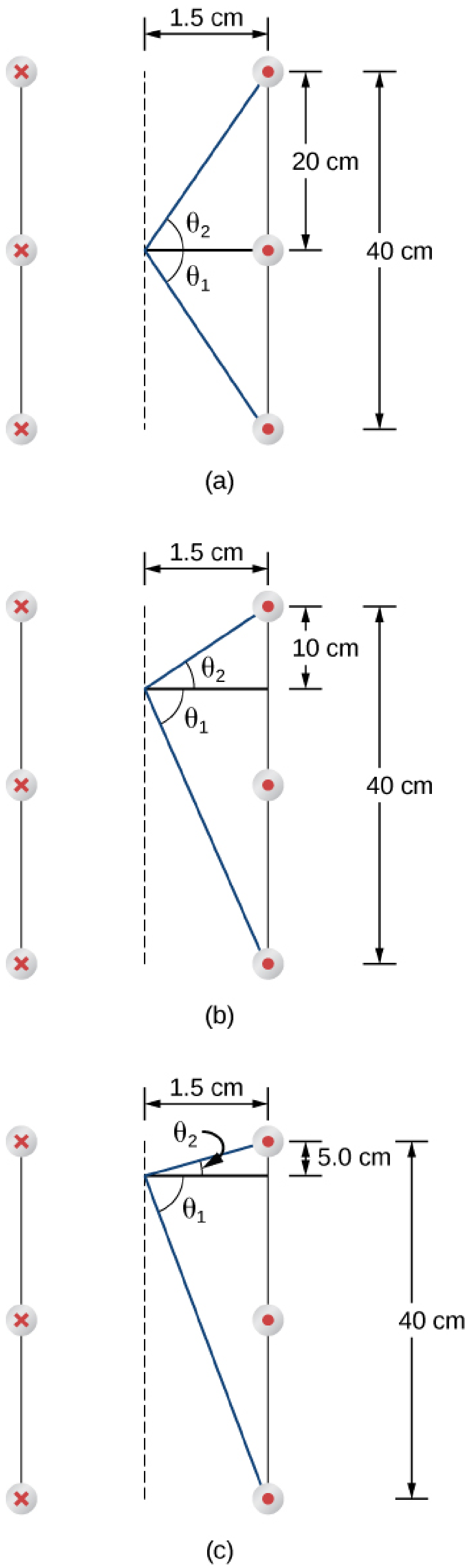
Solution:

roughly eight turns per cm

Exercise:

Problem:

A solenoid is 40 cm long, has a diameter of 3.0 cm, and is wound with 500 turns. If the current through the windings is 4.0 A, what is the magnetic field at a point on the axis of the solenoid that is (a) at the center of the solenoid, (b) 10.0 cm from one end of the solenoid, and (c) 5.0 cm from one end of the solenoid? (d) Compare these answers with the infinite-solenoid case.


1.5 cm
20 cm
40 cm
θ_2
θ_1
(a)
1.5 cm
10 cm
40 cm
θ_2
θ_1
(b)
1.5 cm
5.0 cm
40 cm
θ_2
θ_1
(c)

Exercise:

Problem:

Determine the magnetic field on the central axis at the opening of a semi-infinite solenoid. (That is, take the opening to be at $x = 0$ and the other end to be at
$x = \infty$.)

Solution:

$B = \frac{1}{2}\mu_0 nI$

Exercise:

Problem:

By how much is the approximation $B = \mu_0 nI$ in error at the center of a solenoid that is 15.0 cm long, has a diameter of 4.0 cm, is wrapped with *n* turns per meter, and carries a current *I*?

Exercise:

Problem:

A solenoid with 25 turns per centimeter carries a current *I*. An electron moves within the solenoid in a circle that has a radius of 2.0 cm and is perpendicular to the axis of the solenoid. If the speed of the electron is $2.0\ \times\ 10^5 \text{m/s}$, what is *I*?

Solution:

0.0181 A

Exercise:

Problem:

A toroid has 250 turns of wire and carries a current of 20 A. Its inner and outer radii are 8.0 and 9.0 cm. What are the values of its magnetic field at $r = 8.1, 8.5,$ and 8.9 cm?

Exercise:

Problem:

A toroid with a square cross section 3.0 cm $\times$ 3.0 cm has an inner radius of 25.0 cm. It is wound with 500 turns of wire, and it carries a current of 2.0 A. What is the strength of the magnetic field at the center of the square cross section?

Solution:

0.0008 T

Glossary

solenoid
: thin wire wound into a coil that produces a magnetic field when an electric current is passed through it

toroid
: donut-shaped coil closely wound around that is one continuous wire

Introduction class="introduction"

The black strip found on the back of credit cards and driver’s licenses is a very thin layer of magnetic material with information stored on it. Reading and writing the information on the credit card is done with a swiping motion. The physical reason why this is necessary is called electromagnetic induction and is discussed in this chapter. (credit: modification of work by Jane Whitney)

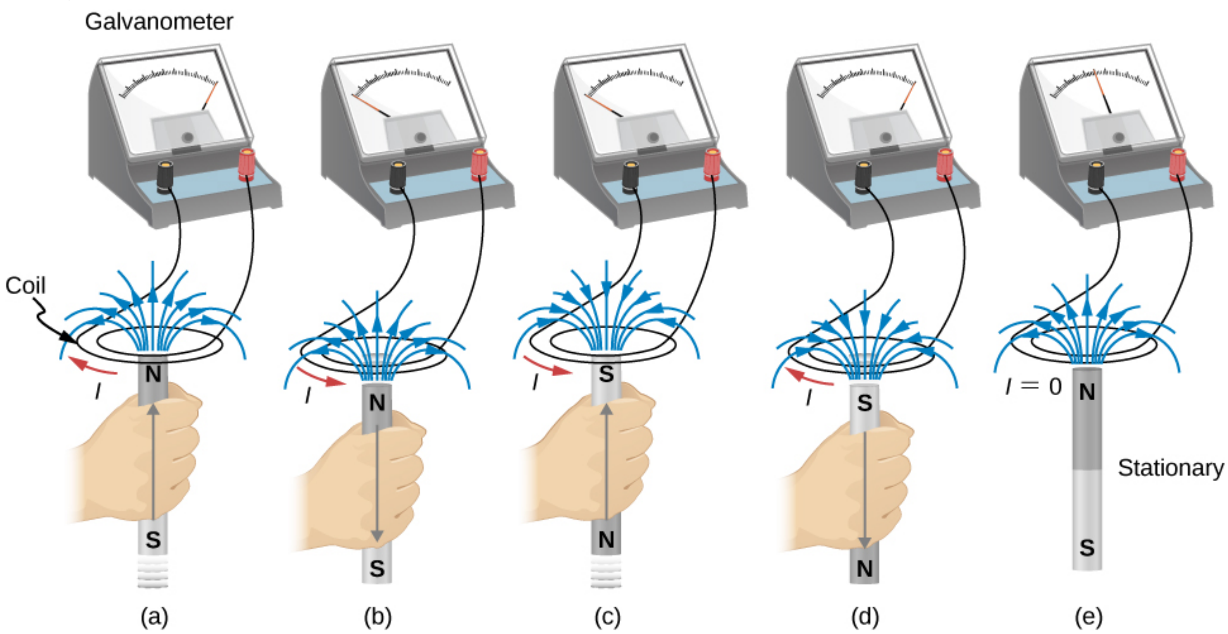
We have been considering electric fields created by fixed charge distributions and magnetic fields produced by constant currents, but electromagnetic phenomena are not restricted to these stationary situations. Most of the interesting applications of electromagnetism are, in fact, time-dependent. To investigate some of these applications, we now remove the time-independent assumption that we have been making and allow the fields to vary with time. In this and the next several chapters, you will see a wonderful symmetry in the behavior exhibited by time-varying electric and magnetic fields. Mathematically, this symmetry is expressed by an additional term in Ampère’s law and by another key equation of electromagnetism called Faraday’s law. We also discuss how moving a wire through a magnetic field produces an emf or voltage. Lastly, we describe applications of these principles, such as the card reader shown above.

Faraday’s Law

By the end of this section, you will be able to:

- Determine the magnetic flux through a surface, knowing the strength of the magnetic field, the surface area, and the angle between the normal to the surface and the magnetic field
- Use Faraday’s law to determine the magnitude of induced emf in a closed loop due to changing magnetic flux through the loop

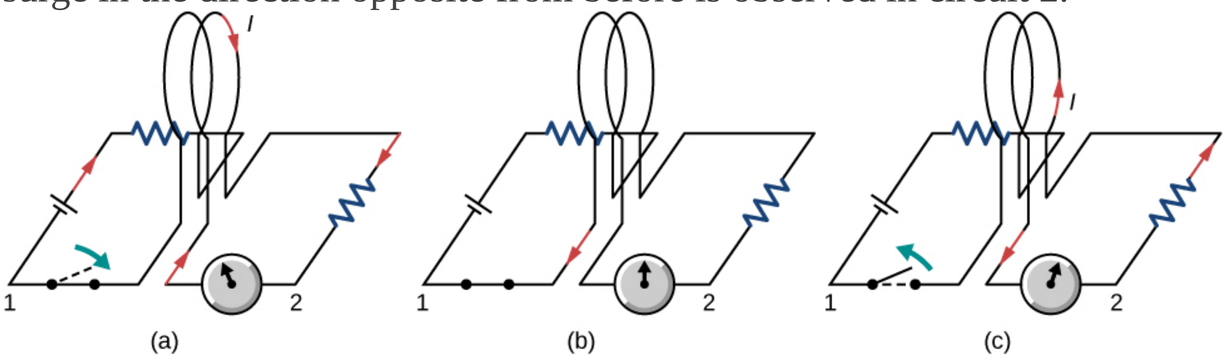
The first productive experiments concerning the effects of time-varying magnetic fields were performed by Michael Faraday in 1831. One of his early experiments is represented in [link]. An emf is induced when the magnetic field in the coil is changed by pushing a bar magnet into or out of the coil. Emfs of opposite signs are produced by motion in opposite directions, and the directions of emfs are also reversed by reversing poles. The same results are produced if the coil is moved rather than the magnet—it is the relative motion that is important. The faster the motion, the greater the emf, and there is no emf when the magnet is stationary relative to the coil.



Movement of a magnet relative to a coil produces emfs as shown (a–d). The same emfs are produced if the coil is moved relative to the magnet. This short-lived emf is only present during the motion. The

greater the speed, the greater the magnitude of the emf, and the emf is zero when there is no motion, as shown in (e).

Faraday also discovered that a similar effect can be produced using two circuits—a changing current in one circuit induces a current in a second, nearby circuit. For example, when the switch is closed in circuit 1 of [link] (a), the ammeter needle of circuit 2 momentarily deflects, indicating that a short-lived current surge has been induced in that circuit. The ammeter needle quickly returns to its original position, where it remains. However, if the switch of circuit 1 is now suddenly opened, another short-lived current surge in the direction opposite from before is observed in circuit 2.



(a) Closing the switch of circuit 1 produces a short-lived current surge in circuit 2. (b) If the switch remains closed, no current is observed in circuit 2. (c) Opening the switch again produces a short-lived current in circuit 2 but in the opposite direction from before.

Faraday realized that in both experiments, a current flowed in the circuit containing the ammeter only when the magnetic field in the region occupied by that circuit was *changing*. As the magnet of the figure was moved, the strength of its magnetic field at the loop changed; and when the current in circuit 1 was turned on or off, the strength of its magnetic field at circuit 2 changed. Faraday was eventually able to interpret these and all other experiments involving magnetic fields that vary with time in terms of the following law:

Note:
Faraday’s Law
The emf ε induced is the negative change in the magnetic flux Φ_{m} per unit time. Any change in the magnetic field or change in orientation of the area of the coil with respect to the magnetic field induces a voltage (emf).

The **magnetic flux** is a measurement of the amount of magnetic field lines through a given surface area, as seen in [link]. This definition is similar to the electric flux studied earlier. This means that if we have

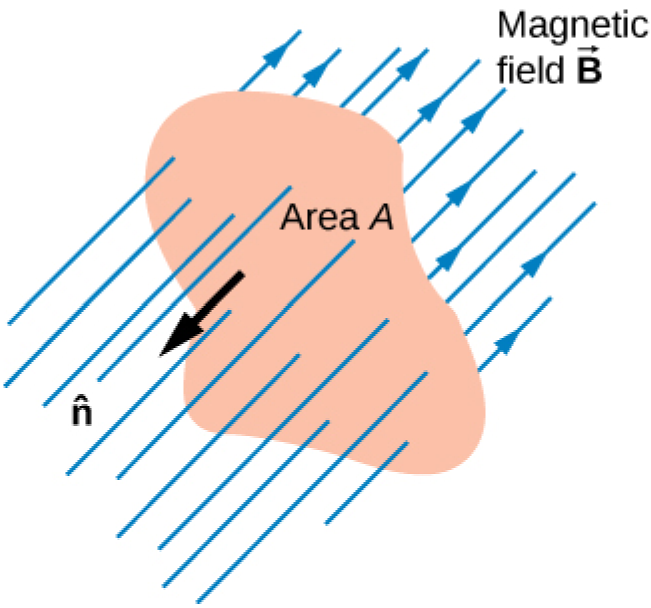
Note:
Equation:

$$\Phi_{\mathrm{m}} = \int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA,$$

then the **induced emf** or the voltage generated by a conductor or coil moving in a magnetic field is
Equation:

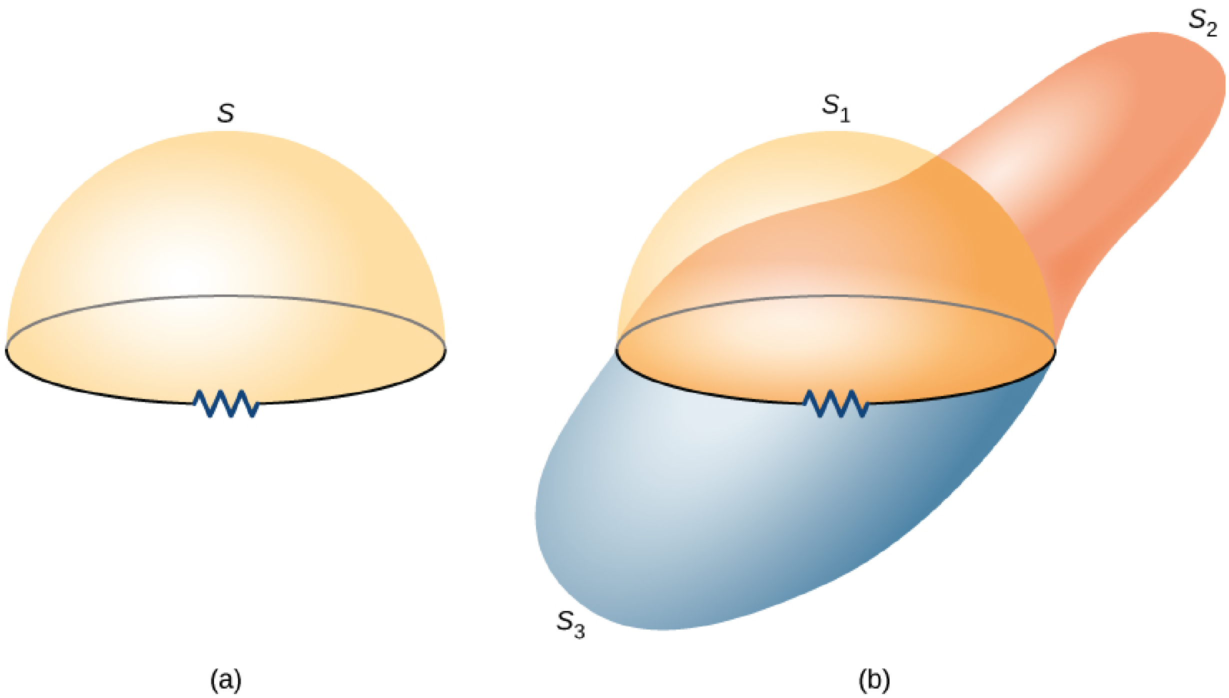
$$\varepsilon = -\frac{d}{dt}\int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA = -\frac{d\Phi_{\mathrm{m}}}{dt}.$$

The negative sign describes the direction in which the induced emf drives current around a circuit. However, that direction is most easily determined with a rule known as Lenz’s law, which we will discuss shortly.



The magnetic flux is the amount of magnetic field lines cutting through a surface area *A* defined by the unit area vector $\hat{\mathbf{n}}$. If the angle between the unit area $\hat{\mathbf{n}}$ and magnetic field vector $\vec{\mathbf{B}}$ are parallel or antiparallel, as shown in the diagram, the magnetic flux is the highest possible value given the values of area and magnetic field.

Part (a) of [link] depicts a circuit and an arbitrary surface *S* that it bounds. Notice that *S* is an *open surface*. It can be shown that *any* open surface bounded by the circuit in question can be used to evaluate Φ_{m}. For example, Φ_{m} is the same for the various surfaces $S_1, S_2, \ldots$ of part (b) of the figure.



(a) A circuit bounding an arbitrary open surface *S*. The planar area bounded by the circuit is not part of *S*. (b) Three arbitrary open surfaces bounded by the same circuit. The value of Φ_{m} is the same for all these surfaces.

The SI unit for magnetic flux is the weber (Wb),

Equation:

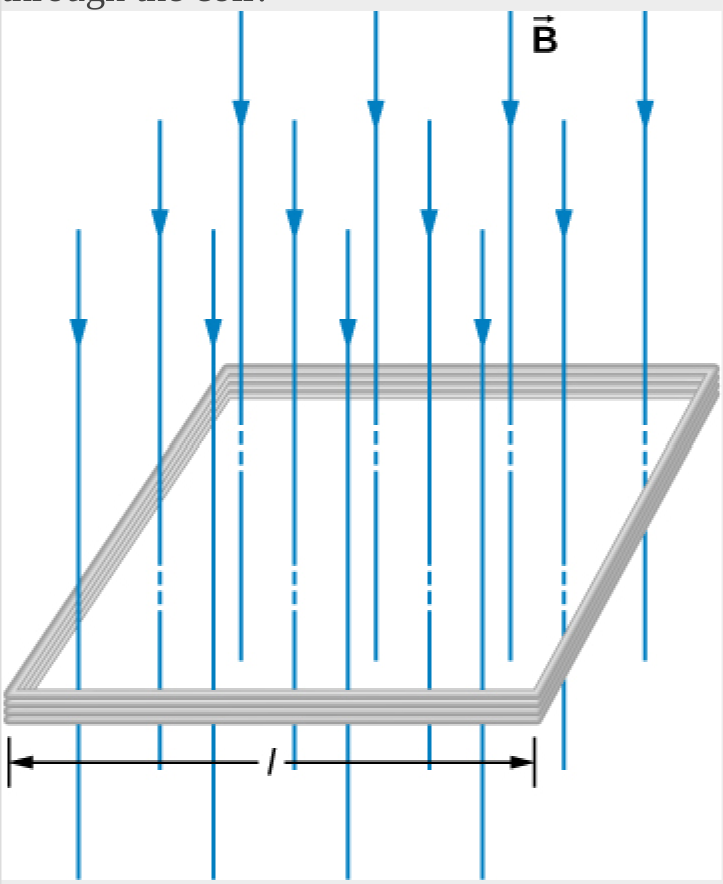
$$1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2.$$

Occasionally, the magnetic field unit is expressed as webers per square meter ($\mathrm{Wb/m^2}$) instead of teslas, based on this definition. In many practical applications, the circuit of interest consists of a number *N* of tightly wound turns (see [link]). Each turn experiences the same magnetic flux. Therefore, the net magnetic flux through the circuits is *N* times the flux through one turn, and Faraday's law is written as

Note:
Equation:

$$\varepsilon = -\frac{d}{dt}(N\Phi_{\text{m}}) = -N\frac{d\Phi_{\text{m}}}{dt}.$$

Example:
A Square Coil in a Changing Magnetic Field
The square coil of [link] has sides $l = 0.25\,\text{m}$ long and is tightly wound with $N = 200$ turns of wire. The resistance of the coil is $R = 5.0\,\Omega$. The coil is placed in a spatially uniform magnetic field that is directed perpendicular to the face of the coil and whose magnitude is decreasing at a rate $dB/dt = -0.040\,\text{T/s}$. (a) What is the magnitude of the emf induced in the coil? (b) What is the magnitude of the current circulating through the coil?



A square coil with N turns of wire with uniform magnetic field $\vec{\mathbf{B}}$ directed in the downward direction, perpendicular to the coil.

Strategy

The area vector, or $\hat{\mathbf{n}}$ direction, is perpendicular to area covering the loop. We will choose this to be pointing downward so that $\vec{\mathbf{B}}$ is parallel to $\hat{\mathbf{n}}$ and that the flux turns into multiplication of magnetic field times area. The area of the loop is not changing in time, so it can be factored out of the time derivative, leaving the magnetic field as the only quantity varying in time. Lastly, we can apply Ohm's law once we know the induced emf to find the current in the loop.

Solution

a. The flux through one turn is

   Equation:

$$\Phi_{\mathrm{m}} = BA = Bl^2,$$

   so we can calculate the magnitude of the emf from Faraday's law. The sign of the emf will be discussed in the next section, on Lenz's law:

   Equation:

$$\begin{aligned} |\varepsilon| &= \left|-N\frac{d\Phi_{\mathrm{m}}}{dt}\right| = Nl^2\frac{dB}{dt} \\ &= (200)(0.25\ \mathrm{m})^2(0.040\ \mathrm{T/s}) = 0.50\ \mathrm{V}. \end{aligned}$$

b. The magnitude of the current induced in the coil is

   Equation:

$$I = \frac{\varepsilon}{R} = \frac{0.50\ \mathrm{V}}{5.0\ \Omega} = 0.10\ \mathrm{A}.$$

Significance

If the area of the loop were changing in time, we would not be able to pull it out of the time derivative. Since the loop is a closed path, the result of this current would be a small amount of heating of the wires until the magnetic field stops changing. This may increase the area of the loop slightly as the wires are heated.

Note:
Exercise:

Problem:

Check Your Understanding A closely wound coil has a radius of 4.0 cm, 50 turns, and a total resistance of $40\ \Omega$. At what rate must a magnetic field perpendicular to the face of the coil change in order to produce Joule heating in the coil at a rate of 2.0 mW?

Solution:

1.1 T/s

Summary

- The magnetic flux through an enclosed area is defined as the amount of field lines cutting through a surface area *A* defined by the unit area vector.
- The units for magnetic flux are webers, where $1\ \text{Wb} = 1\ \text{T} \cdot \text{m}^2$.
- The induced emf in a closed loop due to a change in magnetic flux through the loop is known as Faraday's law. If there is no change in magnetic flux, no induced emf is created.

Conceptual Questions

Exercise:

Problem:

A stationary coil is in a magnetic field that is changing with time. Does the emf induced in the coil depend on the actual values of the magnetic field?

Solution:

The emf depends on the rate of change of the magnetic field.

Exercise:

Problem:

In Faraday’s experiments, what would be the advantage of using coils with many turns?

Exercise:

Problem:

A copper ring and a wooden ring of the same dimensions are placed in magnetic fields so that there is the same change in magnetic flux through them. Compare the induced electric fields and currents in the rings.

Solution:

Both have the same induced electric fields; however, the copper ring has a much higher induced emf because it conducts electricity better than the wooden ring.

Exercise:

Problem:

Discuss the factors determining the induced emf in a closed loop of wire.

Exercise:

Problem:

(a) Does the induced emf in a circuit depend on the resistance of the circuit? (b) Does the induced current depend on the resistance of the circuit?

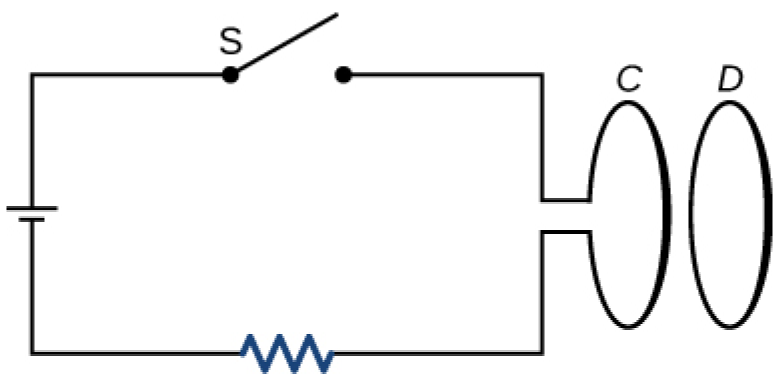
Solution:

a. no; b. yes

Exercise:

Problem:

How would changing the radius of loop *D* shown below affect its emf, assuming *C* and *D* are much closer together compared to their radii?



Exercise:

Problem:

Can there be an induced emf in a circuit at an instant when the magnetic flux through the circuit is zero?

Solution:

As long as the magnetic flux is changing from positive to negative or negative to positive, there could be an induced emf.

Exercise:

Problem:

Does the induced emf always act to decrease the magnetic flux through a circuit?

Exercise:

Problem:

How would you position a flat loop of wire in a changing magnetic field so that there is no induced emf in the loop?

Solution:

Position the loop so that the field lines run perpendicular to the area vector or parallel to the surface.

Exercise:

Problem:

The normal to the plane of a single-turn conducting loop is directed at an angle θ to a spatially uniform magnetic field $\vec{\mathbf{B}}$. It has a fixed area and orientation relative to the magnetic field. Show that the emf induced in the loop is given by $\varepsilon = (dB/dt)(A\cos\theta)$, where A is the area of the loop.

Problems

Exercise:

Problem:

A 50-turn coil has a diameter of 15 cm. The coil is placed in a spatially uniform magnetic field of magnitude 0.50 T so that the face of the coil and the magnetic field are perpendicular. Find the magnitude of the emf induced in the coil if the magnetic field is reduced to zero uniformly in (a) 0.10 s, (b) 1.0 s, and (c) 60 s.

Exercise:

Problem:

Repeat your calculations of the preceding problem's time of 0.1 s with the plane of the coil making an angle of (a) 30°, (b) 60°, and (c) 90° with the magnetic field.

Solution:

a. 3.8 V; b. 2.2 V; c. 0 V

Exercise:

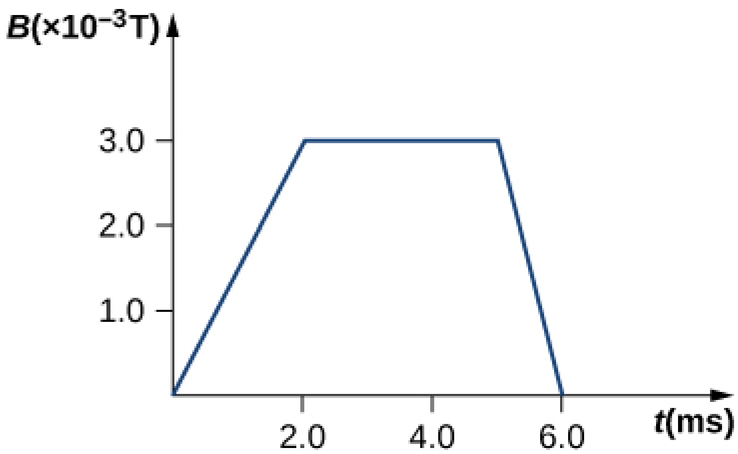
Problem:

A square loop whose sides are 6.0-cm long is made with copper wire of radius 1.0 mm. If a magnetic field perpendicular to the loop is changing at a rate of 5.0 mT/s, what is the current in the loop?

Exercise:

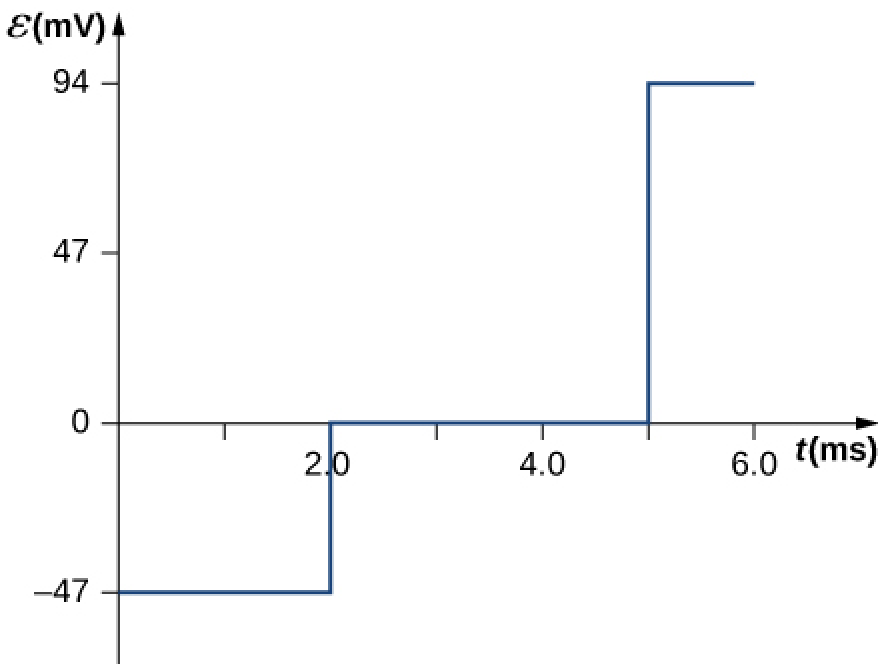
Problem:

The magnetic field through a circular loop of radius 10.0 cm varies with time as shown below. The field is perpendicular to the loop. Plot the magnitude of the induced emf in the loop as a function of time.



Solution:

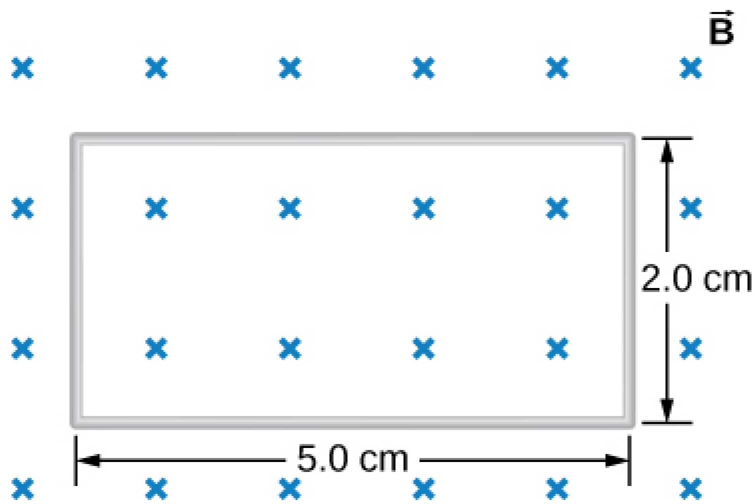
$$\begin{aligned}
B &= 1.5t, 0 \leq t < 2.0\,\mathrm{ms}, B = 3.0\,\mathrm{mT}, 2.0\,\mathrm{ms} \leq t \leq 5.0\,\mathrm{ms}, \\
B &= -3.0t + 18\,\mathrm{mT}, 5.0\,\mathrm{ms} < t \leq 6.0\,\mathrm{ms}, \\
\varepsilon &= -\frac{d\Phi_{\mathrm{m}}}{dt} = -\frac{d(BA)}{dt} = -A\frac{dB}{dt}, \\
\varepsilon &= -\pi(0.100\,\mathrm{m})^2\,(1.5\,\mathrm{T/s}) \\
&= -47\,\mathrm{mV}\,(0 \leq t < 2.0\,\mathrm{ms}), \\
\varepsilon &= \pi(0.100\,\mathrm{m})^2\,(0) = 0\,(2.0\,\mathrm{ms} \leq t \leq 5.0\,\mathrm{ms}), \\
\varepsilon &= -\pi(0.100\,\mathrm{m})^2\,(-3.0\,\mathrm{T/s}) = 94\,\mathrm{mV}\;\;(5.0\,\mathrm{ms} < t < 6.0\,\mathrm{ms}).
\end{aligned}$$



Exercise:

Problem:

The accompanying figure shows a single-turn rectangular coil that has a resistance of $2.0\,\Omega$. The magnetic field at all points inside the coil varies according to $B = B_0 e^{-\alpha t}$, where $B_0 = 0.25\,\mathrm{T}$ and $\alpha = 200\,\mathrm{Hz}$. What is the current induced in the coil at (a) $t = 0.001\,\mathrm{s}$, (b) 0.002 s, (c) 2.0 s?



Exercise:

Problem:

How would the answers to the preceding problem change if the coil consisted of 20 closely spaced turns?

Solution:

Each answer is 20 times the previously given answers.

Exercise:

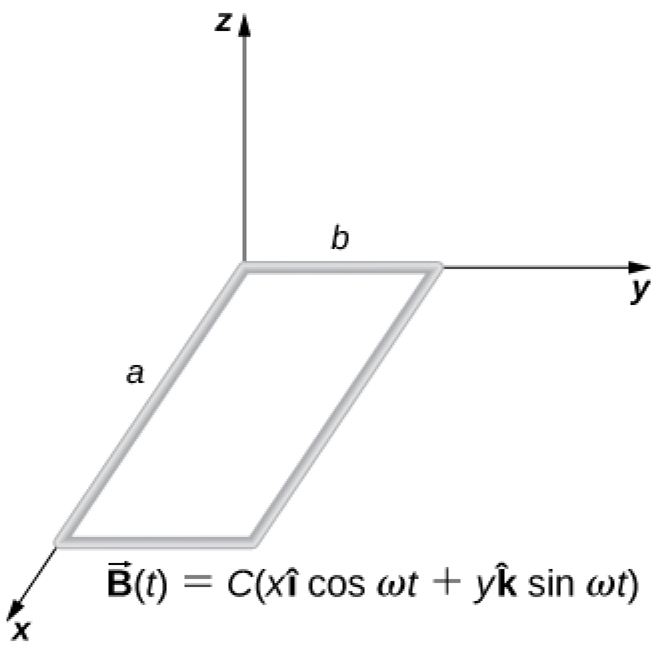
Problem:

A long solenoid with $n = 10$ turns per centimeter has a cross-sectional area of 5.0 cm^2 and carries a current of 0.25 A. A coil with five turns encircles the solenoid. When the current through the solenoid is turned off, it decreases to zero in 0.050 s. What is the average emf induced in the coil?

Exercise:

Problem:

A rectangular wire loop with length *a* and width *b* lies in the *xy*-plane, as shown below. Within the loop there is a time-dependent magnetic field given by $\vec{\mathbf{B}}(t) = C\left((x\cos\omega t)\hat{\mathbf{i}} + (y\sin\omega t)\widehat{\mathbf{k}}\right)$, with $\vec{\mathbf{B}}(t)$ in tesla. Determine the emf induced in the loop as a function of time.



Solution:

$$\begin{aligned}
\hat{\mathbf{n}} &= \widehat{k},\ d\Phi_{\mathrm{m}} = \mathrm{C}y\sin(\omega t)dxdy, \\
\Phi_{\mathrm{m}} &= \frac{Cab^2\sin(\omega t)}{2}, \\
\varepsilon &= -\frac{Cab^2\omega\cos(\omega t)}{2}.
\end{aligned}$$

Exercise:

Problem:

The magnetic field perpendicular to a single wire loop of diameter 10.0 cm decreases from 0.50 T to zero. The wire is made of copper and has a diameter of 2.0 mm and length 1.0 cm. How much charge moves through the wire while the field is changing?

Glossary

Faraday’s law
: induced emf is created in a closed loop due to a change in magnetic flux through the loop

induced emf
: short-lived voltage generated by a conductor or coil moving in a magnetic field

magnetic flux
: measurement of the amount of magnetic field lines through a given area

Lenz's Law

By the end of this section, you will be able to:

- Use Lenz's law to determine the direction of induced emf whenever a magnetic flux changes
- Use Faraday's law with Lenz's law to determine the induced emf in a coil and in a solenoid

The direction in which the induced emf drives current around a wire loop can be found through the negative sign. However, it is usually easier to determine this direction with **Lenz's law**, named in honor of its discoverer, Heinrich Lenz (1804–1865). (Faraday also discovered this law, independently of Lenz.) We state Lenz's law as follows:

Note:
Lenz's Law
The direction of the induced emf drives current around a wire loop to always *oppose* the change in magnetic flux that causes the emf.

Lenz's law can also be considered in terms of conservation of energy. If pushing a magnet into a coil causes current, the energy in that current must have come from somewhere. If the induced current causes a magnetic field opposing the increase in field of the magnet we pushed in, then the situation is clear. We pushed a magnet against a field and did work on the system, and that showed up as current. If it were not the case that the induced field opposes the change in the flux, the magnet would be pulled in produce a current without anything having done work. Electric potential energy would have been created, violating the conservation of energy.

To determine an induced emf ε, you first calculate the magnetic flux Φ_{m} and then obtain $d\Phi_{\mathrm{m}}/dt$. The magnitude of ε is given by $\varepsilon = |d\Phi_{\mathrm{m}}/dt|$. Finally, you can apply Lenz's law to determine the sense of ε. This will be developed through examples that illustrate the following problem-solving strategy.

Note:

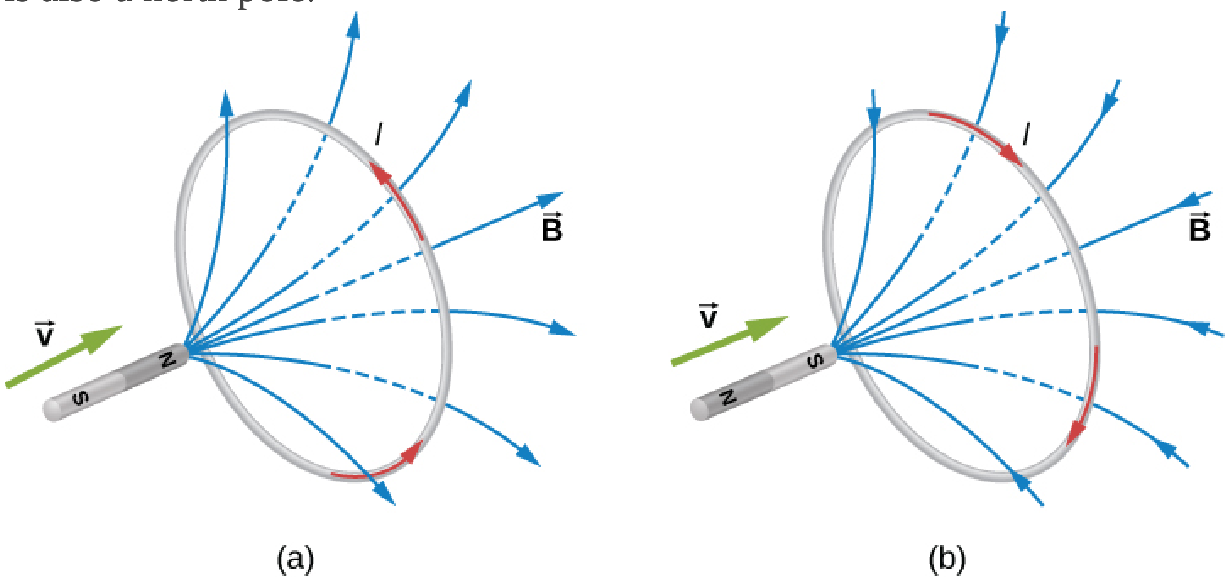
Problem-Solving Strategy: Lenz's Law

To use Lenz's law to determine the directions of induced magnetic fields, currents, and emfs:

1. Make a sketch of the situation for use in visualizing and recording directions.
2. Determine the direction of the applied magnetic field $\vec{\mathbf{B}}$.
3. Determine whether its magnetic flux is increasing or decreasing.
4. Now determine the direction of the induced magnetic field $\vec{\mathbf{B}}$. The induced magnetic field tries to reinforce a magnetic flux that is decreasing or opposes a magnetic flux that is increasing. Therefore, the induced magnetic field adds or subtracts to the applied magnetic field, depending on the change in magnetic flux.
5. Use right-hand rule 2 (RHR-2; see Magnetic Forces and Fields) to determine the direction of the induced current I that is responsible for the induced magnetic field $\vec{\mathbf{B}}$.
6. The direction (or polarity) of the induced emf can now drive a conventional current in this direction.

Let's apply Lenz's law to the system of [link](a). We designate the "front" of the closed conducting loop as the region containing the approaching bar magnet, and the "back" of the loop as the other region. As the north pole of the magnet moves toward the loop, the flux through the loop due to the field of the magnet increases because the strength of field lines directed from the front to the back of the loop is increasing. A current is therefore induced in the loop. By Lenz's law, the direction of the induced current must be such that its own magnetic field is directed in a way to *oppose* the changing flux caused by the field of the approaching magnet. Hence, the induced current circulates so that its magnetic field lines through the loop are directed from the back to the front of the loop. By RHR-2, place your thumb pointing against the magnetic field lines, which is toward the bar magnet. Your fingers wrap in a counterclockwise direction as viewed from the bar magnet. Alternatively, we can determine the direction of the induced current

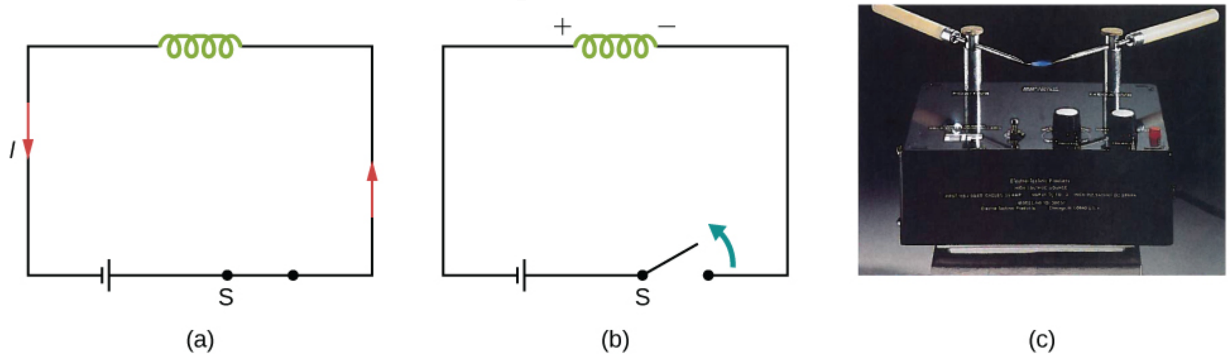
by treating the current loop as an electromagnet that *opposes* the approach of the north pole of the bar magnet. This occurs when the induced current flows as shown, for then the face of the loop nearer the approaching magnet is also a north pole.



The change in magnetic flux caused by the approaching magnet induces a current in the loop. (a) An approaching north pole induces a counterclockwise current with respect to the bar magnet. (b) An approaching south pole induces a clockwise current with respect to the bar magnet.

Part (b) of the figure shows the south pole of a magnet moving toward a conducting loop. In this case, the flux through the loop due to the field of the magnet increases because the number of field lines directed from the back to the front of the loop is increasing. To oppose this change, a current is induced in the loop whose field lines through the loop are directed from the front to the back. Equivalently, we can say that the current flows in a direction so that the face of the loop nearer the approaching magnet is a south pole, which then repels the approaching south pole of the magnet. By RHR-2, your thumb points away from the bar magnet. Your fingers wrap in a clockwise fashion, which is the direction of the induced current.

Another example illustrating the use of Lenz's law is shown in [link]. When the switch is opened, the decrease in current through the solenoid causes a decrease in magnetic flux through its coils, which induces an emf in the solenoid. This emf must oppose the change (the termination of the current) causing it. Consequently, the induced emf has the polarity shown and drives in the direction of the original current. This may generate an arc across the terminals of the switch as it is opened.



(a) A solenoid connected to a source of emf. (b) Opening switch S terminates the current, which in turn induces an emf in the solenoid. (c) A potential difference between the ends of the sharply pointed rods is produced by inducing an emf in a coil. This potential difference is large enough to produce an arc between the sharp points.

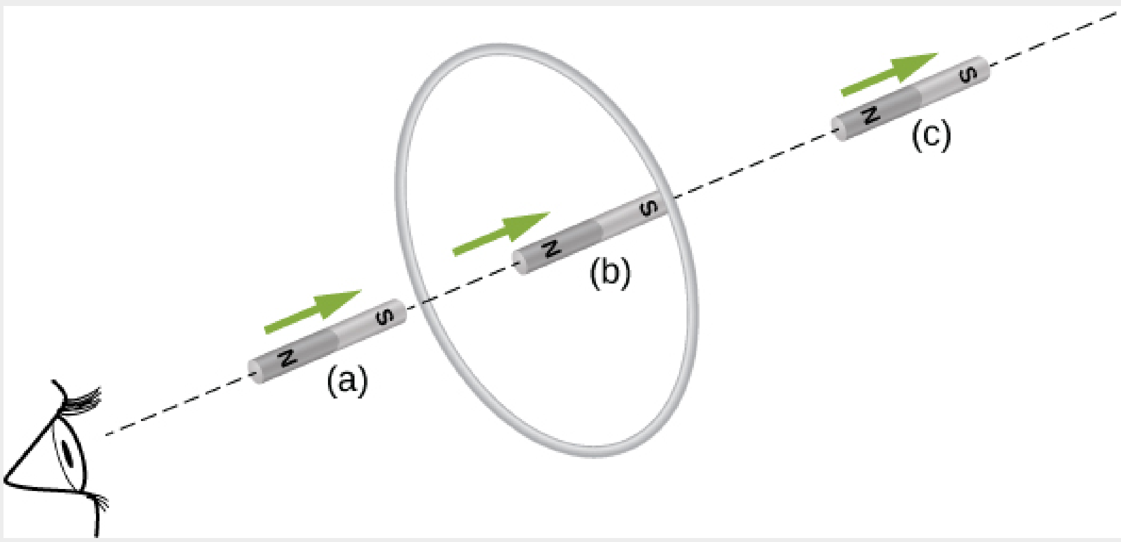
Note:
Exercise:

Problem:

Check Your Understanding Find the direction of the induced current in the wire loop shown below as the magnet enters, passes through, and leaves the loop.

Solution:

To the observer shown, the current flows clockwise as the magnet approaches, decreases to zero when the magnet is centered in the plane of the coil, and then flows counterclockwise as the magnet leaves the coil.



Note:
Exercise:

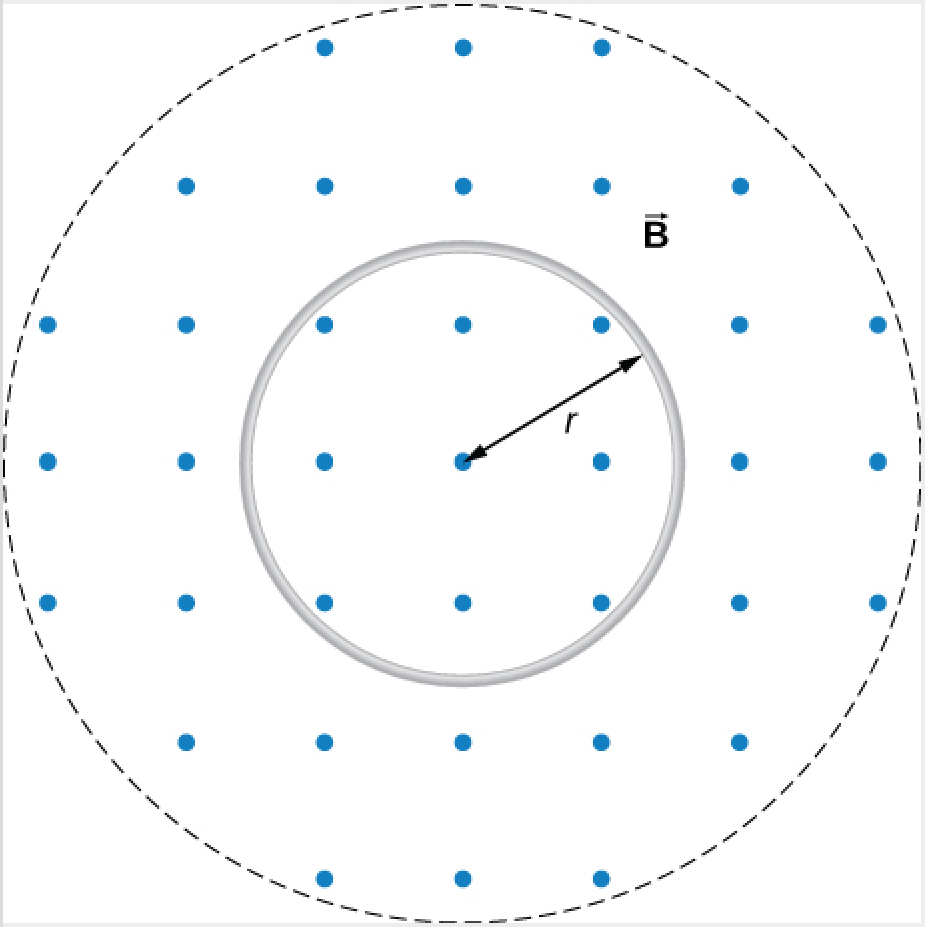
Problem:

Check Your Understanding Verify the directions of the induced currents in [link].

Example:
A Circular Coil in a Changing Magnetic Field

A magnetic field $\vec{\mathbf{B}}$ is directed outward perpendicular to the plane of a circular coil of radius $r = 0.50\ \text{m}$ ([link]). The field is cylindrically symmetrical with respect to the center of the coil, and its magnitude decays exponentially according to $B = (1.5T)e^{-(5.0\text{s}^{-1})t}$, where B is in teslas and

t is in seconds. (a) Calculate the emf induced in the coil at the times $t_1 = 0$, $t_2 = 5.0 \times 10^{-2}$s, and $t_3 = 1.0$ s. (b) Determine the current in the coil at these three times if its resistance is $10\ \Omega$.



A circular coil in a decreasing magnetic field.

Strategy

Since the magnetic field is perpendicular to the plane of the coil and constant over each spot in the coil, the dot product of the magnetic field $\vec{\mathbf{B}}$ and normal to the area unit vector $\hat{\mathbf{n}}$ turns into a multiplication. The magnetic field can be pulled out of the integration, leaving the flux as the product of the magnetic field times area. We need to take the time derivative of the exponential function to calculate the emf using Faraday's law. Then we use Ohm's law to calculate the current.

Solution

a. Since $\vec{\mathbf{B}}$ is perpendicular to the plane of the coil, the magnetic flux is given by
Equation:

$$\begin{aligned}\Phi_{\mathrm{m}} &= B\pi r^2 = (1.5e^{-5.0t}\ \mathrm{T})\pi(0.50\ \mathrm{m})^2 \\ &= 1.2e^{-(5.0\mathrm{s}^{-1})t}\ \mathrm{Wb}.\end{aligned}$$

From Faraday’s law, the magnitude of the induced emf is
Equation:

$$\varepsilon = \left|\frac{d\Phi_m}{dt}\right| = \left|\frac{d}{dt}(1.2e^{-(5.0\mathrm{s}^{-1})t}\ \mathrm{Wb})\right| = 6.0\, e^{-(5.0\mathrm{s}^{-1})t}\ \mathrm{V}.$$

Since $\vec{\mathbf{B}}$ is directed out of the page and is decreasing, the induced current must flow counterclockwise when viewed from above so that the magnetic field it produces through the coil also points out of the page. For all three times, the sense of ε is counterclockwise; its magnitudes are
Equation:

$$\varepsilon\,(t_1) = 6.0\ \mathrm{V};\ \ \varepsilon\,(t_2) = 4.7\ \mathrm{V};\ \ \varepsilon\,(t_3) = 0.040\ \mathrm{V}.$$

b. From Ohm’s law, the respective currents are
Equation:

$$\begin{aligned}I(t_1) &= \frac{\varepsilon(t_1)}{R} = \frac{6.0\ \mathrm{V}}{10\ \Omega} = 0.60\ \mathrm{A}; \\ I(t_2) &= \frac{4.7\ \mathrm{V}}{10\ \Omega} = 0.47\ \mathrm{A};\end{aligned}$$

and
Equation:

$$I(t_3) = \frac{0.040\ \mathrm{V}}{10\ \Omega} = 4.0\ \times\ 10^{-3}\ \mathrm{A}.$$

Significance

An emf voltage is created by a changing magnetic flux over time. If we know how the magnetic field varies with time over a constant area, we can take its time derivative to calculate the induced emf.

Example:

Changing Magnetic Field Inside a Solenoid

The current through the windings of a solenoid with $n = 2000$ turns per meter is changing at a rate $dI/dt = 3.0 \text{ A/s}$. (See Sources of Magnetic Fields for a discussion of solenoids.) The solenoid is 50-cm long and has a cross-sectional diameter of 3.0 cm. A small coil consisting of $N = 20$ closely wound turns wrapped in a circle of diameter 1.0 cm is placed in the middle of the solenoid such that the plane of the coil is perpendicular to the central axis of the solenoid. Assuming that the infinite-solenoid approximation is valid at the location of the small coil, determine the magnitude of the emf induced in the coil.

Strategy

The magnetic field in the middle of the solenoid is a uniform value of $\mu_0 nI$. This field is producing a maximum magnetic flux through the coil as it is directed along the length of the solenoid. Therefore, the magnetic flux through the coil is the product of the solenoid's magnetic field times the area of the coil. Faraday's law involves a time derivative of the magnetic flux. The only quantity varying in time is the current, the rest can be pulled out of the time derivative. Lastly, we include the number of turns in the coil to determine the induced emf in the coil.

Solution

Since the field of the solenoid is given by $B = \mu_0 nI$, the flux through each turn of the small coil is

Equation:

$$\Phi_{\text{m}} = \mu_0 nI \left(\frac{\pi d^2}{4} \right),$$

where d is the diameter of the coil. Now from Faraday's law, the magnitude of the emf induced in the coil is

Equation:

$$\begin{aligned} \varepsilon &= \left| N \frac{d\Phi_{\mathrm{m}}}{dt} \right| = \left| N \mu_0 n \frac{\pi d^2}{4} \; \frac{dI}{dt} \right| \\ &= 20 \left(4\pi \times 10^{-7}\, \mathrm{T \cdot m/s} \right) \left(2000\, \mathrm{m}^{-1} \right) \frac{\pi (0.010\, \mathrm{m})^2}{4} (3.0\, \mathrm{A/s}) \\ &= 1.2 \times 10^{-5}\, \mathrm{V}. \end{aligned}$$

Significance

When the current is turned on in a vertical solenoid, as shown in [link], the ring has an induced emf from the solenoid's changing magnetic flux that opposes the change. The result is that the ring is fired vertically into the air.

The jumping ring. When a current is turned on in the vertical solenoid, a current is induced in the metal ring. The stray field produced by the solenoid causes the ring to jump off the solenoid.

> **Note:**
> Visit this website for a demonstration of the jumping ring from MIT.

Summary

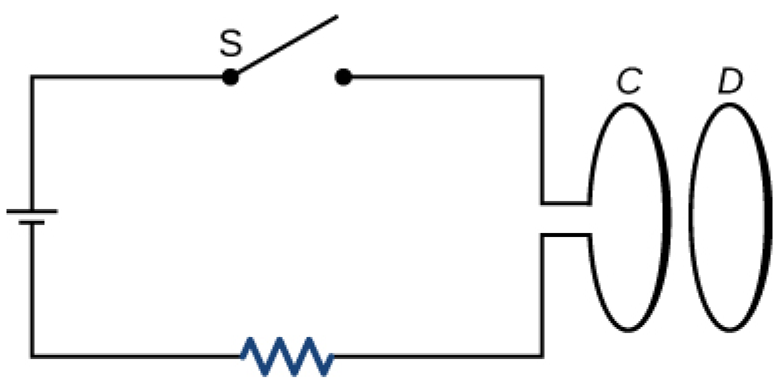
- We can use Lenz's law to determine the directions of induced magnetic fields, currents, and emfs.
- The direction of an induced emf always opposes the change in magnetic flux that causes the emf, a result known as Lenz's law.

Conceptual Questions

Exercise:

Problem:

The circular conducting loops shown in the accompanying figure are parallel, perpendicular to the plane of the page, and coaxial. (a) When the switch S is closed, what is the direction of the current induced in *D*? (b) When the switch is opened, what is the direction of the current induced in loop *D*?



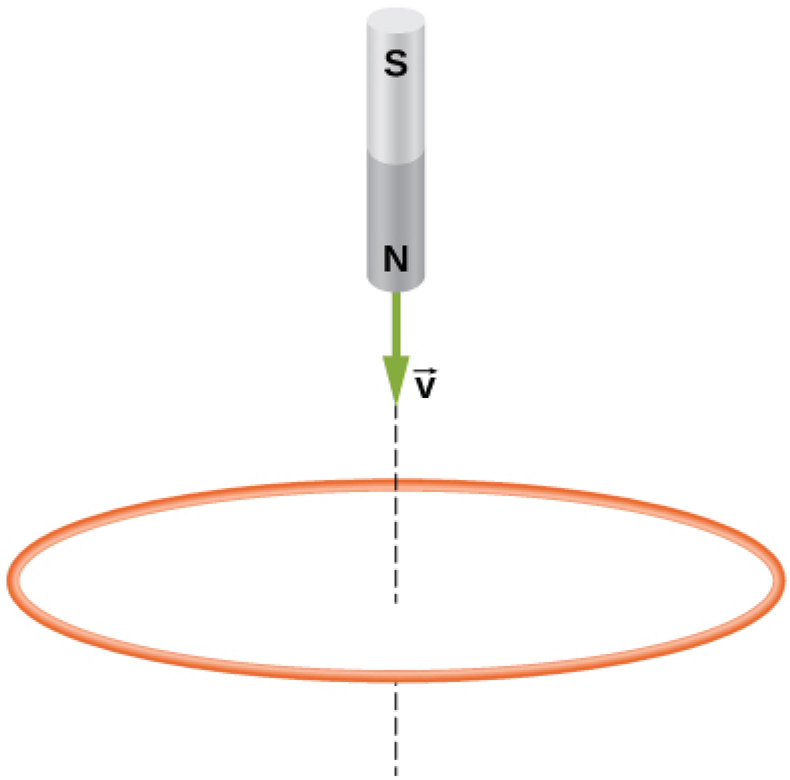
Solution:

a. CW as viewed from the circuit; b. CCW as viewed from the circuit

Exercise:

Problem:

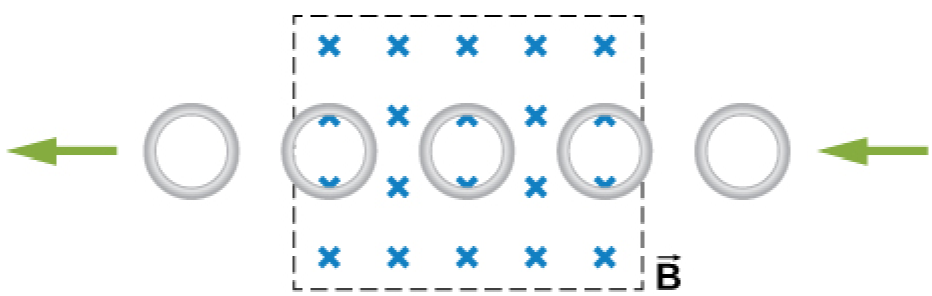
The north pole of a magnet is moved toward a copper loop, as shown below. If you are looking at the loop from above the magnet, will you say the induced current is circulating clockwise or counterclockwise?



Exercise:

Problem:

The accompanying figure shows a conducting ring at various positions as it moves through a magnetic field. What is the sense of the induced emf for each of those positions?



Solution:

As the loop enters, the induced emf creates a CCW current while as the loop leaves the induced emf creates a CW current. While the loop is fully inside the magnetic field, there is no flux change and therefore no induced current.

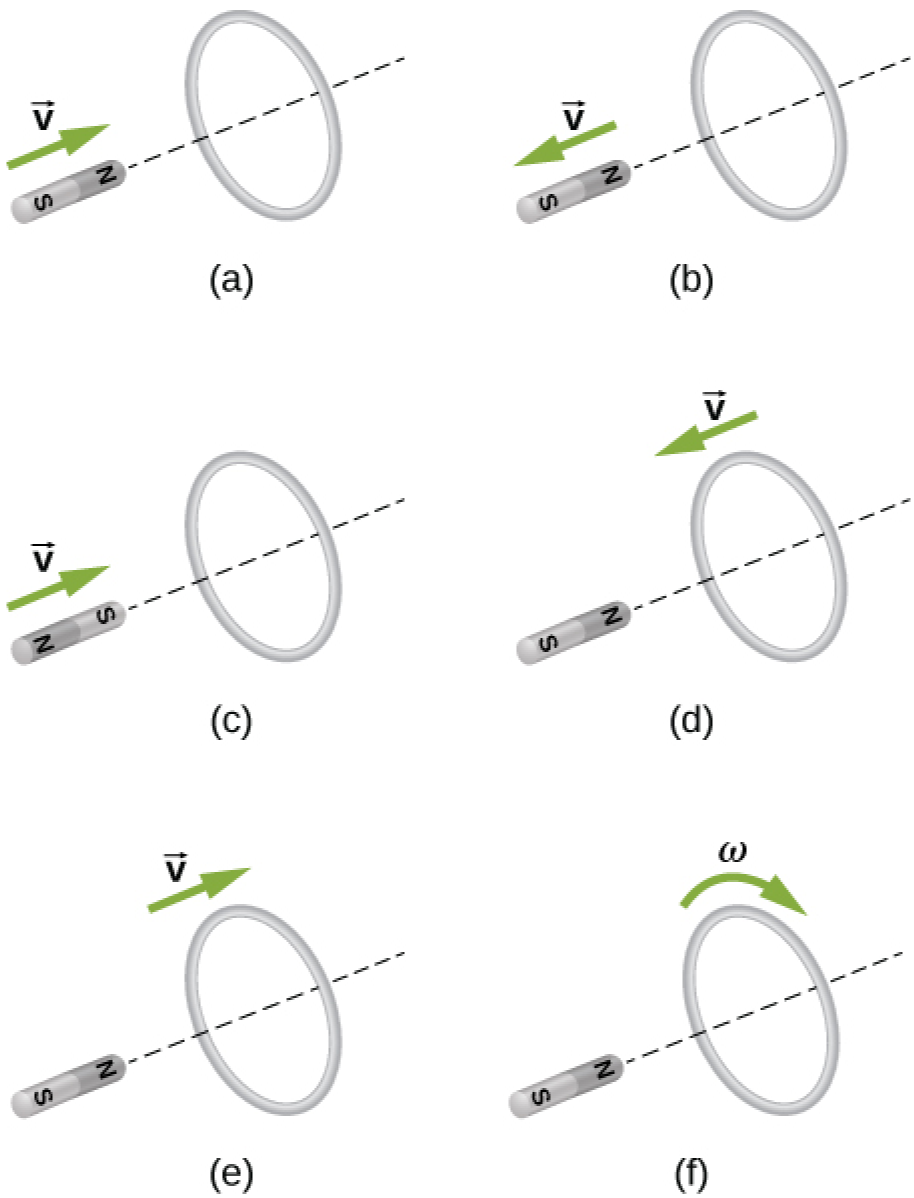
Exercise:

Problem: Show that ε and $d\Phi_{\mathrm{m}}/dt$ have the same units.

Exercise:

Problem:

State the direction of the induced current for each case shown below, observing from the side of the magnet.



Solution:

a. CCW viewed from the magnet; b. CW viewed from the magnet; c. CW viewed from the magnet; d. CCW viewed from the magnet; e. CW viewed from the magnet; f. no current

Problems

Exercise:

Problem:

A single-turn circular loop of wire of radius 50 mm lies in a plane perpendicular to a spatially uniform magnetic field. During a 0.10-s time interval, the magnitude of the field increases uniformly from 200 to 300 mT. (a) Determine the emf induced in the loop. (b) If the magnetic field is directed out of the page, what is the direction of the current induced in the loop?

Solution:

a. 7.8×10^{-3}V; b. CCW from the same view as the magnetic field

Exercise:

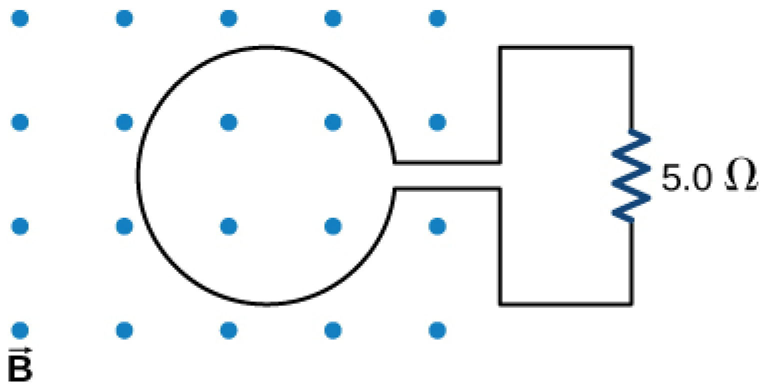
Problem:

When a magnetic field is first turned on, the flux through a 20-turn loop varies with time according to $\Phi_{\mathrm{m}} = 5.0t^2 - 2.0t$, where Φ_{m} is in milliwebers, *t* is in seconds, and the loop is in the plane of the page with the unit normal pointing outward. (a) What is the emf induced in the loop as a function of time? What is the direction of the induced current at (b) *t* = 0, (c) 0.10, (d) 1.0, and (e) 2.0 s?

Exercise:

Problem:

The magnetic flux through the loop shown in the accompanying figure varies with time according to $\Phi_{\mathrm{m}} = 2.00e^{-3t}\sin(120\pi t)$, where Φ_{m} is in milliwebers. What are the direction and magnitude of the current through the 5.00-Ω resistor at (a) $t = 0$; (b) $t = 2.17 \times 10^{-2}$ s, and (c) $t = 3.00$ s?



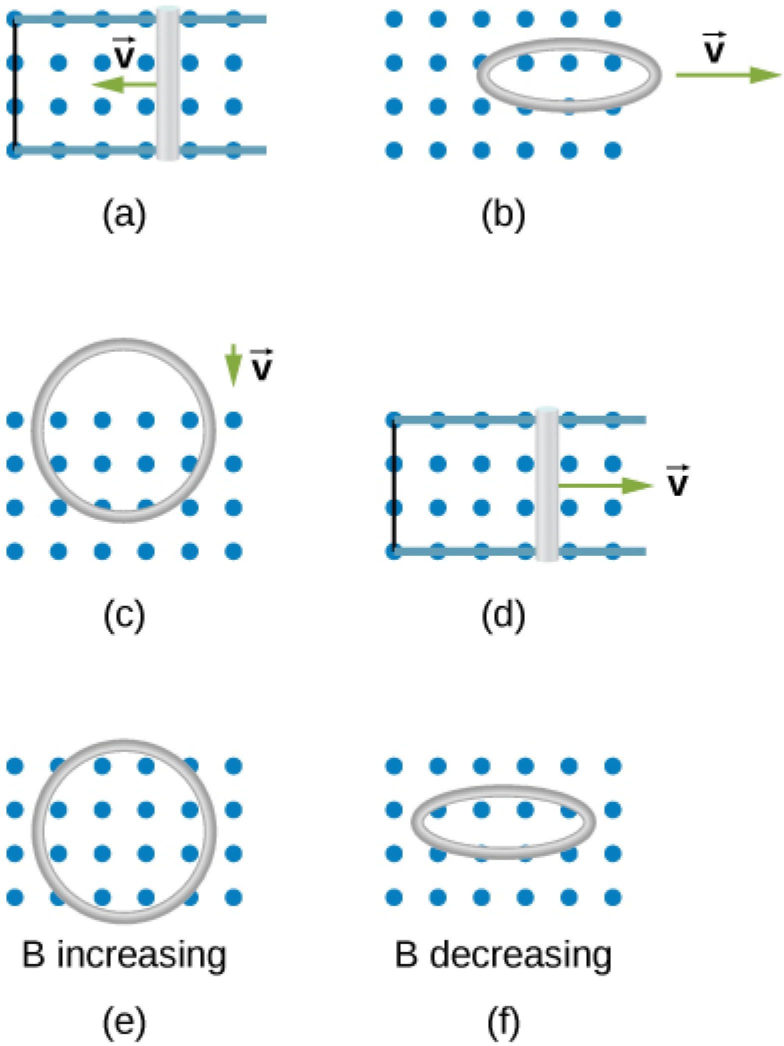
Solution:

a. 150 A downward through the resistor; b. 46 A upward through the resistor; c. 0.019 A downward through the resistor

Exercise:

Problem:

Use Lenz’s law to determine the direction of induced current in each case.



Glossary

Lenz's law
: direction of an induced emf opposes the change in magnetic flux that produced it; this is the negative sign in Faraday's law

Induced Electric Fields

By the end of this section, you will be able to:

- Connect the relationship between an induced emf from Faraday's law to an electric field, thereby showing that a changing magnetic flux creates an electric field
- Solve for the electric field based on a changing magnetic flux in time

The fact that emfs are induced in circuits implies that work is being done on the conduction electrons in the wires. What can possibly be the source of this work? We know that it's neither a battery nor a magnetic field, for a battery does not have to be present in a circuit where current is induced, and magnetic fields never do work on moving charges. The answer is that the source of the work is an electric field $\vec{\mathbf{E}}$ that is induced in the wires. The work done by $\vec{\mathbf{E}}$ in moving a unit charge completely around a circuit is the induced emf ε; that is,

Equation:

$$\varepsilon = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}},$$

where $\oint$ represents the line integral around the circuit. Faraday's law can be written in terms of the **induced electric field** as

Equation:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_{\mathrm{m}}}{dt}.$$

There is an important distinction between the electric field induced by a changing magnetic field and the electrostatic field produced by a fixed charge distribution. Specifically, the induced electric field is nonconservative because it does net work in moving a charge over a closed path, whereas the electrostatic field is conservative and does no net work over a closed path. Hence, electric potential can be associated with the

electrostatic field, but not with the induced field. The following equations represent the distinction between the two types of electric field:
Equation:

$$\begin{aligned}\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &\neq 0 \ \text{(induced)};\\ \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &= 0 \ \text{(electrostatic)}.\end{aligned}$$

Our results can be summarized by combining these equations:

Note:
Equation:

$$\varepsilon = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_{\text{m}}}{dt}.$$

Example:
Induced Electric Field in a Circular Coil
What is the induced electric field in the circular coil of [link] (and [link]) at the three times indicated?
Strategy
Using cylindrical symmetry, the electric field integral simplifies into the electric field times the circumference of a circle. Since we already know the induced emf, we can connect these two expressions by Faraday's law to solve for the induced electric field.
Solution
The induced electric field in the coil is constant in magnitude over the cylindrical surface, similar to how Ampere's law problems with cylinders are solved. Since $\vec{\mathbf{E}}$ is tangent to the coil,
Equation:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \oint E dl = 2\pi r E.$$

When combined with [link], this gives
Equation:

$$E = \frac{\varepsilon}{2\pi r}.$$

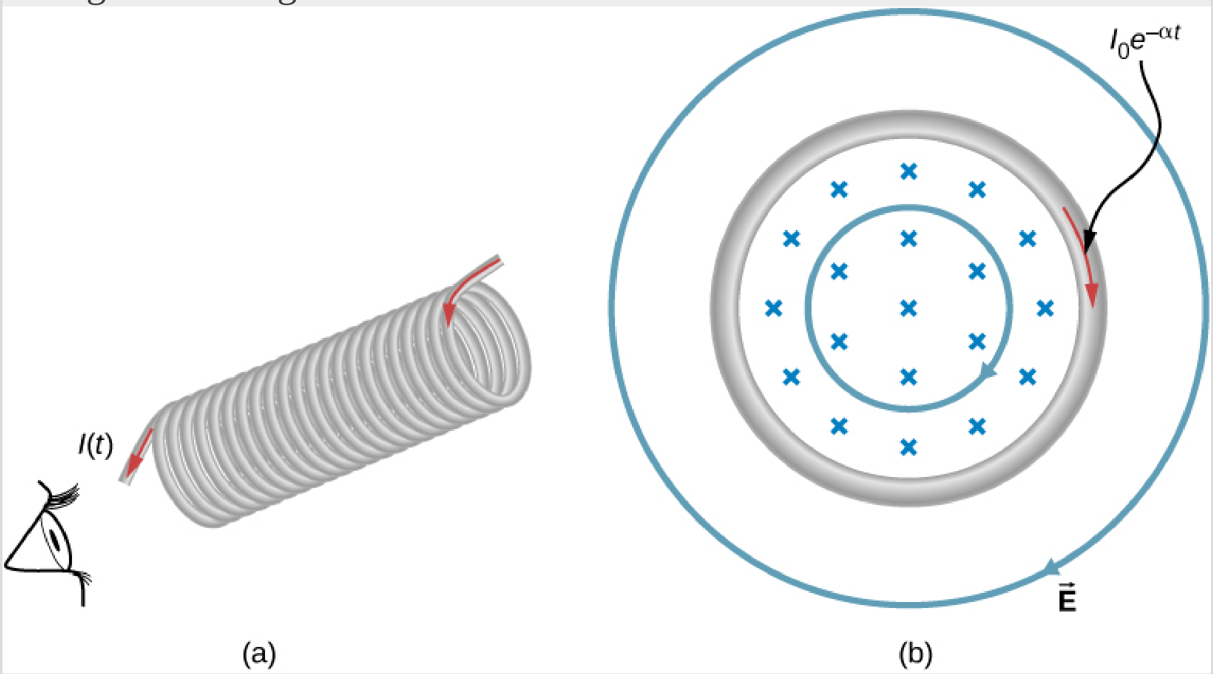
The direction of ε is counterclockwise, and $\vec{\mathbf{E}}$ circulates in the same direction around the coil. The values of E are
Equation:

$$\begin{aligned} E(t_1) &= \frac{6.0 \text{ V}}{2\pi\,(0.50 \text{ m})} = 1.9 \text{ V/m}; \\ E(t_2) &= \frac{4.7 \text{ V}}{2\pi\,(0.50 \text{ m})} = 1.5 \text{ V/m}; \\ E(t_3) &= \frac{0.040 \text{ V}}{2\pi\,(0.50 \text{ m})} = 0.013 \text{ V/m}. \end{aligned}$$

Significance
When the magnetic flux through a circuit changes, a nonconservative electric field is induced, which drives current through the circuit. But what happens if $dB/dt \neq 0$ in free space where there isn't a conducting path? The answer is that this case can be treated *as if a conducting path were present*; that is, nonconservative electric fields are induced wherever $dB/dt \neq 0$, whether or not there is a conducting path present.
These nonconservative electric fields always satisfy [link]. For example, if the circular coil of [link] were removed, an electric field *in free space* at $r = 0.50 \text{ m}$ would still be directed counterclockwise, and its magnitude would still be 1.9 V/m at $t = 0$, 1.5 V/m at $t = 5.0 \times 10^{-2}$ s, etc. The existence of induced electric fields is certainly *not* restricted to wires in circuits.

Example:
Electric Field Induced by the Changing Magnetic Field of a Solenoid

Part (a) of [link] shows a long solenoid with radius R and n turns per unit length; its current decreases with time according to $I = I_0 e^{-\alpha t}$. What is the magnitude of the induced electric field at a point a distance r from the central axis of the solenoid (a) when $r > R$ and (b) when $r < R$ [see part (b) of [link]]. (c) What is the direction of the induced field at both locations? Assume that the infinite-solenoid approximation is valid throughout the regions of interest.



(a) The current in a long solenoid is decreasing exponentially. (b) A cross-sectional view of the solenoid from its left end. The cross-section shown is near the middle of the solenoid. An electric field is induced both inside and outside the solenoid.

Strategy
Using the formula for the magnetic field inside an infinite solenoid and Faraday's law, we calculate the induced emf. Since we have cylindrical symmetry, the electric field integral reduces to the electric field times the circumference of the integration path. Then we solve for the electric field.
Solution

a. The magnetic field is confined to the interior of the solenoid where

Equation:

$$B = \mu_0 nI = \mu_0 nI_0 e^{-\alpha t}.$$

Thus, the magnetic flux through a circular path whose radius *r* is greater than *R*, the solenoid radius, is
Equation:

$$\Phi_{\text{m}} = BA = \mu_0 nI_0 \pi R^2 e^{-\alpha t}.$$

The induced field $\vec{\mathbf{E}}$ is tangent to this path, and because of the cylindrical symmetry of the system, its magnitude is constant on the path. Hence, we have
Equation:

$$\begin{aligned}
\left| \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \right| &= \left| \frac{d\Phi_{\text{m}}}{dt} \right|, \\
E(2\pi r) &= \left| \frac{d}{dt} \left(\mu_0 nI_0 \pi R^2 e^{-\alpha t} \right) \right| = \alpha \mu_0 nI_0 \pi R^2 e^{-\alpha t}, \\
E &= \frac{\alpha \mu_0 nI_0 R^2}{2r} e^{-\alpha t} \ \ (r > R).
\end{aligned}$$

b. For a path of radius *r* inside the solenoid, $\Phi_{\text{m}} = B\pi r^2$, so
Equation:

$$E(2\pi r) = \left| \frac{d}{dt} \left(\mu_0 nI_0 \pi r^2 e^{-\alpha t} \right) \right| = \alpha \mu_0 nI_0 \pi r^2 e^{-\alpha t},$$

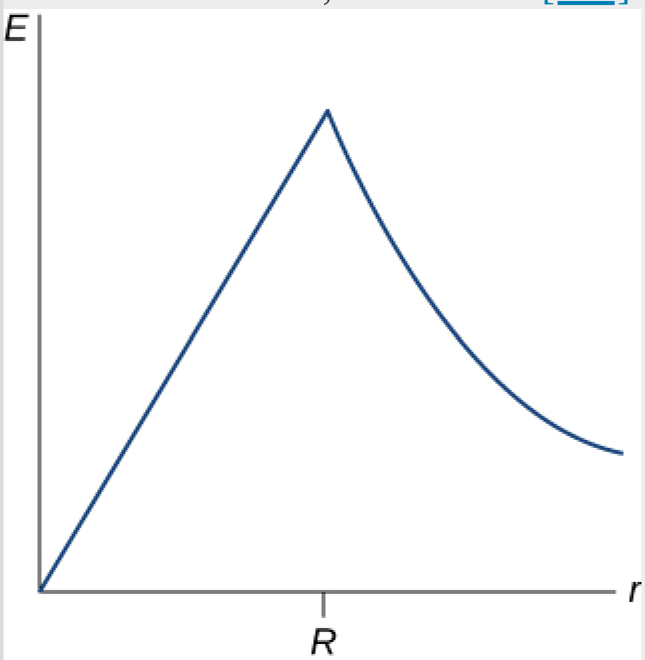
and the induced field is
Equation:

$$E = \frac{\alpha \mu_0 nI_0 r}{2} e^{-\alpha t} \ (r < R).$$

c. The magnetic field points into the page as shown in part (b) and is decreasing. If either of the circular paths were occupied by conducting rings, the currents induced in them would circulate as shown, in conformity with Lenz's law. The induced electric field must be so directed as well.

Significance

In part (b), note that $\left|\vec{\mathbf{E}}\right|$ increases with *r* inside and decreases as 1/*r* outside the solenoid, as shown in [link].



The electric field vs. distance *r*. When $r < R$, the electric field rises linearly, whereas when $r > R$, the electric field falls of proportional to 1/*r*.

Note:
Exercise:

Problem:

Check Your Understanding Suppose that the coil of [link] is a square rather than circular. Can [link] be used to calculate (a) the induced emf and (b) the induced electric field?

Solution:

a. yes; b. Yes; however there is a lack of symmetry between the electric field and coil, making $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ a more complicated relationship that can't be simplified as shown in the example.

Note:
Exercise:

Problem:

Check Your Understanding What is the magnitude of the induced electric field in [link] at $t = 0$ if $r = 6.0\,\text{cm}$, $R = 2.0\,\text{cm}$, $n = 2000$ turns per meter, $I_0 = 2.0\,\text{A}$, and $\alpha = 200\,\text{s}^{-1}$?

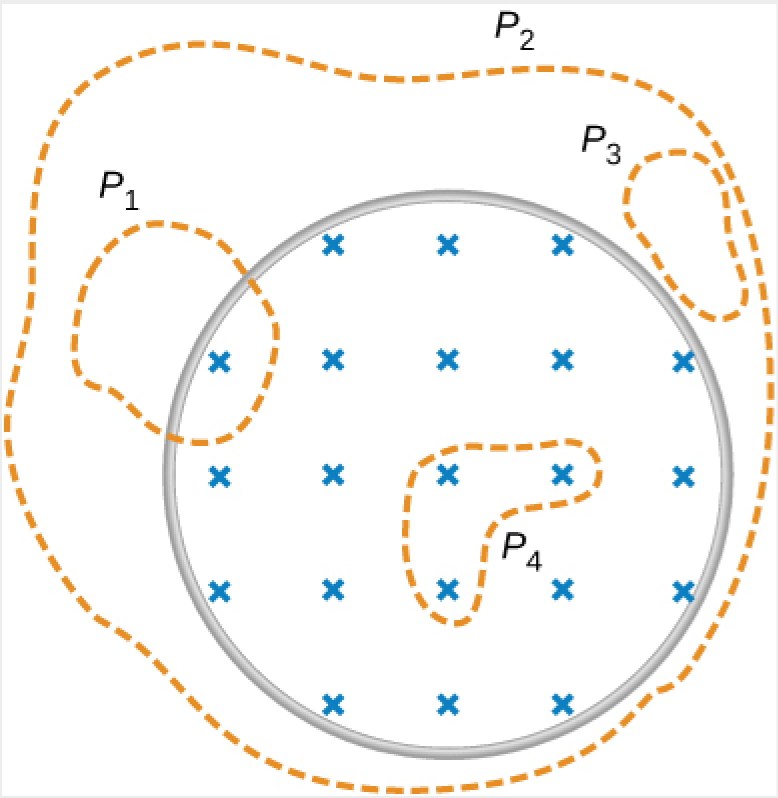
Solution:

$3.4\ \times\ 10^{-3}\ \text{V/m}$

Note:
Exercise:

Problem:

Check Your Understanding The magnetic field shown below is confined to the cylindrical region shown and is changing with time. Identify those paths for which $\varepsilon = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \neq 0$.



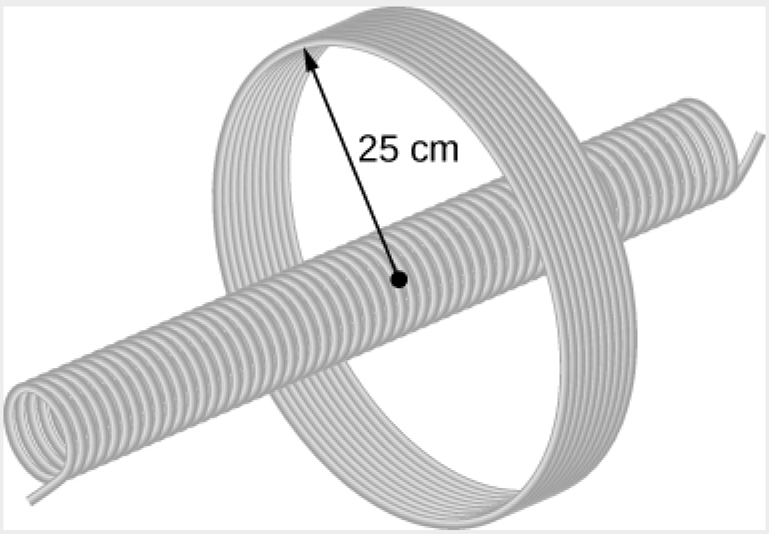
Solution:

P_1, P_2, P_4

Note:
Exercise:

Problem:

Check Your Understanding A long solenoid of cross-sectional area $5.0\ \text{cm}^2$ is wound with 25 turns of wire per centimeter. It is placed in the middle of a closely wrapped coil of 10 turns and radius 25 cm, as shown below. (a) What is the emf induced in the coil when the current through the solenoid is decreasing at a rate $dI/dt = -0.20\ \text{A/s}$? (b) What is the electric field induced in the coil?



Solution:

a. $3.1 \times 10^{-6}\text{V}$; b. $2.0 \times 10^{-7}\ \text{V/m}$

Summary

- A changing magnetic flux induces an electric field.
- Both the changing magnetic flux and the induced electric field are related to the induced emf from Faraday's law.

Conceptual Questions

Exercise:

Problem:

Is the work required to accelerate a rod from rest to a speed *v* in a magnetic field greater than the final kinetic energy of the rod? Why?

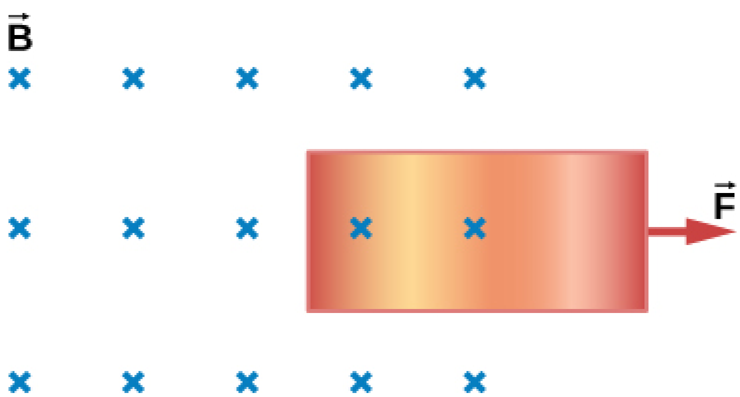
Solution:

The work is greater than the kinetic energy because it takes energy to counteract the induced emf.

Exercise:

Problem:

The copper sheet shown below is partially in a magnetic field. When it is pulled to the right, a resisting force pulls it to the left. Explain. What happen if the sheet is pushed to the left?



Problems

Exercise:

Problem:

Calculate the induced electric field in a 50-turn coil with a diameter of 15 cm that is placed in a spatially uniform magnetic field of magnitude 0.50 T so that the face of the coil and the magnetic field are perpendicular. This magnetic field is reduced to zero in 0.10 seconds. Assume that the magnetic field is cylindrically symmetric with respect to the central axis of the coil.

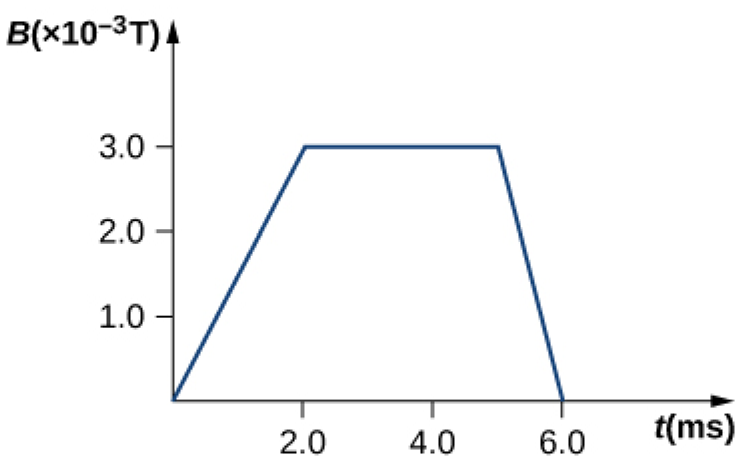
Solution:

9.375 V/m

Exercise:

Problem:

The magnetic field through a circular loop of radius 10.0 cm varies with time as shown in the accompanying figure. The field is perpendicular to the loop. Assuming cylindrical symmetry with respect to the central axis of the loop, plot the induced electric field in the loop as a function of time.



Exercise:

Problem:

The current I through a long solenoid with n turns per meter and radius R is changing with time as given by dI/dt. Calculate the induced electric field as a function of distance r from the central axis of the solenoid.

Solution:

Inside, $B = \mu_0 nI$, $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \left(\pi r^2\right)\mu_0 n \frac{dI}{dt}$, so, $E = \frac{\mu_0 nr}{2} \cdot \frac{dI}{dt}$ (inside). Outside, $E\left(2\pi r\right) = \pi R^2 \mu_0 n \frac{dI}{dt}$, so, $E = \frac{\mu_0 nR^2}{2r} \cdot \frac{dI}{dt}$ (outside)

Exercise:

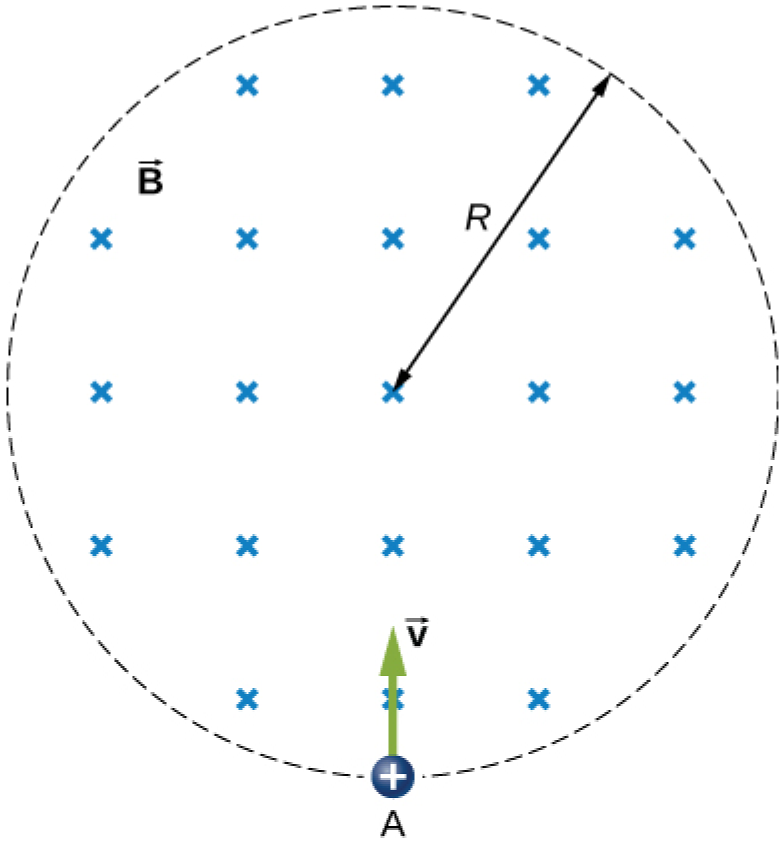
Problem:

Calculate the electric field induced both inside and outside the solenoid of the preceding problem if $I = I_0 \sin \omega t$.

Exercise:

Problem:

Over a region of radius R, there is a spatially uniform magnetic field $\vec{\mathbf{B}}$. (See below.) At $t = 0$, $B = 1.0 \text{ T}$, after which it decreases at a constant rate to zero in 30 s. (a) What is the electric field in the regions where $r \leq R$ and $r \geq R$ during that 30-s interval? (b) Assume that $R = 10.0 \text{ cm}$. How much work is done by the electric field on a proton that is carried once clock wise around a circular path of radius 5.0 cm? (c) How much work is done by the electric field on a proton that is carried once counterclockwise around a circular path of any radius $r \geq R$? (d) At the instant when $B = 0.50 \text{ T}$, a proton enters the magnetic field at A, moving a velocity $\vec{\mathbf{v}}$ $\left(v = 5.0 \times 10^6 \text{ m/s}\right)$ as shown. What are the electric and magnetic forces on the proton at that instant?



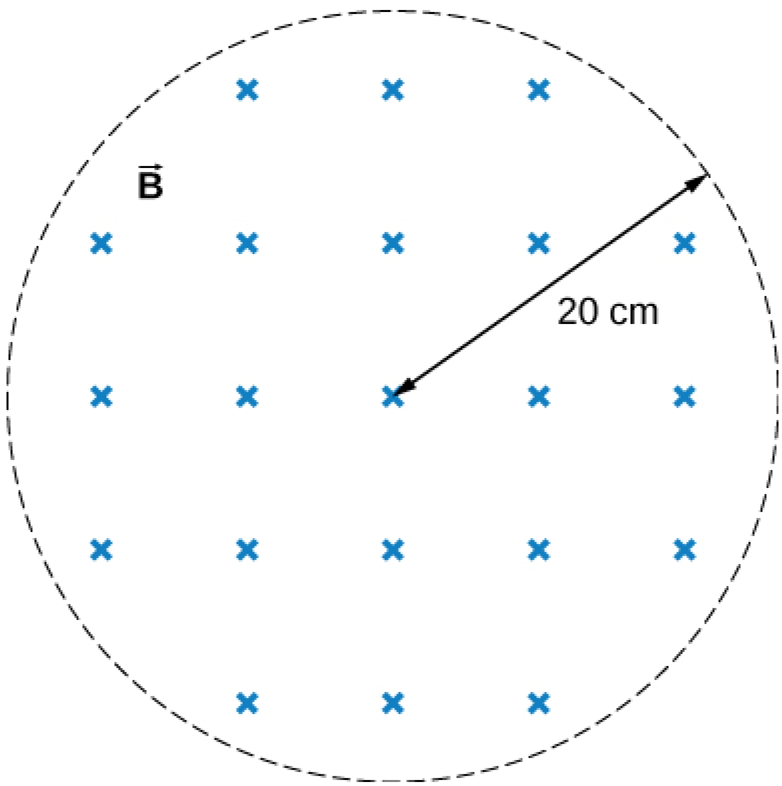
Solution:

a. $E_{\text{inside}} = \frac{r}{2}\frac{dB}{dt}$, $E_{\text{outside}} = \frac{r^2}{2R}\frac{dB}{dt}$; b. $W = 4.19 \times 10^{-23}$ J; c. 0 J; d. $F_{\text{mag}} = 4 \times 10^{-13}$ N, $F_{\text{elec}} = 2.7 \times 10^{-22}$ N

Exercise:

Problem:

The magnetic field at all points within the cylindrical region whose cross-section is indicated in the accompanying figure starts at 1.0 T and decreases uniformly to zero in 20 s. What is the electric field (both magnitude and direction) as a function of *r*, the distance from the geometric center of the region?



Exercise:

Problem:

The current in a long solenoid with 20 turns per centimeter of radius 3 cm is varied with time at a rate of 2 A/s. A circular loop of wire of radius 5 cm and resistance $2\,\Omega$ surrounds the solenoid. Find the electrical current induced in the loop.

Solution:

$7.1\ \mu\text{A}$

Exercise:

Problem:

The current in a long solenoid of radius 3 cm and 20 turns/cm is varied with time at a rate of 2 A/s. Find the electric field at a distance of 4 cm from the center of the solenoid.

Glossary

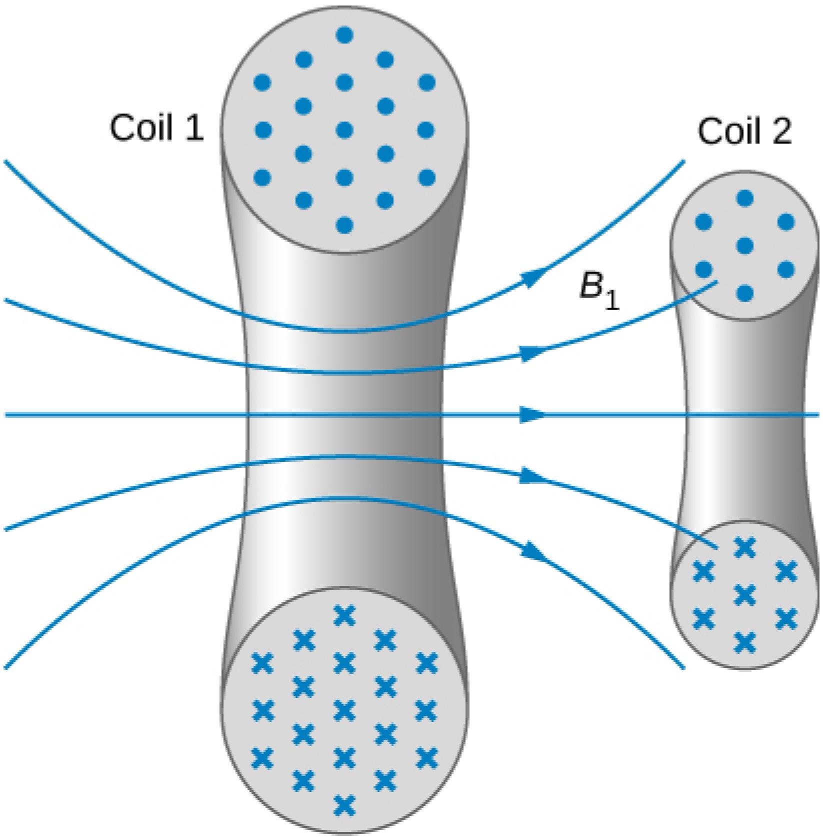
induced electric field
: created based on the changing magnetic flux with time

Mutual Inductance
By the end of this section, you will be able to:

- Correlate two nearby circuits that carry time-varying currents with the emf induced in each circuit
- Describe examples in which mutual inductance may or may not be desirable

Inductance is the property of a device that tells us how effectively it induces an emf in another device. In other words, it is a physical quantity that expresses the effectiveness of a given device.

When two circuits carrying time-varying currents are close to one another, the magnetic flux through each circuit varies because of the changing current *I* in the other circuit. Consequently, an emf is induced in each circuit by the changing current in the other. This type of emf is therefore called a *mutually induced emf*, and the phenomenon that occurs is known as **mutual inductance (*M*)**. As an example, let's consider two tightly wound coils ([link]). Coils 1 and 2 have N_1 and N_2 turns and carry currents I_1 and I_2, respectively. The flux through a single turn of coil 2 produced by the magnetic field of the current in coil 1 is Φ_{21}, whereas the flux through a single turn of coil 1 due to the magnetic field of I_2 is Φ_{12}.



Some of the magnetic field lines produced by the current in coil 1 pass through coil 2.

The mutual inductance M_{21} of coil 2 with respect to coil 1 is the ratio of the flux through the N_2 turns of coil 2 produced by the magnetic field of the current in coil 1, divided by that current, that is,

Equation:

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}.$$

Similarly, the mutual inductance of coil 1 with respect to coil 2 is

Equation:

$$M_{12} = \frac{N_1 \Phi_{12}}{I_2}.$$

Like capacitance, mutual inductance is a geometric quantity. It depends on the shapes and relative positions of the two coils, and it is independent of the currents in the coils. The SI unit for mutual inductance *M* is called the **henry (H)** in honor of Joseph Henry (1799–1878), an American scientist who discovered induced emf independently of Faraday. Thus, we have $1 \text{ H} = 1 \text{ V} \cdot \text{s/A}$. From [link] and [link], we can show that $M_{21} = M_{12}$, so we usually drop the subscripts associated with mutual inductance and write

Note:
Equation:

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}.$$

The emf developed in either coil is found by combining Faraday's law and the definition of mutual inductance. Since $N_2 \Phi_{21}$ is the total flux through coil 2 due to I_1, we obtain

Equation:

$$\varepsilon_2 = -\frac{d}{dt}(N_2 \Phi_{21}) = -\frac{d}{dt}(M I_1) = -M \frac{dI_1}{dt}$$

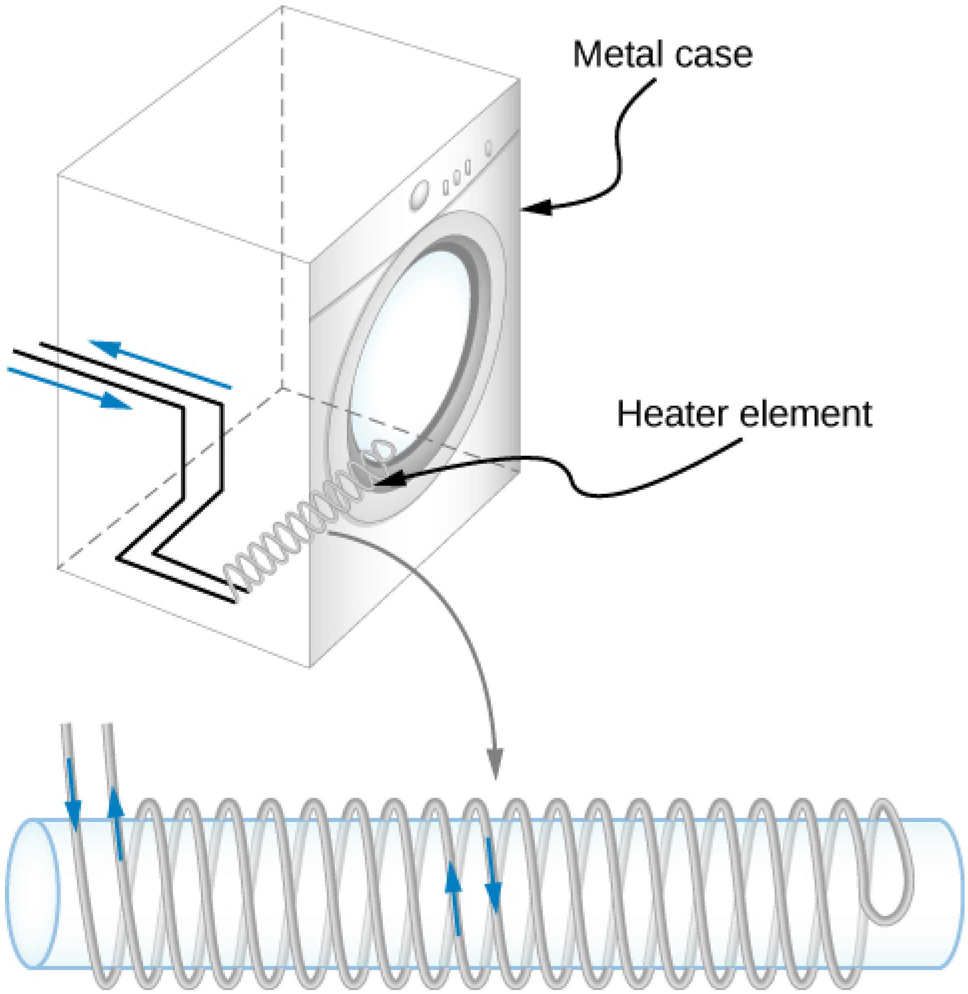
where we have used the fact that *M* is a time-independent constant because the geometry is time-independent. Similarly, we have

Note:
Equation:

$$\varepsilon_1 = -M\frac{dI_2}{dt}.$$

In [link], we can see the significance of the earlier description of mutual inductance (*M*) as a geometric quantity. The value of *M* neatly encapsulates the physical properties of circuit elements and allows us to separate the physical layout of the circuit from the dynamic quantities, such as the emf and the current. [link] defines the mutual inductance in terms of properties in the circuit, whereas the previous definition of mutual inductance in [link] is defined in terms of the magnetic flux experienced, regardless of circuit elements. You should be careful when using [link] and [link] because ε_1 and ε_2 do not necessarily represent the total emfs in the respective coils. Each coil can also have an emf induced in it because of its *self-inductance* (self-inductance will be discussed in more detail in a later section).

A large mutual inductance *M* may or may not be desirable. We want a transformer to have a large mutual inductance. But an appliance, such as an electric clothes dryer, can induce a dangerous emf on its metal case if the mutual inductance between its coils and the case is large. One way to reduce mutual inductance is to counter-wind coils to cancel the magnetic field produced ([link]).



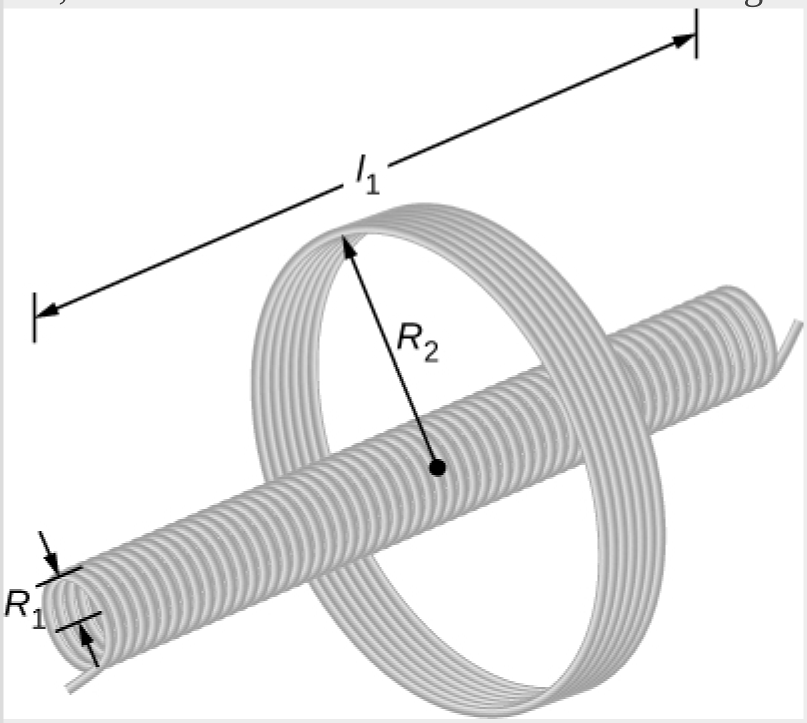
The heating coils of an electric clothes dryer can be counter-wound so that their magnetic fields cancel one another, greatly reducing the mutual inductance with the case of the dryer.

Digital signal processing is another example in which mutual inductance is reduced by counter-winding coils. The rapid on/off emf representing 1s and 0s in a digital circuit creates a complex time-dependent magnetic field. An emf can be generated in neighboring conductors. If that conductor is also carrying a digital signal, the induced emf may be large enough to switch 1s and 0s, with consequences ranging from inconvenient to disastrous.

Example:

Mutual Inductance

[link] shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1, radius R_1, and N_1 turns. (a) What is the mutual inductance of the two coils? (b) If $N_1 = 500\,\text{turns}$, $N_2 = 10\,\text{turns}$, $R_1 = 3.10\,\text{cm}$, $l_1 = 75.0\,\text{cm}$, and the current in the solenoid is changing at a rate of 200 A/s, what is the emf induced in the surrounding coil?



A solenoid surrounded by a coil.

Strategy

There is no magnetic field outside the solenoid, and the field inside has magnitude $B_1 = \mu_0 (N_1/l_1) I_1$ and is directed parallel to the solenoid's axis. We can use this magnetic field to find the magnetic flux through the surrounding coil and then use this flux to calculate the mutual inductance for part (a), using [link]. We solve part (b) by calculating the mutual inductance from the given quantities and using [link] to calculate the induced emf.

Solution

a. The magnetic flux Φ_{21} through the surrounding coil is
Equation:

$$\Phi_{21} = B_1 \pi R_1^2 = \frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2.$$

Now from [link], the mutual inductance is
Equation:

$$M = \frac{N_2 \Phi_{21}}{I_1} = \left(\frac{N_2}{I_1}\right)\left(\frac{\mu_0 N_1 I_1}{l_1}\right)\pi R_1^2 = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}.$$

b. Using the previous expression and the given values, the mutual inductance is
Equation:

$$\begin{aligned} M &= \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(500)(10)\pi(0.0310\,\text{m})^2}{0.750\,\text{m}} \\ &= 2.53 \times 10^{-5}\,\text{H}. \end{aligned}$$

Thus, from [link], the emf induced in the surrounding coil is
Equation:

$$\begin{aligned} \varepsilon_2 &= -M\frac{dI_1}{dt} = -(2.53 \times 10^{-5}\,\text{H})(200\,\text{A/s}) \\ &= -5.06 \times 10^{-3}\,\text{V}. \end{aligned}$$

Significance
Notice that M in part (a) is independent of the radius R_2 of the surrounding coil because the solenoid's magnetic field is confined to its interior. In principle, we can also calculate M by finding the magnetic flux through the solenoid produced by the current in the surrounding coil. This approach is much more difficult because Φ_{12} is so complicated. However, since $M_{12} = M_{21}$, we do know the result of this calculation.

Note:
Exercise:

Problem:

Check Your Understanding A current $I(t) = (5.0\text{ A})\sin((120\pi\text{ rad/s})t)$ flows through the solenoid of part (b) of [link]. What is the maximum emf induced in the surrounding coil?

Solution:

$4.77 \times 10^{-2}\text{ V}$

Summary

- Inductance is the property of a device that expresses how effectively it induces an emf in another device.
- Mutual inductance is the effect of two devices inducing emfs in each other.
- A change in current dI_1/dt in one circuit induces an emf (ε_2) in the second:
  Equation:
  $$\varepsilon_2 = -M\frac{dI1}{dt},$$
  where M is defined to be the mutual inductance between the two circuits and the minus sign is due to Lenz's law.
- Symmetrically, a change in current dI_2/dt through the second circuit induces an emf (ε_1) in the first:
  Equation:
  $$\varepsilon_1 = -M\frac{dI_2}{dt},$$

where M is the same mutual inductance as in the reverse process.

Conceptual Questions

Exercise:

Problem:

Show that $N\Phi_{\mathrm{m}}/I$ and $\varepsilon/(dI/dt)$, which are both expressions for self-inductance, have the same units.

Solution:

$$\frac{\mathrm{Wb}}{\mathrm{A}} = \frac{\mathrm{T{\cdot}m^2}}{\mathrm{A}} = \frac{\mathrm{V{\cdot}s}}{\mathrm{A}} = \frac{\mathrm{V}}{\mathrm{A/s}}$$

Exercise:

Problem:

A 10-H inductor carries a current of 20 A. Describe how a 50-V emf can be induced across it.

Exercise:

Problem:

The ignition circuit of an automobile is powered by a 12-V battery. How are we able to generate large voltages with this power source?

Solution:

The induced current from the 12-V battery goes through an inductor, generating a large voltage.

Exercise:

Problem:

When the current through a large inductor is interrupted with a switch, an arc appears across the open terminals of the switch. Explain.

Problems

Exercise:

Problem:

When the current in one coil changes at a rate of 5.6 A/s, an emf of 6.3×10^{-3} V is induced in a second, nearby coil. What is the mutual inductance of the two coils?

Exercise:

Problem:

An emf of 9.7×10^{-3} V is induced in a coil while the current in a nearby coil is decreasing at a rate of 2.7 A/s. What is the mutual inductance of the two coils?

Solution:

$M = 3.6 \times 10^{-3}$ H

Exercise:

Problem:

Two coils close to each other have a mutual inductance of 32 mH. If the current in one coil decays according to $I = I_0 e^{-\alpha t}$, where $I_0 = 5.0$ A and $\alpha = 2.0 \times 10^3$ s^{-1}, what is the emf induced in the second coil immediately after the current starts to decay? At $t = 1.0 \times 10^{-3}$ s?

Exercise:

Problem:

A coil of 40 turns is wrapped around a long solenoid of cross-sectional area $7.5 \times 10^{-3}\ \mathrm{m}^2$. The solenoid is 0.50 m long and has 500 turns. (a) What is the mutual inductance of this system? (b) The outer coil is replaced by a coil of 40 turns whose radius is three times that of the solenoid. What is the mutual inductance of this configuration?

Solution:

a. $3.8 \times 10^{-4}\ \mathrm{H}$; b. $3.8 \times 10^{-4}\ \mathrm{H}$

Exercise:

Problem:

A 600-turn solenoid is 0.55 m long and 4.2 cm in diameter. Inside the solenoid, a small $(1.1\ \mathrm{cm} \times 1.4\ \mathrm{cm})$, single-turn rectangular coil is fixed in place with its face perpendicular to the long axis of the solenoid. What is the mutual inductance of this system?

Exercise:

Problem:

A toroidal coil has a mean radius of 16 cm and a cross-sectional area of $0.25\ \mathrm{cm}^2$; it is wound uniformly with 1000 turns. A second toroidal coil of 750 turns is wound uniformly over the first coil. Ignoring the variation of the magnetic field within a toroid, determine the mutual inductance of the two coils.

Solution:

$M_{21} = 2.3 \times 10^{-5}\ \mathrm{H}$

Exercise:

Problem:

A solenoid of N_1 turns has length l_1 and radius R_1, and a second smaller solenoid of N_2 turns has length l_2 and radius R_2. The smaller solenoid is placed completely inside the larger solenoid so that their long axes coincide. What is the mutual inductance of the two solenoids?

Glossary

henry (H)
: unit of inductance, $1\ \text{H} = 1\ \Omega \cdot \text{s}$; it is also expressed as a volt second per ampere

inductance
: property of a device that tells how effectively it induces an emf in another device

mutual inductance
: geometric quantity that expresses how effective two devices are at inducing emfs in one another

Self-Inductance and Inductors
By the end of this section, you will be able to:

- Correlate the rate of change of current to the induced emf created by that current in the same circuit
- Derive the self-inductance for a cylindrical solenoid
- Derive the self-inductance for a rectangular toroid

Mutual inductance arises when a current in one circuit produces a changing magnetic field that induces an emf in another circuit. But can the magnetic field affect the current in the original circuit that produced the field? The answer is yes, and this is the phenomenon called *self-inductance*.

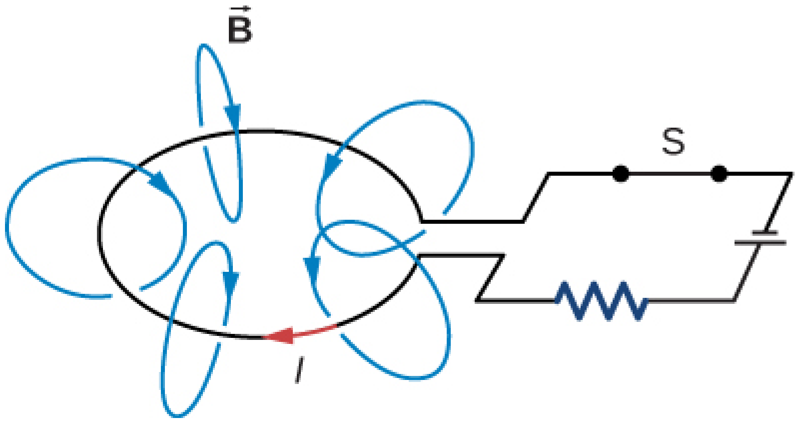
Inductors

[link] shows some of the magnetic field lines due to the current in a circular loop of wire. If the current is constant, the magnetic flux through the loop is also constant. However, if the current *I* were to vary with time—say, immediately after switch S is closed—then the magnetic flux Φ_{m} would correspondingly change. Then Faraday's law tells us that an emf ε would be induced in the circuit, where
Equation:

$$\varepsilon = -\frac{d\Phi_{\mathrm{m}}}{dt}.$$

Since the magnetic field due to a current-carrying wire is directly proportional to the current, the flux due to this field is also proportional to the current; that is,
Equation:

$$\Phi_{\mathrm{m}} \propto I.$$



A magnetic field is produced by the current *I* in the loop. If *I* were to vary with time, the magnetic flux through the loop would also vary and an emf would be induced in the loop.

This can also be written as

Equation:

$$\Phi_{\mathrm{m}} = LI$$

where the constant of proportionality L is known as the **self-inductance** of the wire loop. If the loop has N turns, this equation becomes

Note:

Equation:

$$N\Phi_{\mathrm{m}} = LI.$$

By convention, the positive sense of the normal to the loop is related to the current by the right-hand rule, so in [link], the normal points downward.

With this convention, Φ_{m} is positive in [link], so L *always has a positive value*.

For a loop with N turns, $\varepsilon = -N d\Phi_{\mathrm{m}}/dt$, so the induced emf may be written in terms of the self-inductance as

Note:
Equation:

$$\varepsilon = -L\frac{dI}{dt}.$$

When using this equation to determine L, it is easiest to ignore the signs of ε and dI/dt, and calculate L as
Equation:

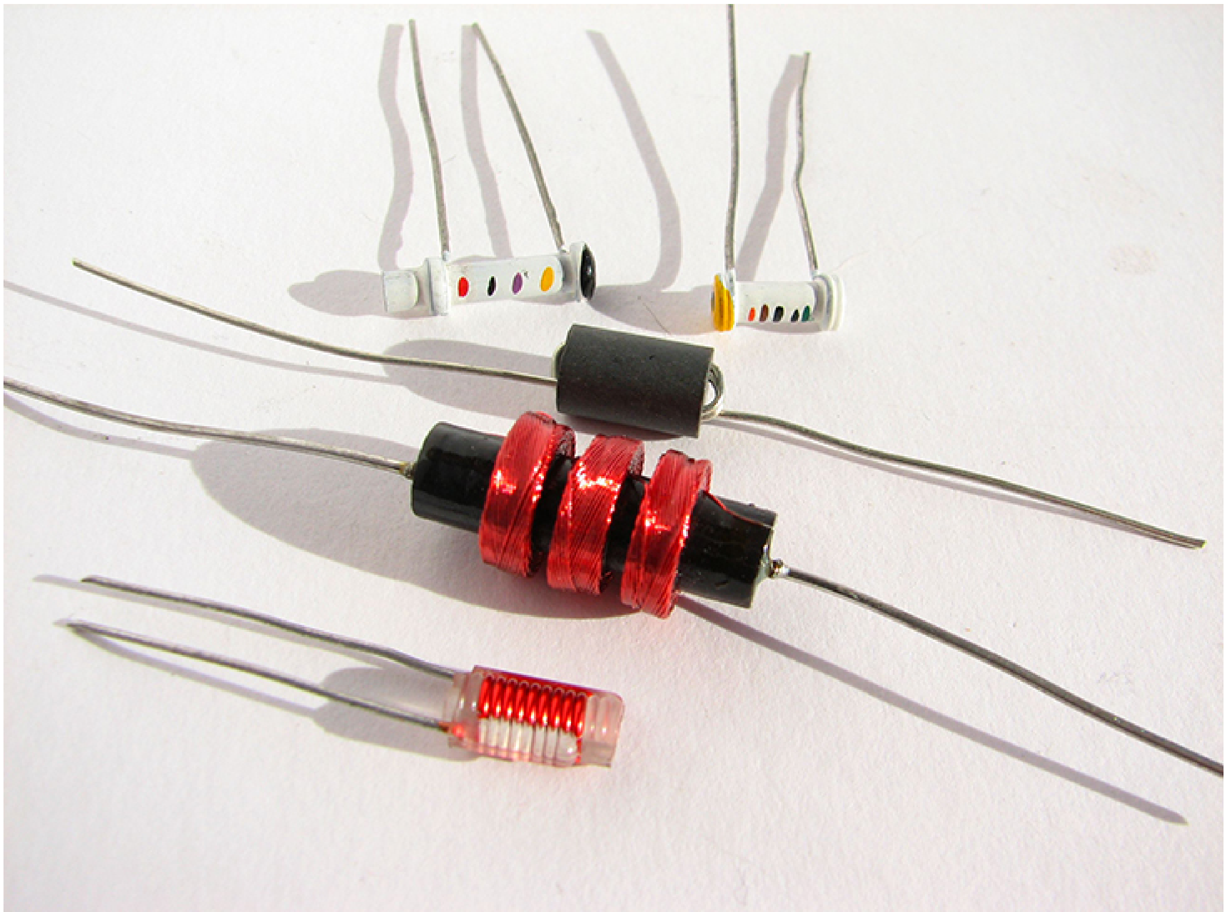
$$L = \frac{|\varepsilon|}{|dI/dt|}.$$

Since self-inductance is associated with the magnetic field produced by a current, any configuration of conductors possesses self-inductance. For example, besides the wire loop, a long, straight wire has self-inductance, as does a coaxial cable. A coaxial cable is most commonly used by the cable television industry and may also be found connecting to your cable modem. Coaxial cables are used due to their ability to transmit electrical signals with minimal distortions. Coaxial cables have two long cylindrical conductors that possess current and a self-inductance that may have undesirable effects.

A circuit element used to provide self-inductance is known as an **inductor**. It is represented by the symbol shown in [link], which resembles a coil of

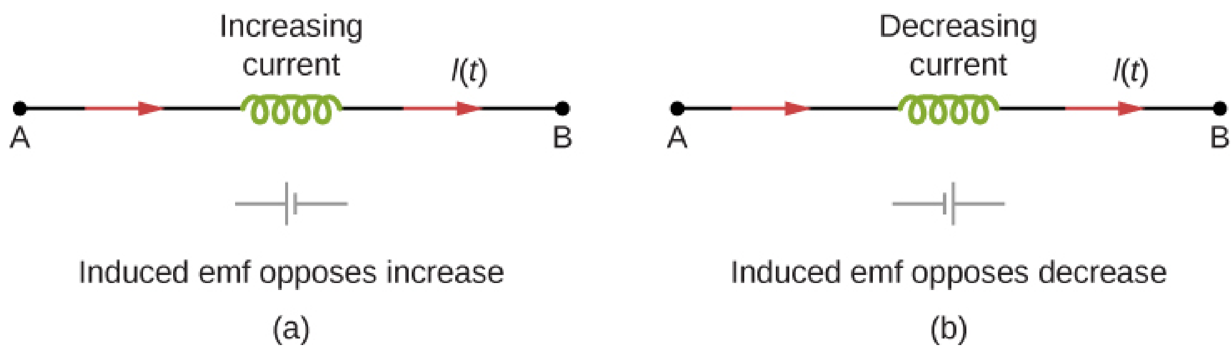
wire, the basic form of the inductor. [link] shows several types of inductors commonly used in circuits.

Symbol used to represent an inductor in a circuit.



A variety of inductors. Whether they are encapsulated like the top three shown or wound around in a coil like the bottom-most one, each is simply a relatively long coil of wire. (credit: Windell Oskay)

In accordance with Lenz's law, the negative sign in [link] indicates that the induced emf across an inductor always has a polarity that *opposes* the change in the current. For example, if the current flowing from *A* to *B* in [link](a) were increasing, the induced emf (represented by the imaginary battery) would have the polarity shown in order to oppose the increase. If the current from *A* to *B* were decreasing, then the induced emf would have the opposite polarity, again to oppose the change in current ([link](b)). Finally, if the current through the inductor were constant, no emf would be induced in the coil.



The induced emf across an inductor always acts to oppose the change in the current. This can be visualized as an imaginary battery causing current to flow to oppose the change in (a) and reinforce the change in (b).

One common application of inductance is to allow traffic signals to sense when vehicles are waiting at a street intersection. An electrical circuit with an inductor is placed in the road underneath the location where a waiting car will stop. The body of the car increases the inductance and the circuit changes, sending a signal to the traffic lights to change colors. Similarly, metal detectors used for airport security employ the same technique. A coil or inductor in the metal detector frame acts as both a transmitter and a receiver. The pulsed signal from the transmitter coil induces a signal in the receiver. The self-inductance of the circuit is affected by any metal object in the path ([link]). Metal detectors can be adjusted for sensitivity and can also sense the presence of metal on a person.

The familiar security gate at an airport not only detects metals, but can also indicate their approximate height above the floor. (credit: “Alexbuirds”/Wikimedia Commons)

Large induced voltages are found in camera flashes. Camera flashes use a battery, two inductors that function as a transformer, and a switching system or *oscillator* to induce large voltages. Recall from Oscillations on oscillations that “oscillation” is defined as the fluctuation of a quantity, or repeated regular fluctuations of a quantity, between two extreme values around an average value. Also recall (from Electromagnetic Induction on electromagnetic induction) that we need a changing magnetic field, brought about by a changing current, to induce a voltage in another coil. The oscillator system does this many times as the battery voltage is boosted to over 1000 volts. (You may hear the high-pitched whine from the

transformer as the capacitor is being charged.) A capacitor stores the high voltage for later use in powering the flash.

Example:

Self-Inductance of a Coil

An induced emf of 2.0 V is measured across a coil of 50 closely wound turns while the current through it increases uniformly from 0.0 to 5.0 A in 0.10 s. (a) What is the self-inductance of the coil? (b) With the current at 5.0 A, what is the flux through each turn of the coil?

Strategy

Both parts of this problem give all the information needed to solve for the self-inductance in part (a) or the flux through each turn of the coil in part (b). The equations needed are [link] for part (a) and [link] for part (b).

Solution

a. Ignoring the negative sign and using magnitudes, we have, from [link],
   Equation:

$$L = \frac{\varepsilon}{dI/dt} = \frac{2.0\ \text{V}}{5.0\ \text{A}/0.10\ \text{s}} = 4.0 \times 10^{-2}\ \text{H}.$$

b. From [link], the flux is given in terms of the current by $\Phi_\text{m} = LI/N$, so
   Equation:

$$\Phi_\text{m} = \frac{(4.0 \times 10^{-2}\ \text{H})(5.0\ \text{A})}{50\ \text{turns}} = 4.0 \times 10^{-3}\ \text{Wb}.$$

Significance

The self-inductance and flux calculated in parts (a) and (b) are typical values for coils found in contemporary devices. If the current is not changing over time, the flux is not changing in time, so no emf is induced.

Note:
Exercise:

Problem:

Check Your Understanding Current flows through the inductor in [link] from *B* to *A* instead of from *A* to *B* as shown. Is the current increasing or decreasing in order to produce the emf given in diagram (a)? In diagram (b)?

Solution:

a. decreasing; b. increasing; Since the current flows in the opposite direction of the diagram, in order to get a positive emf on the left-hand side of diagram (a), we need to decrease the current to the left, which creates a reinforced emf where the positive end is on the left-hand side. To get a positive emf on the right-hand side of diagram (b), we need to increase the current to the left, which creates a reinforced emf where the positive end is on the right-hand side.

Note:
Exercise:

Problem:

Check Your Understanding A changing current induces an emf of 10 V across a 0.25-H inductor. What is the rate at which the current is changing?

Solution:

40 A/s

A good approach for calculating the self-inductance of an inductor consists of the following steps:

Note:
Problem-Solving Strategy: Self-Inductance

1. Assume a current I is flowing through the inductor.
2. Determine the magnetic field $\vec{\mathbf{B}}$ produced by the current. If there is appropriate symmetry, you may be able to do this with Ampère’s law.
3. Obtain the magnetic flux, Φ_{m}.
4. With the flux known, the self-inductance can be found from [link], $L = N\Phi_{\text{m}}/I$.

To demonstrate this procedure, we now calculate the self-inductances of two inductors.

Cylindrical Solenoid

Consider a long, cylindrical solenoid with length l, cross-sectional area A, and N turns of wire. We assume that the length of the solenoid is so much larger than its diameter that we can take the magnetic field to be $B = \mu_0 nI$ throughout the interior of the solenoid, that is, we ignore end effects in the solenoid. With a current I flowing through the coils, the magnetic field produced within the solenoid is
Equation:

$$B = \mu_0 \left(\frac{N}{l}\right) I,$$

so the magnetic flux through one turn is
Equation:

$$\Phi_{\mathrm{m}} = BA = \frac{\mu_0 NA}{l} I.$$

Using [link], we find for the self-inductance of the solenoid,

Note:
Equation:

$$L_{\text{solenoid}} = \frac{N\Phi_{\mathrm{m}}}{I} = \frac{\mu_0 N^2 A}{l}.$$

If $n = N/l$ is the number of turns per unit length of the solenoid, we may write [link] as

Equation:

$$L = \mu_0 \left(\frac{N}{l}\right)^2 Al = \mu_0 n^2 Al = \mu_0 n^2 (V),$$

where $V = Al$ is the volume of the solenoid. Notice that *the self-inductance of a long solenoid depends only on its physical properties* (such as the number of turns of wire per unit length and the volume), and not on the magnetic field or the current. This is true for inductors in general.

Rectangular Toroid

A toroid with a rectangular cross-section is shown in [link]. The inner and outer radii of the toroid are R_1 and $R_2,$ and h is the height of the toroid. Applying Ampère's law in the same manner as we did in [link] for a toroid with a circular cross-section, we find the magnetic field inside a rectangular toroid is also given by

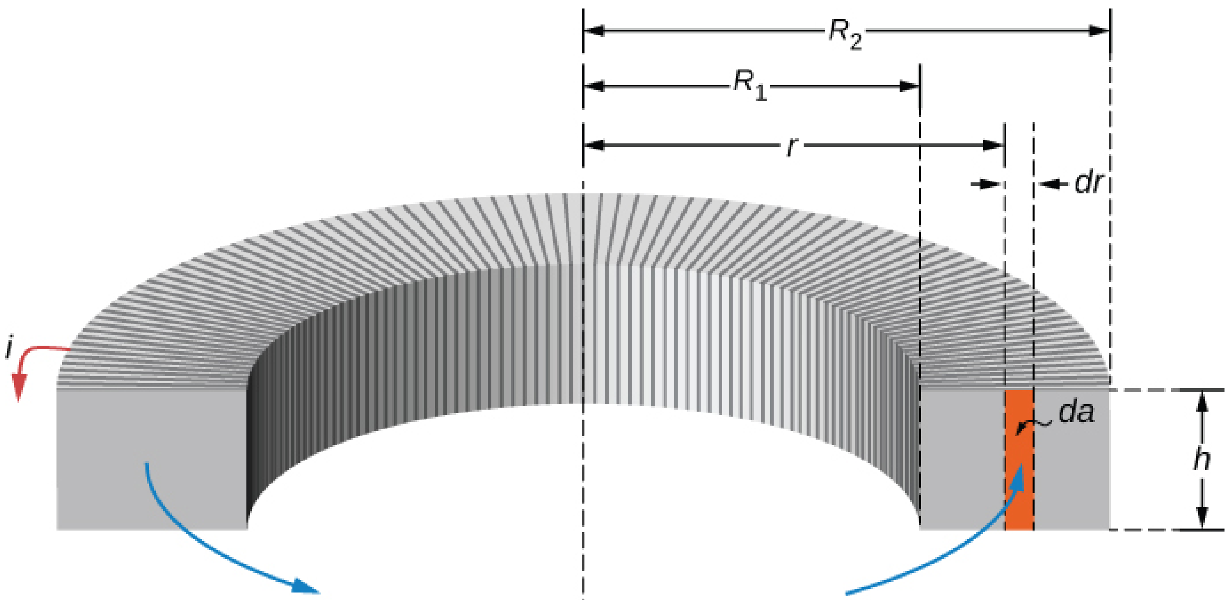
Equation:

$$B = \frac{\mu_0 NI}{2\pi r},$$

where r is the distance from the central axis of the toroid. Because the field changes within the toroid, we must calculate the flux by integrating over the toroid's cross-section. Using the infinitesimal cross-sectional area element $da = h\,dr$ shown in [link], we obtain

Equation:

$$\Phi_{\mathrm{m}} = \int B\,da = \int_{R_1}^{R_2} \left(\frac{\mu_0 NI}{2\pi r}\right)(hdr) = \frac{\mu_0 NhI}{2\pi} \ln \frac{R_2}{R_1}.$$



Calculating the self-inductance of a rectangular toroid.

Now from [link], we obtain for the self-inductance of a rectangular toroid

Note:
Equation:

$$L = \frac{N\Phi_{\mathrm{m}}}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}.$$

As expected, the self-inductance is a constant determined by only the physical properties of the toroid.

Note:
Exercise:

Problem:

Check Your Understanding (a) Calculate the self-inductance of a solenoid that is tightly wound with wire of diameter 0.10 cm, has a cross-sectional area of 0.90 cm^2, and is 40 cm long. (b) If the current through the solenoid decreases uniformly from 10 to 0 A in 0.10 s, what is the emf induced between the ends of the solenoid?

Solution:

a. 4.5×10^{-5} H; b. 4.5×10^{-3} V

Note:
Exercise:

Problem:

Check Your Understanding (a) What is the magnetic flux through one turn of a solenoid of self-inductance 8.0×10^{-5} H when a current of 3.0 A flows through it? Assume that the solenoid has 1000 turns and is wound from wire of diameter 1.0 mm. (b) What is the cross-sectional area of the solenoid?

Solution:

a. 2.4×10^{-7} Wb; b. $6.4 \times 10^{-5}\ \text{m}^2$

Summary

- Current changes in a device induce an emf in the device itself, called self-inductance,
  Equation:
  $$\varepsilon = -L\frac{dI}{dt},$$
  where L is the self-inductance of the inductor and dI/dt is the rate of change of current through it. The minus sign indicates that emf opposes the change in current, as required by Lenz's law. The unit of self-inductance and inductance is the henry (H), where $1\ \text{H} = 1\ \Omega \cdot \text{s}$.
- The self-inductance of a solenoid is
  Equation:
  $$L = \frac{\mu_0 N^2 A}{l},$$
  where N is its number of turns in the solenoid, A is its cross-sectional

area, l is its length, and $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ is the permeability of free space.
- The self-inductance of a toroid is
  Equation:

  $$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1},$$

  where N is its number of turns in the toroid, R_1 and R_2 are the inner and outer radii of the toroid, h is the height of the toroid, and $\mu_0 = 4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$ is the permeability of free space.

Conceptual Questions

Exercise:

Problem:

Does self-inductance depend on the value of the magnetic flux? Does it depend on the current through the wire? Correlate your answers with the equation $N\Phi_{\text{m}} = LI$.

Solution:

Self-inductance is proportional to the magnetic flux and inversely proportional to the current. However, since the magnetic flux depends on the current I, these effects cancel out. This means that the self-inductance does not depend on the current. If the emf is induced across an element, it does depend on how the current changes with time.

Exercise:

Problem:

Would the self-inductance of a 1.0 m long, tightly wound solenoid differ from the self-inductance per meter of an infinite, but otherwise identical, solenoid?

Exercise:

Problem:

Discuss how you might determine the self-inductance per unit length of a long, straight wire.

Solution:

Consider the ends of a wire a part of an *RL* circuit and determine the self-inductance from this circuit.

Exercise:

Problem:

The self-inductance of a coil is zero if there is no current passing through the windings. True or false?

Exercise:

Problem:

How does the self-inductance per unit length near the center of a solenoid (away from the ends) compare with its value near the end of the solenoid?

Solution:

The magnetic field will flare out at the end of the solenoid so there is less flux through the last turn than through the middle of the solenoid.

Problems

Exercise:

Problem:

An emf of 0.40 V is induced across a coil when the current through it changes uniformly from 0.10 to 0.60 A in 0.30 s. What is the self-inductance of the coil?

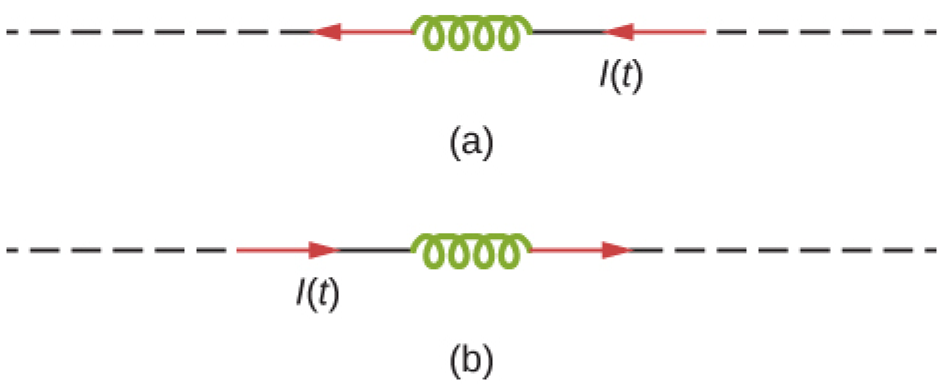
Solution:

0.24 H

Exercise:

Problem:

The current shown in part (a) below is increasing, whereas that shown in part (b) is decreasing. In each case, determine which end of the inductor is at the higher potential.



Exercise:

Problem:

What is the rate at which the current though a 0.30-H coil is changing if an emf of 0.12 V is induced across the coil?

Solution:

0.4 A/s

Exercise:

Problem:

When a camera uses a flash, a fully charged capacitor discharges through an inductor. In what time must the 0.100-A current through a 2.00-mH inductor be switched on or off to induce a 500-V emf?

Exercise:

Problem:

A coil with a self-inductance of 2.0 H carries a current that varies with time according to $I(t) = (2.0\text{ A})\sin 120\pi t$. Find an expression for the emf induced in the coil.

Solution:

$\varepsilon = 480\pi \sin(120\pi t - \pi/2)\text{ V}$

Exercise:

Problem:

A solenoid 50 cm long is wound with 500 turns of wire. The cross-sectional area of the coil is 2.0 cm^2 What is the self-inductance of the solenoid?

Exercise:

Problem:

A coil with a self-inductance of 3.0 H carries a current that decreases at a uniform rate $dI/dt = -0.050\text{ A/s}$. What is the emf induced in the coil? Describe the polarity of the induced emf.

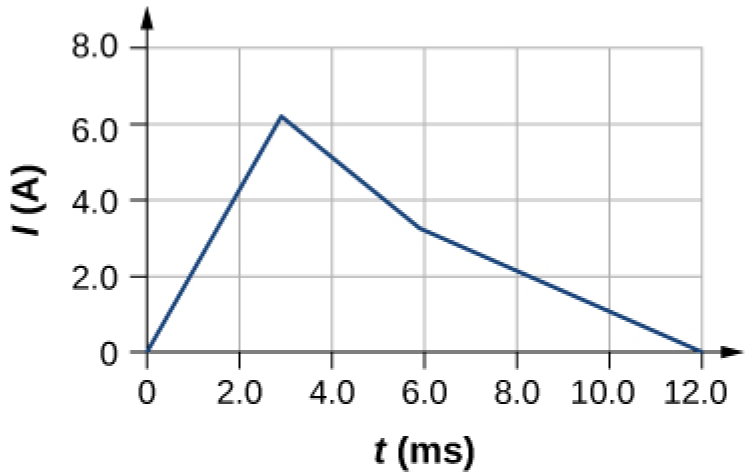
Solution:

0.15 V. This is the same polarity as the emf driving the current.

Exercise:

Problem:

The current *I(t)* through a 5.0-mH inductor varies with time, as shown below. The resistance of the inductor is $5.0\ \Omega$. Calculate the voltage across the inductor at $t = 2.0\ \text{ms}, t = 4.0\ \text{ms},$ and $t = 8.0\ \text{ms}.$



Exercise:

Problem:

A long, cylindrical solenoid with 100 turns per centimeter has a radius of 1.5 cm. (a) Neglecting end effects, what is the self-inductance per unit length of the solenoid? (b) If the current through the solenoid changes at the rate 5.0 A/s, what is the emf induced per unit length?

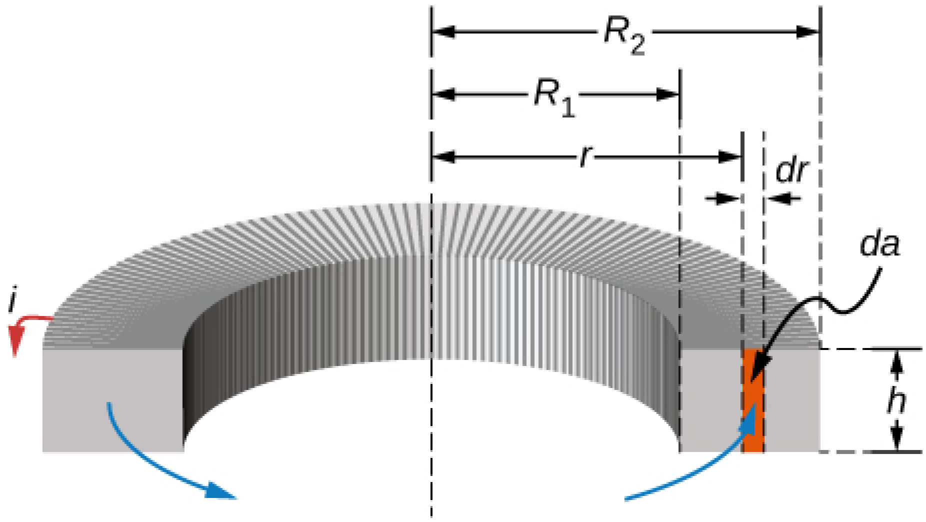
Solution:

a. 0.089 H/m; b. 0.44 V/m

Exercise:

Problem:

Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If $h = 0.10\ \text{m}$, what is the ratio of its outer radius to its inner radius?



Exercise:

Problem:

What is the self-inductance per meter of a coaxial cable whose inner radius is 0.50 mm and whose outer radius is 4.00 mm?

Solution:

$\frac{L}{l} = 4.16\ \times\ 10^{-7}\,\text{H/m}$

Glossary

inductor
: part of an electrical circuit to provide self-inductance, which is symbolized by a coil of wire

self-inductance
: effect of the device inducing emf in itself

Energy in a Magnetic Field
By the end of this section, you will be able to:

- Explain how energy can be stored in a magnetic field
- Derive the equation for energy stored in a coaxial cable given the magnetic energy density

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability to store energy, but in its magnetic field. This energy can be found by integrating the **magnetic energy density**,
Equation:

$$u_{\text{m}}=\frac{B^2}{2\mu_0}$$

over the appropriate volume. To understand where this formula comes from, let's consider the long, cylindrical solenoid of the previous section. Again using the infinite solenoid approximation, we can assume that the magnetic field is essentially constant and given by $B=\mu_0 nI$ everywhere inside the solenoid. Thus, the energy stored in a solenoid or the magnetic energy density times volume is equivalent to
Equation:

$$U=u_{\text{m}}(V)=\frac{(\mu_0 nI)^2}{2\mu_0}(Al)=\frac{1}{2}(\mu_0 n^2 Al)I^2.$$

With the substitution of [link], this becomes

Note:
Equation:

$$U=\frac{1}{2}LI^2.$$

Although derived for a special case, this equation gives the energy stored in the magnetic field of *any* inductor. We can see this by considering an arbitrary inductor through which a changing current is passing. At any instant, the magnitude of the induced emf is $\varepsilon = Ldi/dt$, where i is the induced current at that instance. Therefore, the power absorbed by the inductor is

Equation:

$$P = \varepsilon i = L\frac{di}{dt}i.$$

The total energy stored in the magnetic field when the current increases from 0 to *I* in a time interval from 0 to *t* can be determined by integrating this expression:

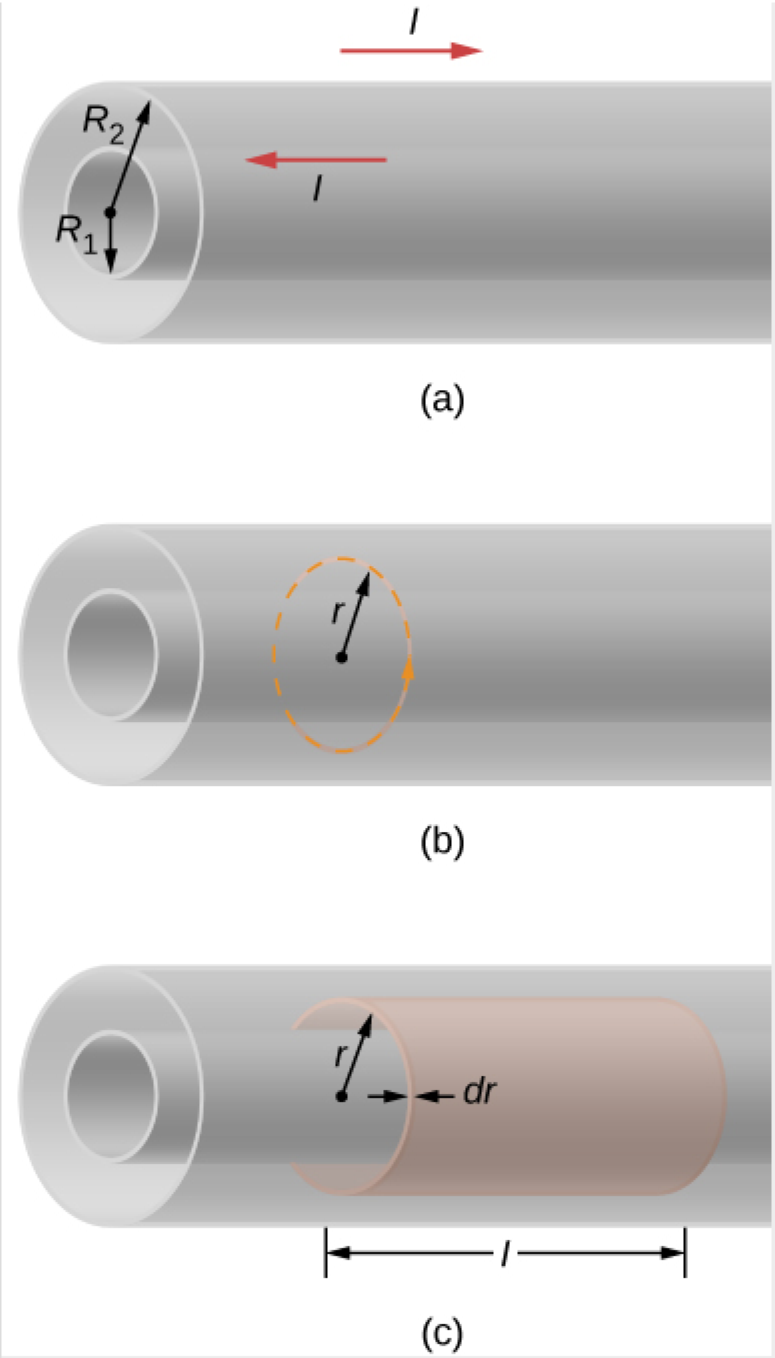
Equation:

$$U = \int_0^t Pdt' = \int_0^t L\frac{di}{dt'}idt' = L\int_0^I idi = \frac{1}{2}LI^2.$$

Example:

Self-Inductance of a Coaxial Cable

[link] shows two long, concentric cylindrical shells of radii R_1 and R_2. As discussed in Capacitance on capacitance, this configuration is a simplified representation of a coaxial cable. The capacitance per unit length of the cable has already been calculated. Now (a) determine the magnetic energy stored per unit length of the coaxial cable and (b) use this result to find the self-inductance per unit length of the cable.



(a) A coaxial cable is represented here by two hollow, concentric cylindrical conductors along which electric current flows in opposite directions. (b) The magnetic field between the conductors can be found by applying Ampère’s law to the dashed path. (c) The cylindrical shell is

used to find the magnetic energy stored in a length *l* of the cable.

Strategy

The magnetic field both inside and outside the coaxial cable is determined by Ampère's law. Based on this magnetic field, we can use [link] to calculate the energy density of the magnetic field. The magnetic energy is calculated by an integral of the magnetic energy density times the differential volume over the cylindrical shell. After the integration is carried out, we have a closed-form solution for part (a). The self-inductance per unit length is determined based on this result and [link].

Solution

a. We determine the magnetic field between the conductors by applying Ampère's law to the dashed circular path shown in [link](b). Because of the cylindrical symmetry, $\vec{\mathbf{B}}$ is constant along the path, and

Equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r) = \mu_0 I.$$

This gives us

Equation:

$$B = \frac{\mu_0 I}{2\pi r}.$$

In the region outside the cable, a similar application of Ampère's law shows that $B = 0$, since no net current crosses the area bounded by a circular path where $r > R_2$. This argument also holds when $r < R_1$; that is, in the region within the inner cylinder. All the magnetic energy of the cable is therefore stored between the two conductors. Since the energy density of the magnetic field is

Equation:

$$u_{\mathrm{m}} = \frac{B^2}{2\mu_0}$$

the energy stored in a cylindrical shell of inner radius *r*, outer radius $r + dr$, and length *l* (see part (c) of the figure) is

Equation:

$$u_{\mathrm{m}} = \frac{\mu_0 I^2}{8\pi^2 r^2}.$$

Thus, the total energy of the magnetic field in a length *l* of the cable is

Equation:

$$U = \int_{R_1}^{R_2} dU = \int_{R_1}^{R_2} \frac{\mu_0 I^2}{8\pi^2 r^2}(2\pi r l)dr = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{R_2}{R_1},$$

and the energy per unit length is $(\mu_0 I^2/4\pi)\ln(R_2/R_1)$.

b. From [link],

Equation:

$$U = \frac{1}{2}LI^2,$$

where *L* is the self-inductance of a length *l* of the coaxial cable. Equating the previous two equations, we find that the self-inductance per unit length of the cable is

Equation:

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}.$$

Significance

The inductance per unit length depends only on the inner and outer radii as seen in the result. To increase the inductance, we could either increase the outer radius (R_2) or decrease the inner radius (R_1). In the limit as the two radii become equal, the inductance goes to zero. In this limit, there is no coaxial cable. Also, the magnetic energy per unit length from part (a) is proportional to the square of the current.

Note:
Exercise:

Problem:

Check Your Understanding How much energy is stored in the inductor of [link] after the current reaches its maximum value?

Solution:

0.50 J

Summary

- The energy stored in an inductor U is
  Equation:

$$U = \frac{1}{2} L I^2.$$

- The self-inductance per unit length of coaxial cable is
  Equation:

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}.$$

Conceptual Questions

Exercise:

Problem: Show that $LI^2/2$ has units of energy.

Problems

Exercise:

Problem:

At the instant a current of 0.20 A is flowing through a coil of wire, the energy stored in its magnetic field is $6.0 \times 10^{-3}\,\mathrm{J}$. What is the self-inductance of the coil?

Exercise:

Problem:

Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If $h = 0.10\,\mathrm{m}$, what is the current flowing through a rectangular toroid when the energy in its magnetic field is 2.0×10^{-6} J?

Solution:

0.01 A

Exercise:

Problem:

Solenoid *A* is tightly wound while solenoid *B* has windings that are evenly spaced with a gap equal to the diameter of the wire. The solenoids are otherwise identical. Determine the ratio of the energies stored per unit length of these solenoids when the same current flows through each.

Exercise:

Problem:

A 10-H inductor carries a current of 20 A. How much ice at $0^\circ\ \mathrm{C}$ could be melted by the energy stored in the magnetic field of the inductor? (*Hint*: Use the value $L_\mathrm{f} = 334\ \mathrm{J/g}$ for ice.)

Solution:

6.0 g

Exercise:

Problem:

A coil with a self-inductance of 3.0 H and a resistance of $100\ \Omega$ carries a steady current of 2.0 A. (a) What is the energy stored in the magnetic field of the coil? (b) What is the energy per second dissipated in the resistance of the coil?

Exercise:

Problem:

A current of 1.2 A is flowing in a coaxial cable whose outer radius is five times its inner radius. What is the magnetic field energy stored in a 3.0-m length of the cable?

Solution:

$U_\mathrm{m} = 7.0\ \times\ 10^{-7}\ \mathrm{J}$

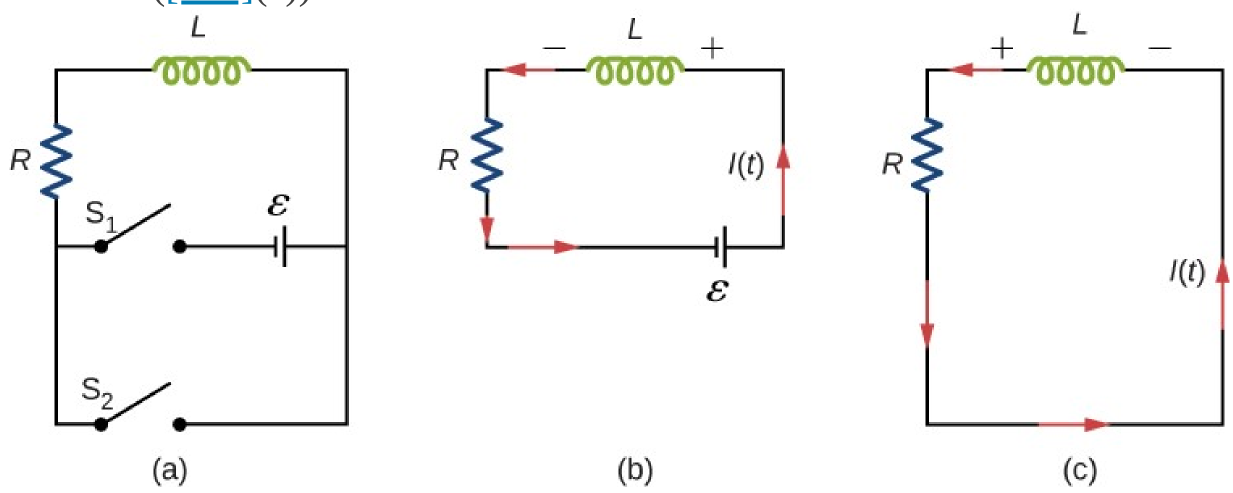
Glossary

magnetic energy density
: energy stored per volume in a magnetic field

RL Circuits
By the end of this section, you will be able to:

- Analyze circuits that have an inductor and resistor in series
- Describe how current and voltage exponentially grow or decay based on the initial conditions

A circuit with resistance and self-inductance is known as an *RL* circuit. [link](a) shows an *RL* circuit consisting of a resistor, an inductor, a constant source of emf, and switches S_1 and S_2. When S_1 is closed, the circuit is equivalent to a single-loop circuit consisting of a resistor and an inductor connected across a source of emf ([link](b)). When S_1 is opened and S_2 is closed, the circuit becomes a single-loop circuit with only a resistor and an inductor ([link](c)).



(a) An *RL* circuit with switches S_1 and S_2. (b) The equivalent circuit with S_1 closed and S_2 open. (c) The equivalent circuit after S_1 is opened and S_2 is closed.

We first consider the *RL* circuit of [link](b). Once S_1 is closed and S_2 is open, the source of emf produces a current in the circuit. If there were no self-inductance in the circuit, the current would rise immediately to a steady value of ε/R. However, from Faraday's law, the increasing current produces an emf $V_L = -L(dI/dt)$ across the inductor. In accordance with Lenz's law, the induced emf counteracts the increase in the current and is

directed as shown in the figure. As a result, *I(t)* starts at zero and increases asymptotically to its final value.

Applying Kirchhoff’s loop rule to this circuit, we obtain

Equation:

$$\varepsilon - L\frac{dI}{dt} - IR = 0,$$

which is a first-order differential equation for *I(t)*. Notice its similarity to the equation for a capacitor and resistor in series (See *RC Circuits*). Similarly, the solution to [link] can be found by making substitutions in the equations relating the capacitor to the inductor. This gives

Note:

Equation:

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau_L}\right),$$

where

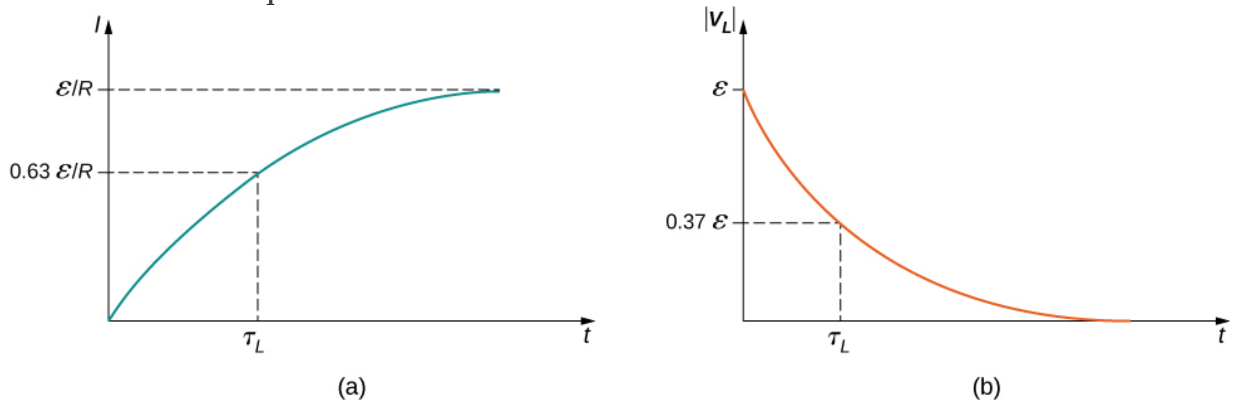
Note:

Equation:

$$\tau_L = L/R$$

is the **inductive time constant** of the circuit.

The current *I(t)* is plotted in [link](a). It starts at zero, and as $t \to \infty$, *I(t)* approaches ε/R asymptotically. The induced emf $V_L(t)$ is directly proportional to *dI*/*dt*, or the slope of the curve. Hence, while at its greatest immediately after the switches are thrown, the induced emf decreases to zero with time as the current approaches its final value of ε/R. The circuit then becomes equivalent to a resistor connected across a source of emf.



Time variation of (a) the electric current and (b) the magnitude of the induced voltage across the coil in the circuit of [link](b).

The energy stored in the magnetic field of an inductor is

Equation:

$$U_L = \frac{1}{2}LI^2.$$

Thus, as the current approaches the maximum current ε/R, the stored energy in the inductor increases from zero and asymptotically approaches a maximum of $L(\varepsilon/R)^2/2$.

The time constant τ_L tells us how rapidly the current increases to its final value. At $t = \tau_L$, the current in the circuit is, from [link],

Equation:

$$I(\tau_L) = \frac{\varepsilon}{R}(1 - e^{-1}) = 0.63\frac{\varepsilon}{R},$$

which is 63% of the final value ε/R. The smaller the inductive time constant $\tau_L = L/R$, the more rapidly the current approaches ε/R.

We can find the time dependence of the induced voltage across the inductor in this circuit by using $V_L(t) = -L(dI/dt)$ and [link]:
Equation:

$$V_L(t) = -L\frac{dI}{dt} = -\varepsilon e^{-t/\tau_L}.$$

The magnitude of this function is plotted in [link](b). The greatest value of $L\,(dI/dt)$ is ε; it occurs when *dI/dt* is greatest, which is immediately after S_1 is closed and S_2 is opened. In the approach to steady state, *dI/dt* decreases to zero. As a result, the voltage across the inductor also vanishes as $t \to \infty$.

The time constant τ_L also tells us how quickly the induced voltage decays. At $t = \tau_L$, the magnitude of the induced voltage is
Equation:

$$|V_L(\tau_L)| = \varepsilon e^{-1} = 0.37\varepsilon = 0.37V(0).$$

The voltage across the inductor therefore drops to about 37% of its initial value after one time constant. The shorter the time constant τ_L, the more rapidly the voltage decreases.

After enough time has elapsed so that the current has essentially reached its final value, the positions of the switches in [link](a) are reversed, giving us the circuit in part (c). At $t = 0$, the current in the circuit is $I(0) = \varepsilon/R$. With Kirchhoff’s loop rule, we obtain
Equation:

$$IR + L\frac{dI}{dt} = 0.$$

The solution to this equation is similar to the solution of the equation for a discharging capacitor, with similar substitutions. The current at time *t* is then

Equation:

$$I(t) = \frac{\varepsilon}{R} e^{-t/\tau_L}.$$

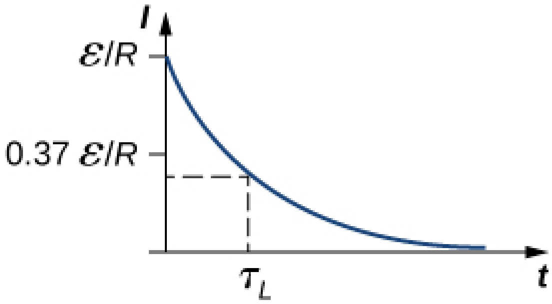
The current starts at $I(0) = \varepsilon/R$ and decreases with time as the energy stored in the inductor is depleted ([link]).

The time dependence of the voltage across the inductor can be determined from $V_L = -L(dI/dt)$:

Equation:

$$V_L(t) = \varepsilon e^{-t/\tau L}.$$

This voltage is initially $V_L(0) = \varepsilon$, and it decays to zero like the current. The energy stored in the magnetic field of the inductor, $LI^2/2$, also decreases exponentially with time, as it is dissipated by Joule heating in the resistance of the circuit.



Time variation of electric current in the *RL* circuit of [link](c). The induced voltage across the coil also decays exponentially.

Example:

An *RL* Circuit with a Source of emf

In the circuit of [link](a), let $\varepsilon = 2.0 V$, $R = 4.0\,\Omega,$ and $L = 4.0\,\text{H}.$ With S_1 closed and S_2 open ([link](b)), (a) what is the time constant of the circuit? (b) What are the current in the circuit and the magnitude of the induced emf across the inductor at $t = 0,$ at $t = 2.0\tau_L$, and as $t \to \infty$?

Strategy

The time constant for an inductor and resistor in a series circuit is calculated using [link]. The current through and voltage across the inductor are calculated by the scenarios detailed from [link] and [link].

Solution

a. The inductive time constant is
   Equation:

$$\tau_L = \frac{L}{R} = \frac{4.0\,\text{H}}{4.0\,\Omega} = 1.0\,\text{s}.$$

b. The current in the circuit of [link](b) increases according to [link]:
   Equation:

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau_L}).$$

At $t = 0,$
Equation:

$$(1 - e^{-t/\tau_L}) = (1 - 1) = 0;\ \text{so}\ I(0) = 0.$$

At $t = 2.0\tau_L$ and $t \to \infty$, we have, respectively,
Equation:

$$I(2.0\tau_L) = \frac{\varepsilon}{R}(1 - e^{-2.0}) = (0.50\,\text{A})(0.86) = 0.43\,\text{A},$$

and
Equation:

$$I(\infty) = \frac{\varepsilon}{R} = 0.50\ \text{A}.$$

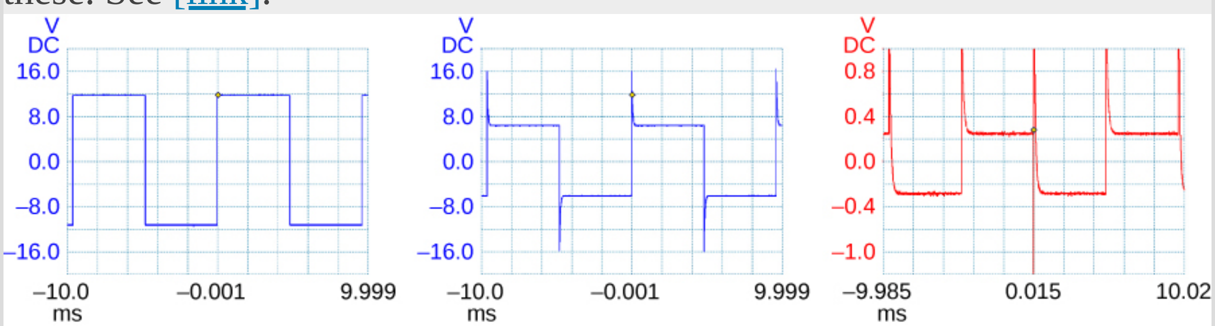
From [link], the magnitude of the induced emf decays as
Equation:

$$|V_L(t)| = \varepsilon e^{-t/\tau_L}.$$

At $t = 0, t = 2.0\tau_L,$ and as $t \to \infty$, we obtain
Equation:

$$\begin{aligned} |V_L(0)| &= \varepsilon = 2.0\ \text{V}, \\ |V_L(2.0\tau_L)| &= (2.0\ \text{V})\, e^{-2.0} = 0.27\ \text{V} \\ &\text{and} \\ |V_L(\infty)| &= 0. \end{aligned}$$

Significance
If the time of the measurement were much larger than the time constant, we would not see the decay or growth of the voltage across the inductor or resistor. The circuit would quickly reach the asymptotic values for both of these. See [link].



(a) Voltage across the source (b) Voltage across the inductor (c) Voltage across the resistor

A generator in an *RL* circuit produces a square-pulse output in which the voltage oscillates between zero and some set value. These oscilloscope traces show (a) the voltage across the source; (b) the voltage across the inductor; (c) the voltage across the resistor.

Example:

An *RL* Circuit without a Source of emf

After the current in the *RL* circuit of [link] has reached its final value, the positions of the switches are reversed so that the circuit becomes the one shown in [link](c). (a) How long does it take the current to drop to half its initial value? (b) How long does it take before the energy stored in the inductor is reduced to 1.0% of its maximum value?

Strategy

The current in the inductor will now decrease as the resistor dissipates this energy. Therefore, the current falls as an exponential decay. We can also use that same relationship as a substitution for the energy in an inductor formula to find how the energy decreases at different time intervals.

Solution

a. With the switches reversed, the current decreases according to

   Equation:

$$I(t) = \frac{\varepsilon}{R} e^{-t/\tau_L} = I(0)e^{-t/\tau_L}.$$

   At a time *t* when the current is one-half its initial value, we have

   Equation:

$$I(t) = 0.50I(0) \text{ so } e^{-t/\tau_L} = 0.50,$$

   and

   Equation:

$$t = -[\ln(0.50)]\tau_L = 0.69(1.0\,\text{s}) = 0.69\,\text{s},$$

where we have used the inductive time constant found in [link].

b. The energy stored in the inductor is given by

Equation:

$$U_L(t) = \frac{1}{2}L[I\,(t)]^2 = \frac{1}{2}L\left(\frac{\varepsilon}{R}e^{-t/\tau_L}\right)^2 = \frac{L\varepsilon^2}{2R^2}e^{-2t/\tau_L}.$$

If the energy drops to 1.0% of its initial value at a time *t*, we have

Equation:

$$U_L(t) = (0.010)U_L(0) \text{ or } \frac{L\varepsilon^2}{2R^2}e^{-2t/\tau_L} = (0.010)\frac{L\varepsilon^2}{2R^2}.$$

Upon canceling terms and taking the natural logarithm of both sides, we obtain

Equation:

$$-\frac{2t}{\tau_L} = \ln(0.010),$$

so

Equation:

$$t = -\frac{1}{2}\tau_L \ln(0.010).$$

Since $\tau_L = 1.0\,\text{s}$, the time it takes for the energy stored in the inductor to decrease to 1.0% of its initial value is

Equation:

$$t = -\frac{1}{2}(1.0\,\text{s})\ln(0.010) = 2.3\,\text{s}.$$

Significance
This calculation only works if the circuit is at maximum current in situation (b) prior to this new situation. Otherwise, we start with a lower initial current, which will decay by the same relationship.

Note:
Exercise:

Problem:

Check Your Understanding Verify that *RC* and *L/R* have the dimensions of time.

Note:
Exercise:

Problem:

Check Your Understanding (a) If the current in the circuit of in [link](b) increases to 90% of its final value after 5.0 s, what is the inductive time constant? (b) If $R = 20\,\Omega$, what is the value of the self-inductance? (c) If the $20\text{-}\Omega$ resistor is replaced with a $100\text{-}\Omega$ resister, what is the time taken for the current to reach 90% of its final value?

Solution:

a. 2.2 s; b. 43 H; c. 1.0 s

Note:
Exercise:

Problem:

Check Your Understanding For the circuit of in [link](b), show that when steady state is reached, the difference in the total energies produced by the battery and dissipated in the resistor is equal to the energy stored in the magnetic field of the coil.

Summary

- When a series connection of a resistor and an inductor—an *RL* circuit—is connected to a voltage source, the time variation of the current is $I(t) = \frac{\varepsilon}{R}(1 - e^{-Rt/L}) = \frac{\varepsilon}{R}(1 - e^{-t/\tau_L})$ (turning on), where the initial current is $I_0 = \varepsilon/R$.
- The characteristic time constant τ is $\tau_L = L/R$, where *L* is the inductance and *R* is the resistance.
- In the first time constant τ, the current rises from zero to $0.632I_0$, and to 0.632 of the remainder in every subsequent time interval τ.
- When the inductor is shorted through a resistor, current decreases as $I(t) = \frac{\varepsilon}{R}e^{-t/\tau_L}$ (turning off). Current falls to $0.368I_0$ in the first time interval τ, and to 0.368 of the remainder toward zero in each subsequent time τ.

Conceptual Questions

Exercise:

Problem:

Use Lenz's law to explain why the initial current in the *RL* circuit of [link](b) is zero.

Solution:

As current flows through the inductor, there is a back current by Lenz's law that is created to keep the net current at zero amps, the initial current.

Exercise:

Problem:

When the current in the *RL* circuit of [link](b) reaches its final value ε/R, what is the voltage across the inductor? Across the resistor?

Exercise:

Problem:

Does the time required for the current in an *RL* circuit to reach any fraction of its steady-state value depend on the emf of the battery?

Solution:

no

Exercise:

Problem:

An inductor is connected across the terminals of a battery. Does the current that eventually flows through the inductor depend on the internal resistance of the battery? Does the time required for the current to reach its final value depend on this resistance?

Exercise:

Problem:

At what time is the voltage across the inductor of the *RL* circuit of [link](b) a maximum?

Solution:

At $t=0$, or when the switch is first thrown.

Exercise:

Problem:

In the simple *RL* circuit of [link](b), can the emf induced across the inductor ever be greater than the emf of the battery used to produce the current?

Exercise:

Problem:

If the emf of the battery of [link](b) is reduced by a factor of 2, by how much does the steady-state energy stored in the magnetic field of the inductor change?

Solution:

1/4

Exercise:

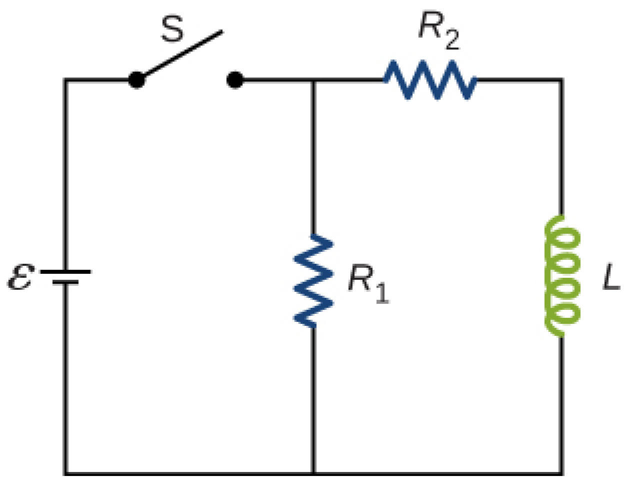
Problem:

A steady current flows through a circuit with a large inductive time constant. When a switch in the circuit is opened, a large spark occurs across the terminals of the switch. Explain.

Exercise:

Problem:

Describe how the currents through R_1 and R_2 shown below vary with time after switch S is closed.



Solution:

Initially, $I_{R1} = \frac{\varepsilon}{R_1}$ and $I_{R2} = 0$, and after a long time has passed, $I_{R1} = \frac{\varepsilon}{R_1}$ and $I_{R2} = \frac{\varepsilon}{R_2}$.

Exercise:

Problem: Discuss possible practical applications of *RL* circuits.

Problems

Exercise:

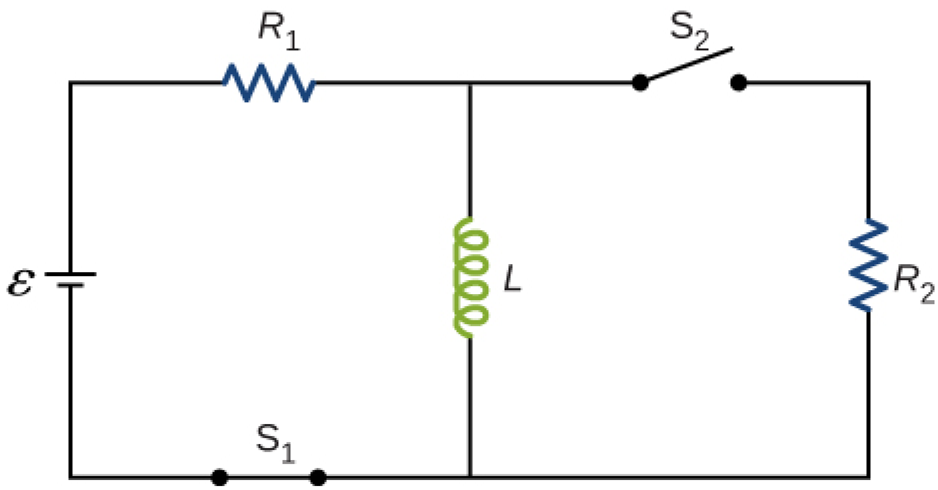
Problem:

In [link], $\varepsilon = 12\text{ V}$, $L = 20\text{ mH}$, and $R = 5.0\,\Omega$. Determine (a) the time constant of the circuit, (b) the initial current through the resistor, (c) the final current through the resistor, (d) the current through the resistor when $t = 2\tau_L$, and (e) the voltages across the inductor and the resistor when $t = 2\tau_L$.

Exercise:

Problem:

For the circuit shown below, $\varepsilon = 20\text{ V}$, $L = 4.0\text{ mH}$, and $R = 5.0\,\Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter $(\text{at } t = 0)$ S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4}$ s, and (c) the voltages across L and R_1 at $t = 4.0 \times 10^{-4}$ s. $R_1 = R_2 = R$.



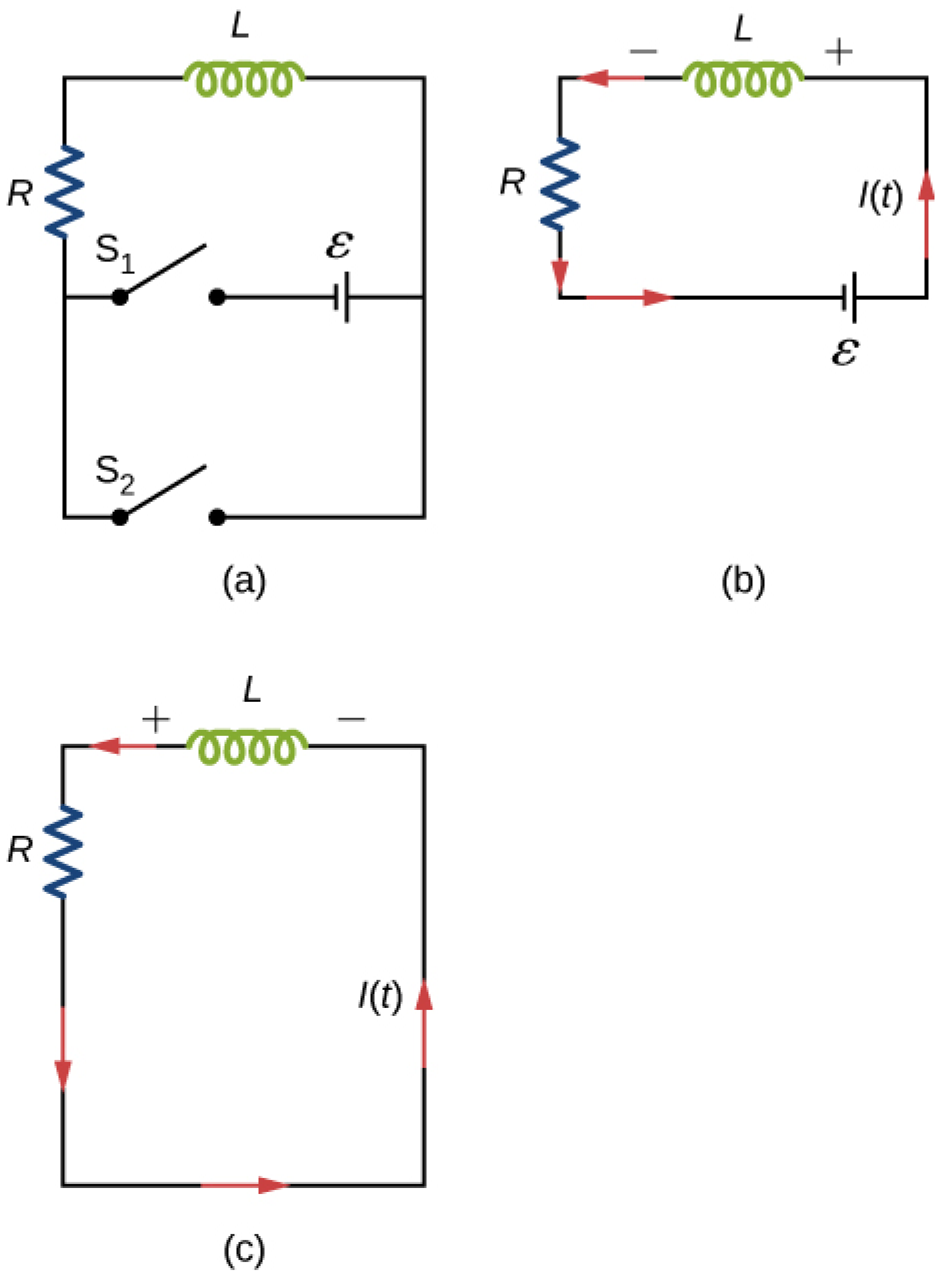
Solution:

a. 4.0 A; b. 2.4 A; c. on R: $V = 12\text{ V}$; on L: $V = 7.9\text{ V}$

Exercise:

Problem:

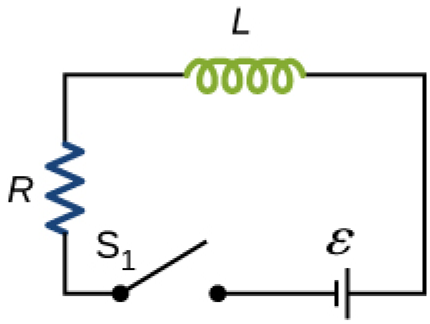
The current in the *RL* circuit shown here increases to 40% of its steady-state value in 2.0 s. What is the time constant of the circuit?



Exercise:

Problem:

How long after switch S_1 is thrown does it take the current in the circuit shown to reach half its maximum value? Express your answer in terms of the time constant of the circuit.



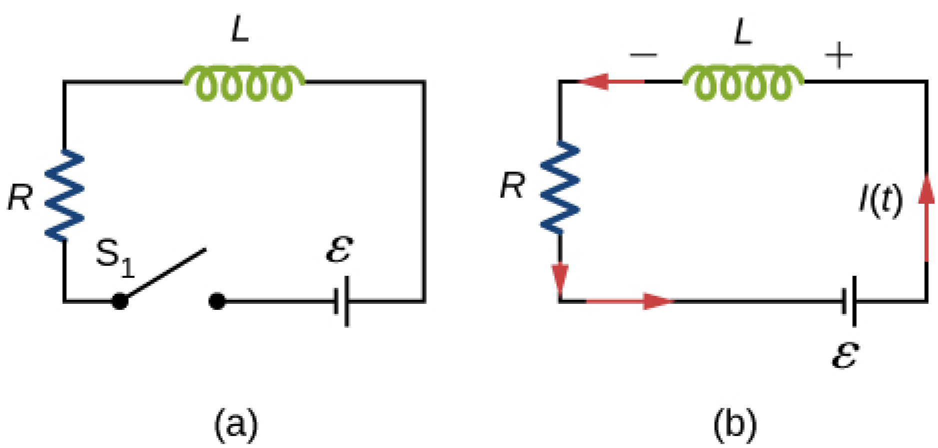
Solution:

0.69τ

Exercise:

Problem:

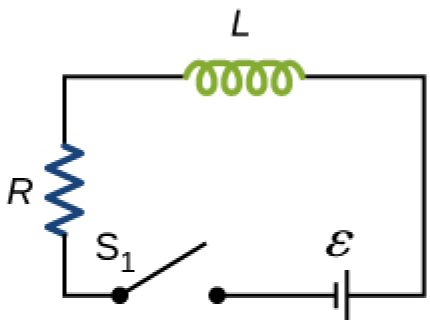
Examine the circuit shown below in part (a). Determine *dI/dt* at the instant after the switch is thrown in the circuit of (a), thereby producing the circuit of (b). Show that if *I* were to continue to increase at this initial rate, it would reach its maximum ε/R in one time constant.



Exercise:

Problem:

The current in the *RL* circuit shown below reaches half its maximum value in 1.75 ms after the switch S_1 is thrown. Determine (a) the time constant of the circuit and (b) the resistance of the circuit if $L = 250\,\mathrm{mH}$.



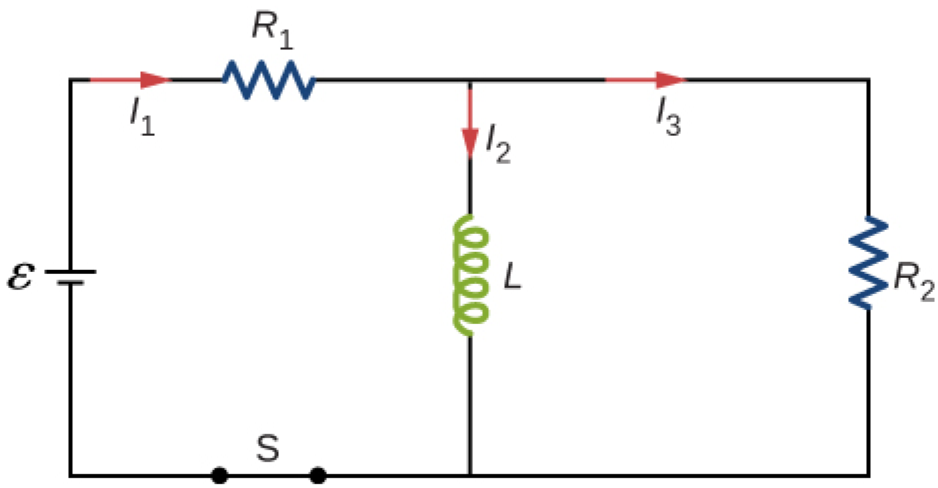
Solution:

a. 2.52 ms; b. 99.2 Ω

Exercise:

Problem:

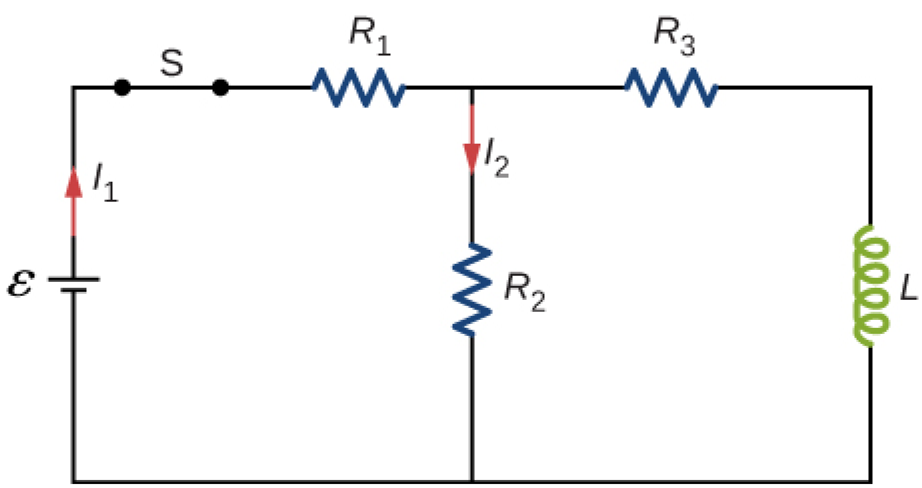
Consider the circuit shown below. Find $I_1, I_2,$ and I_3 when (a) the switch S is first closed, (b) after the currents have reached steady-state values, and (c) at the instant the switch is reopened (after being closed for a long time).



Exercise:

Problem:

For the circuit shown below, $\varepsilon = 50\ \text{V}$, $R_1 = 10\ \Omega$,,
$R_2 = R_3 =\ 19.4\ \Omega$,, and $L = 2.0\ \text{mH}$. Find the values of I_1 and I_2 (a) immediately after switch S is closed, (b) a long time after S is closed, (c) immediately after S is reopened, and (d) a long time after S is reopened.



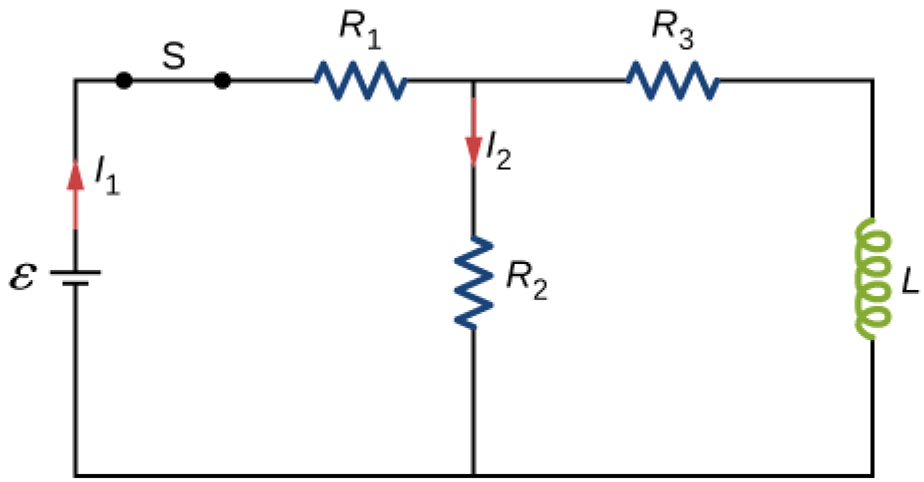
Solution:

a. $I_1 = I_2 = 1.7\ A$; b. $I_1 = 2.73\ A, I_2 = 1.36\ A$; c. $I_1 = 0, I_2 = 0.54\ A$; d. $I_1 = I_2 = 0$

Exercise:

Problem:

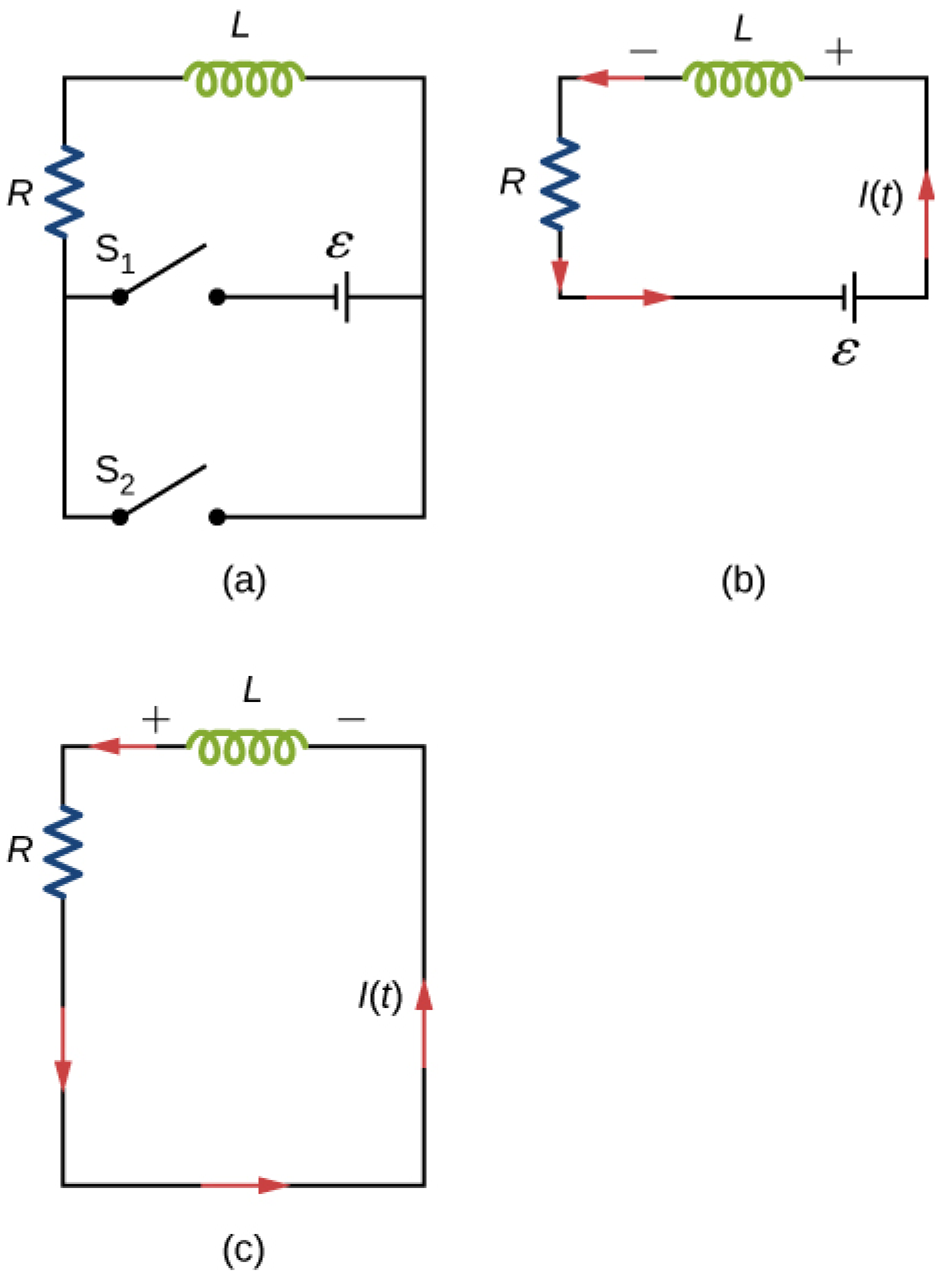
For the circuit shown below, find the current through the inductor 2.0×10^{-5} s after the switch is reopened.



Exercise:

Problem:

Show that for the circuit shown below, the initial energy stored in the inductor, $LI^2(0)/2$, is equal to the total energy eventually dissipated in the resistor, $\int_0^\infty I^2(t)Rdt$.



Solution:

proof

Glossary

inductive time constant

denoted by τ, the characteristic time given by quantity L/R of a particular series RL circuit

Oscillations in an LC Circuit

By the end of this section, you will be able to:

- Explain why charge or current oscillates between a capacitor and inductor, respectively, when wired in series
- Describe the relationship between the charge and current oscillating between a capacitor and inductor wired in series

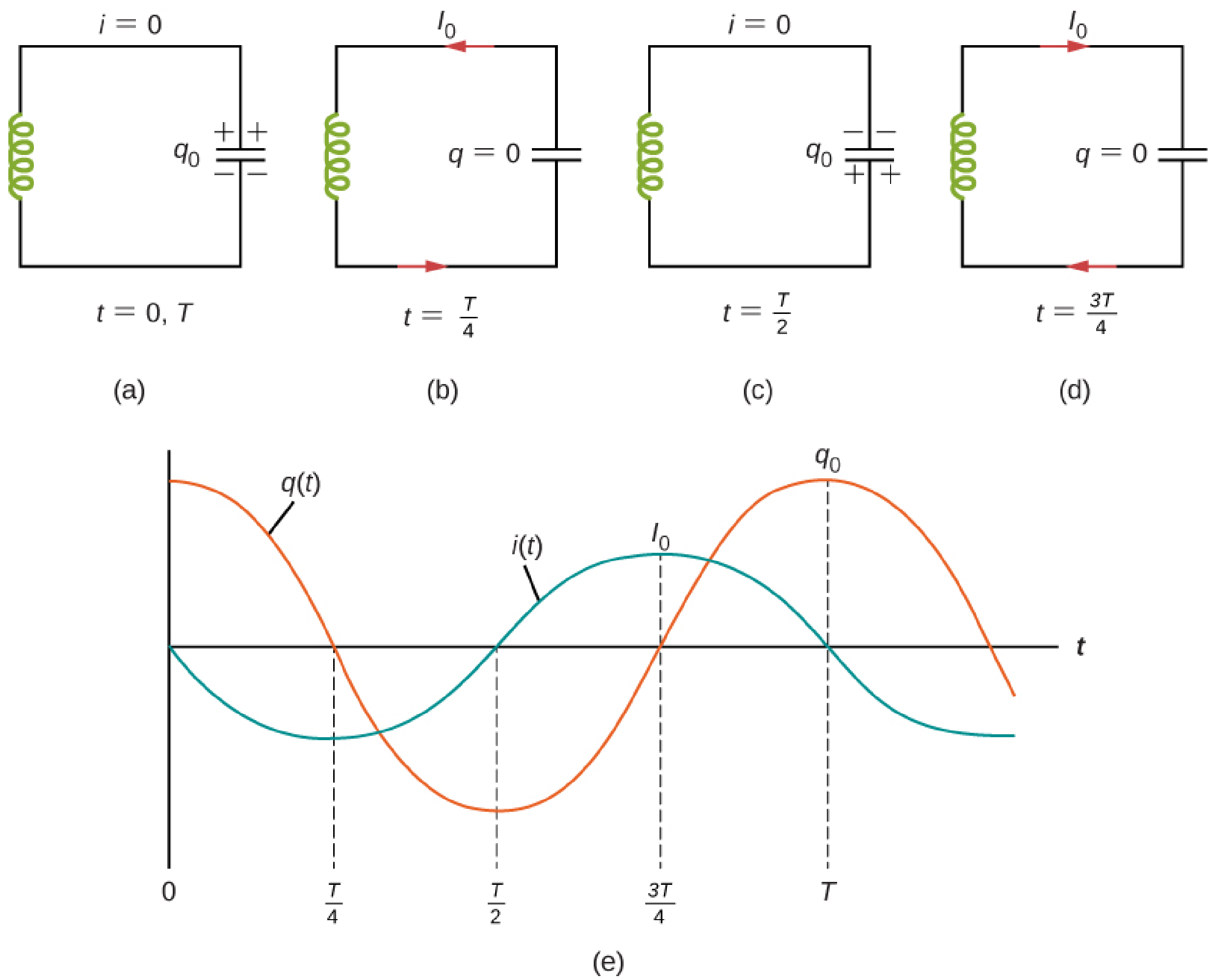
It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (L) and a capacitor (C) can oscillate without a source of emf by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an ***LC* circuit**.

An *LC* circuit is shown in [link]. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor ([link](a)). This energy is

Equation:

$$U_C = \frac{1}{2}\frac{q_0^2}{C}.$$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.



(a–d) The oscillation of charge storage with changing directions of current in an LC circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor.

In [link](b), the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

Equation:

$$U_L = \frac{1}{2} L I_0^2.$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:
Equation:

$$\frac{1}{2}\,\frac{q_0^2}{C} = \frac{1}{2}LI_0^2.$$

At an arbitrary time when the capacitor charge is *q(t)* and the current is *i(t)*, the total energy *U* in the circuit is given by
Equation:

$$\frac{q^2(t)}{2C} + \frac{Li^2(t)}{2}.$$

Because there is no energy dissipation,
Equation:

$$U = \frac{1}{2}\,\frac{q^2}{C} + \frac{1}{2}Li^2 = \frac{1}{2}\,\frac{q_0^2}{C} = \frac{1}{2}LI_0^2.$$

After reaching its maximum I_0, the current *i(t)* continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor *back* to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0. However, as [link](c) shows, the capacitor plates are charged *opposite* to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in [link](d). Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $mv^2/2$, and the spring, which has potential energy $kx^2/2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an *LC* circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance *LC* circuit is not practical, and any *LC* circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free *LC* circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = dq(t)/dt$, the total electromagnetic energy *U* is

Equation:

$$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}.$$

For the mass-spring system, $v(t) = dx(t)/dt$, the total mechanical energy *E* is

Equation:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace *m* by *L*, *v* by *i*, *k* by 1/*C*, and *x* by *q*. Now *x(t)* is given by

Equation:

$$x(t) = A\cos(\omega t + \phi)$$

where $\omega = \sqrt{k/m}$. Hence, the charge on the capacitor in an *LC* circuit is given by

Note:

Equation:

$$q\left(t\right)={q}_{0}\,\text{cos}(\omega t+\phi)$$

where the angular frequency of the oscillations in the circuit is

Note:
Equation:

$$\omega =\sqrt{\frac{1}{LC}}.$$

Finally, the current in the *LC* circuit is found by taking the time derivative of *q(t)*:

Note:
Equation:

$$i(t)=\frac{dq(t)}{dt}=-\omega {q}_{0}\,\text{sin}(\omega t+\phi).$$

The time variations of q and I are shown in [link](e) for $\phi =0$.

Example:
An *LC* Circuit

In an LC circuit, the self-inductance is 2.0×10^{-2} H and the capacitance is 8.0×10^{-6} F. At $t = 0$, all of the energy is stored in the capacitor, which has charge 1.2×10^{-5} C. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents *q(t)*.

Strategy

The angular frequency of the *LC* circuit is given by [link]. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find $q(t)$.

Solution

a. From [link], the angular frequency of the oscillations is

   Equation:

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.0 \times 10^{-2}\ \text{H})(8.0 \times 10^{-6}\ \text{F})}} = 2.5 \times 10^{3}\ \text{rad/s}.$$

b. The current is at its maximum I_0 when all the energy is stored in the inductor. From the law of energy conservation,

   Equation:

$$\frac{1}{2} L I_0^2 = \frac{1}{2} \frac{q_0^2}{C},$$

   so

   Equation:

$$I_0 = \sqrt{\frac{1}{LC}} q_0 = (2.5 \times 10^{3}\ \text{rad/s})(1.2 \times 10^{-5}\ \text{C}) = 3.0 \times 10^{-2}\ \text{A}.$$

   This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.

c. The capacitor becomes completely discharged in one-fourth of a cycle, or during a time *T*/4, where *T* is the period of the oscillations. Since
Equation:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5 \times 10^{3}\,\text{rad/s}} = 2.5 \times 10^{-3}\,\text{s},$$

the time taken for the capacitor to become fully discharged is $(2.5 \times 10^{-3}\,\text{s})/4 = 6.3 \times 10^{-4}\,\text{s}$.

d. The capacitor is completely charged at $t = 0$, so $q(0) = q_0$. Using [link], we obtain
Equation:

$$q(0) = q_0 = q_0 \cos \phi.$$

Thus, $\phi = 0$, and
Equation:

$$q(t) = (1.2 \times 10^{-5}\,\text{C})\cos(2.5 \times 10^{3} t).$$

Significance
The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.

Note:
Exercise:

Problem:

Check Your Understanding The angular frequency of the oscillations in an LC circuit is 2.0×10^3 rad/s. (a) If $L = 0.10\,\text{H}$, what is C? (b) Suppose that at $t = 0$, all the energy is stored in the inductor. What is the value of ϕ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

Solution:

a. $2.5\mu\text{F}$; b. $\pi/2\,\text{rad}$ or $3\pi/2\,\text{rad}$; c. $1.4 \times 10^3\,\text{rad/s}$

Summary

- The energy transferred in an oscillatory manner between the capacitor and inductor in an LC circuit occurs at an angular frequency $\omega = \sqrt{\frac{1}{LC}}$.
- The charge and current in the circuit are given by
  Equation:

$$\begin{aligned} q(t) &= q_0 \cos(\omega t + \phi), \\ i(t) &= -\omega q_0 \sin(\omega t + \phi). \end{aligned}$$

Conceptual Questions

Exercise:

Problem:

Do Kirchhoff's rules apply to circuits that contain inductors and capacitors?

Solution:

yes

Exercise:

Problem: Can a circuit element have both capacitance and inductance?

Exercise:

Problem:

In an *LC* circuit, what determines the frequency and the amplitude of the energy oscillations in either the inductor or capacitor?

Solution:

The amplitude of energy oscillations depend on the initial energy of the system. The frequency in a *LC* circuit depends on the values of inductance and capacitance.

Problems

Exercise:

Problem:

A 5000-pF capacitor is charged to 100 V and then quickly connected to an 80-mH inductor. Determine (a) the maximum energy stored in the magnetic field of the inductor, (b) the peak value of the current, and (c) the frequency of oscillation of the circuit.

Exercise:

Problem:

The self-inductance and capacitance of an *LC* circuit are 0.20 mH and 5.0 pF. What is the angular frequency at which the circuit oscillates?

Solution:

$\omega = 3.2 \times 10^7 \text{ rad/s}$

Exercise:

Problem:

What is the self-inductance of an *LC* circuit that oscillates at 60 Hz when the capacitance is 10 μF?

Exercise:

Problem:

In an oscillating *LC* circuit, the maximum charge on the capacitor is 2.0×10^{-6} C and the maximum current through the inductor is 8.0 mA. (a) What is the period of the oscillations? (b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

Solution:

a. 1.57×10^{-6} s; b. 3.93×10^{-7} s

Exercise:

Problem:

The self-inductance and capacitance of an oscillating *LC* circuit are $L = 20\,\text{mH}$ and $C = 1.0\,\mu\text{F}$, respectively. (a) What is the frequency of the oscillations? (b) If the maximum potential difference between the plates of the capacitor is 50 V, what is the maximum current in the circuit?

Exercise:

Problem:

In an oscillating *LC* circuit, the maximum charge on the capacitor is q_m. Determine the charge on the capacitor and the current through the inductor when energy is shared equally between the electric and magnetic fields. Express your answer in terms of q_m, *L*, and *C*.

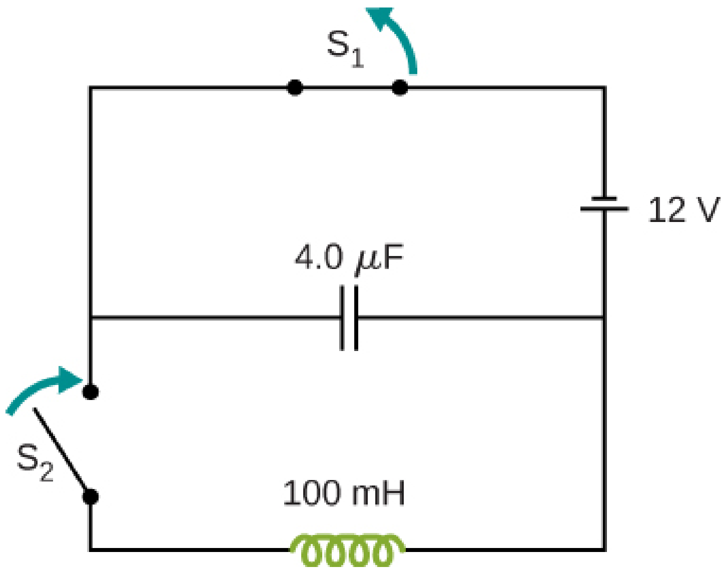
Solution:

$q = \frac{q_m}{\sqrt{2}}, I = \frac{q_m}{\sqrt{2LC}}$

Exercise:

Problem:

In the circuit shown below, S_1 is opened and S_2 is closed simultaneously. Determine (a) the frequency of the resulting oscillations, (b) the maximum charge on the capacitor, (c) the maximum current through the inductor, and (d) the electromagnetic energy of the oscillating circuit.



Exercise:

Problem:

An *LC* circuit in an AM tuner (in a car stereo) uses a coil with an inductance of 2.5 mH and a variable capacitor. If the natural frequency of the circuit is to be adjustable over the range 540 to 1600 kHz (the AM broadcast band), what range of capacitance is required?

Solution:

$$\begin{aligned} C &= \frac{1}{4\pi^2 f^2 L} \\ f_1 &= 540\ \text{Hz}; & C_1 &= 3.5 \times 10^{-11}\ \text{F} \\ f_2 &= 1600\ \text{Hz}; & C_2 &= 4.0 \times 10^{-12}\ \text{F} \end{aligned}$$

Glossary

LC circuit

circuit composed of an ac source, inductor, and capacitor

RLC Series Circuits

By the end of this section, you will be able to:

- Determine the angular frequency of oscillation for a resistor, inductor, capacitor (RLC) series circuit
- Relate the RLC circuit to a damped spring oscillation

When the switch is closed in the ***RLC* circuit** of [link](a), the capacitor begins to discharge and electromagnetic energy is dissipated by the resistor at a rate i^2R. With U given by [link], we have

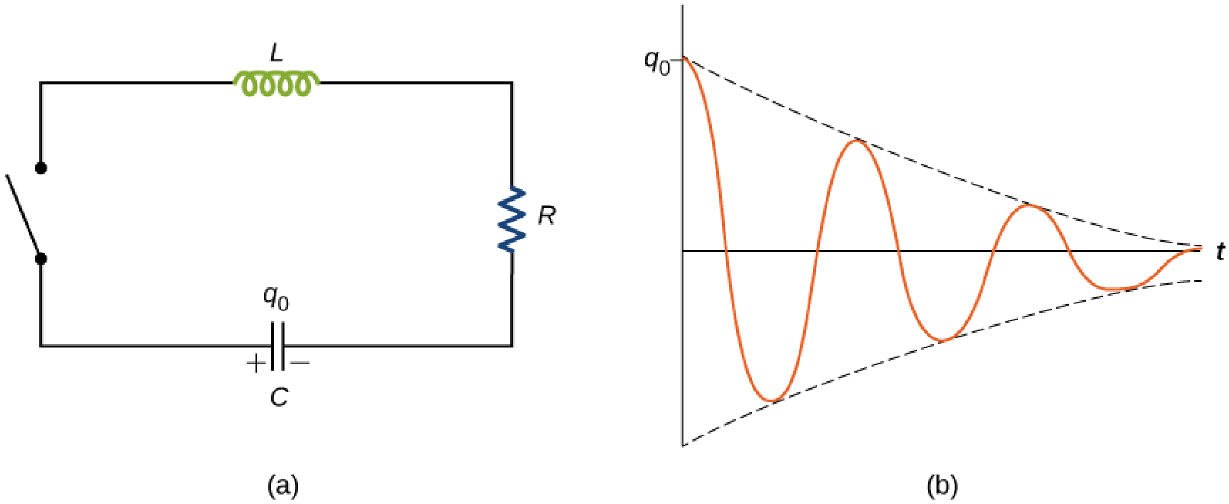
Equation:

$$\frac{dU}{dt} = \frac{q}{C}\,\frac{dq}{dt} + Li\frac{di}{dt} = -i^2R$$

where i and q are time-dependent functions. This reduces to

Equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0.$$



(a) An RLC circuit. Electromagnetic oscillations begin when the switch is closed. The capacitor is fully charged initially. (b) Damped oscillations of the capacitor charge are shown in this curve of charge

versus time, or q versus t. The capacitor contains a charge q_0 before the switch is closed.

This equation is analogous to

Equation:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0,$$

which is the equation of motion for a *damped mass-spring system* (you first encountered this equation in Oscillations). As we saw in that chapter, it can be shown that the solution to this differential equation takes three forms, depending on whether the angular frequency of the undamped spring is greater than, equal to, or less than $b/2m$. Therefore, the result can be underdamped $(\sqrt{k/m}>b/2m)$, critically damped $(\sqrt{k/m}=b/2m)$, or overdamped $(\sqrt{k/m}<b/2m)$. By analogy, the solution $q(t)$ to the *RLC* differential equation has the same feature. Here we look only at the case of under-damping. By replacing m by L, b by R, k by $1/C$, and x by q in [link], and assuming $\sqrt{1/LC}>R/2L$, we obtain

Note:

Equation:

$$q(t)=q_0e^{-Rt/2L}\cos(\omega\prime t+\phi)$$

where the angular frequency of the oscillations is given by

Note:

Equation:

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

This underdamped solution is shown in [link](b). Notice that the amplitude of the oscillations decreases as energy is dissipated in the resistor. [link] can be confirmed experimentally by measuring the voltage across the capacitor as a function of time. This voltage, multiplied by the capacitance of the capacitor, then gives $q(t)$.

Note:
Try an interactive circuit construction kit that allows you to graph current and voltage as a function of time. You can add inductors and capacitors to work with any combination of R, L, and C circuits with both dc and ac sources.

Note:
Try out a circuit-based java applet website that has many problems with both dc and ac sources that will help you practice circuit problems.

Note:
Exercise:

Problem:

Check Your Understanding In an *RLC* circuit, $L = 5.0\,\text{mH}, C = 6.0\mu\text{F},$ and $R = 200\,\Omega.$ (a) Is the circuit underdamped, critically damped, or overdamped? (b) If the circuit starts oscillating with a charge of $3.0\ \times\ 10^{-3}\,\text{C}$ on the capacitor, how much energy has been dissipated in the resistor by the time the oscillations cease?

Solution:

a. overdamped; b. 0.75 J

Summary

- The underdamped solution for the capacitor charge in an *RLC* circuit is
  Equation:

$$q(t) = q_0 e^{-Rt/2L} \cos(\omega' t + \phi).$$

- The angular frequency given in the underdamped solution for the *RLC* circuit is
  Equation:

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

Key Equations

Mutual inductance by flux	$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{N_1 \Phi_{12}}{I_2}$
Mutual inductance in circuits	$\varepsilon_1 = -M \frac{dI_2}{dt}$
Self-inductance in terms of magnetic flux	$N\Phi_{\mathrm{m}} = LI$
Self-inductance in terms of emf	$\varepsilon = -L \frac{dI}{dt}$
Self-inductance of a solenoid	$L_{\text{solenoid}} = \frac{\mu_0 N^2 A}{l}$
Self-inductance of a toroid	$L_{\text{toroid}} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}.$
Energy stored in an inductor	$U = \frac{1}{2} L I^2$
Current as a function of time for a *RL* circuit	$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau_L}\right)$
Time constant for a *RL* circuit	$\tau_L = L/R$
Charge oscillation in *LC* circuits	$q\left(t\right) = q_0 \cos(\omega t + \phi)$
Angular frequency in *LC* circuits	$\omega = \sqrt{\frac{1}{LC}}$
Current oscillations in *LC* circuits	$i(t) = -\omega q_0 \sin(\omega t + \phi)$
Charge as a function of time in *RLC* circuit	$q(t) = q_0 e^{-Rt/2L} \cos(\omega\prime t + \phi)$
Angular frequency in *RLC* circuit	$\omega\prime = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

Conceptual Questions

Exercise:

Problem:

When a wire is connected between the two ends of a solenoid, the resulting circuit can oscillate like an *RLC* circuit. Describe what causes the capacitance in this circuit.

Exercise:

Problem:

Describe what effect the resistance of the connecting wires has on an oscillating *LC* circuit.

Solution:

This creates an *RLC* circuit that dissipates energy, causing oscillations to decrease in amplitude slowly or quickly depending on the value of resistance.

Exercise:

Problem:

Suppose you wanted to design an *LC* circuit with a frequency of 0.01 Hz. What problems might you encounter?

Exercise:

Problem:

A radio receiver uses an *RLC* circuit to pick out particular frequencies to listen to in your house or car without hearing other unwanted frequencies. How would someone design such a circuit?

Solution:

You would have to pick out a resistance that is small enough so that only one station at a time is picked up, but big enough so that the tuner

doesn’t have to be set at exactly the correct frequency. The inductance or capacitance would have to be varied to tune into the station however practically speaking, variable capacitors are a lot easier to build in a circuit.

Problems

Exercise:

Problem:

In an oscillating *RLC* circuit, $R = 5.0\,\Omega, L = 5.0\,\text{mH}, \text{ and } C = 500\,\mu\text{F}$. What is the angular frequency of the oscillations?

Exercise:

Problem:

In an oscillating *RLC* circuit with $L = 10\,\text{mH}, C = 1.5\,\mu\text{F}, \text{ and } R = 2.0\,\Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

Solution:

6.9 ms

Exercise:

Problem:

What resistance *R* must be connected in series with a 200-mH inductor of the resulting *RLC* oscillating circuit is to decay to 50% of its initial value of charge in 50 cycles? To 0.10% of its initial value in 50 cycles?

Additional Problems

Exercise:

Problem:

Show that the self-inductance per unit length of an infinite, straight, thin wire is infinite.

Solution:

Let *a* equal the radius of the long, thin wire, *r* the location where the magnetic field is measured, and *R* the upper limit of the problem where we will take *R* as it approaches infinity.

$$\text{Outside,} \quad B = \frac{\mu_0 I}{2\pi r} \quad \text{Inside,} \quad B = \frac{\mu_0 I r}{2\pi a^2}$$

proof

$$U = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{1}{4} + \ln\frac{R}{a}\right)$$

$$\text{So,} \quad \frac{2U}{I^2} = \frac{\mu_0 l}{2\pi}\left(\frac{1}{4} + \ln\frac{R}{a}\right) \text{ and } \quad L = \infty$$

Exercise:

Problem:

Two long, parallel wires carry equal currents in opposite directions. The radius of each wire is *a*, and the distance between the centers of the wires is *d*. Show that if the magnetic flux within the wires themselves can be ignored, the self-inductance of a length *l* of such a pair of wires is

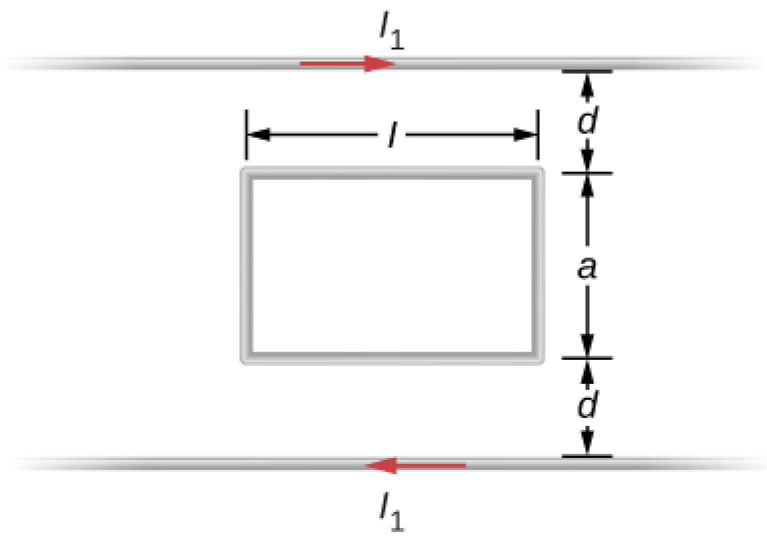
$$L = \frac{\mu_0 l}{\pi} \ln \frac{d-a}{a}.$$

(*Hint*: Calculate the magnetic flux through a rectangle of length *l* between the wires and then use $L = N\Phi/I$.)

Exercise:

Problem:

A small, rectangular single loop of wire with dimensions *l*, and *a* is placed, as shown below, in the plane of a much larger, rectangular single loop of wire. The two short sides of the larger loop are so far from the smaller loop that their magnetic fields over the smaller fields over the smaller loop can be ignored. What is the mutual inductance of the two loops?



Solution:

$$M = \frac{\mu_0 l}{\pi} \ln \frac{d+a}{d}$$

Exercise:

Problem:

Suppose that a cylindrical solenoid is wrapped around a core of iron whose magnetic susceptibility is x. Using [link], show that the self-inductance of the solenoid is given by

$$L = \frac{(1+x)\mu_0 N^2 A}{l},$$

where *l* is its length, *A* its cross-sectional area, and *N* its total number of turns.

Exercise:

Problem:

A solenoid with 4 x 10^7 turns/m has an iron core placed in it whose magnetic susceptibility is 4.0×10^3. (a) If a current of 2.0 A flows through the solenoid, what is the magnetic field in the iron core? (b) What is the effective surface current formed by the aligned atomic current loops in the iron core? (c) What is the self-inductance of the filled solenoid?

Solution:

a. 100 T; b. 2 A; c. 0.50 H

Exercise:

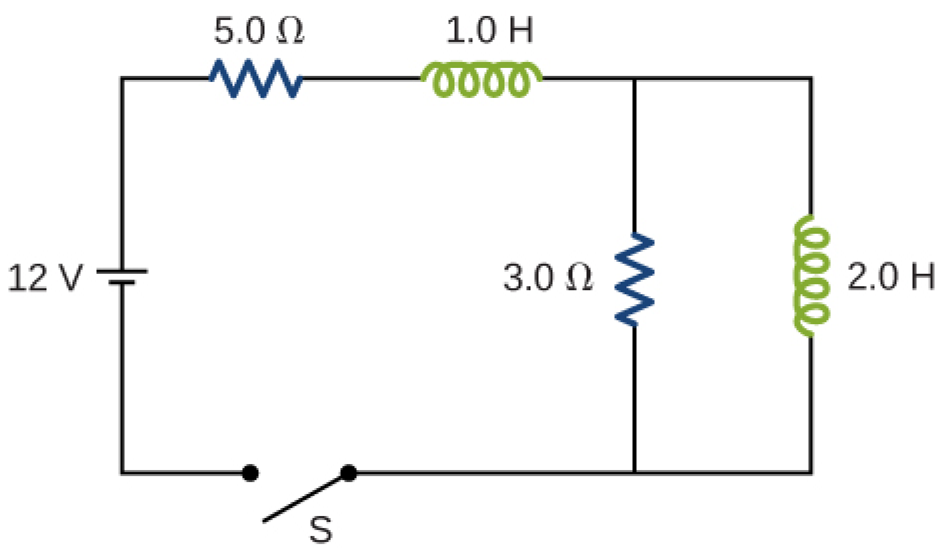
Problem:

A rectangular toroid with inner radius $R_1 = 7.0\,\text{cm}$, outer radius $R_2 = 9.0\,\text{cm}$, height $h = 3.0$, and $N = 3000$ turns is filled with an iron core of magnetic susceptibility 5.2×10^3. (a) What is the self-inductance of the toroid? (b) If the current through the toroid is 2.0 A, what is the magnetic field at the center of the core? (c) For this same 2.0-A current, what is the effective surface current formed by the aligned atomic current loops in the iron core?

Exercise:

Problem:

The switch S of the circuit shown below is closed at $t = 0$. Determine (a) the initial current through the battery and (b) the steady-state current through the battery.



Solution:

a. 0 A; b. 2.4 A

Exercise:

Problem:

In an oscillating *RLC* circuit, $R = 7.0\,\Omega, L = 10\,\text{mH},$ and $C = 3.0\,\mu\text{F}$. Initially, the capacitor has a charge of $8.0\,\mu\text{C}$ and the current is zero. Calculate the charge on the capacitor (a) five cycles later and (b) 50 cycles later.

Exercise:

Problem:

A 25.0-H inductor has 100 A of current turned off in 1.00 ms. (a) What voltage is induced to oppose this? (b) What is unreasonable about this result? (c) Which assumption or premise is responsible?

Solution:

a. 2.50×10^6 V; (b) The voltage is so extremely high that arcing would occur and the current would not be reduced so rapidly. (c) It is

not reasonable to shut off such a large current in such a large inductor in such an extremely short time.

Challenge Problems

Exercise:

Problem:

A coaxial cable has an inner conductor of radius a, and outer thin cylindrical shell of radius b. A current *I* flows in the inner conductor and returns in the outer conductor. The self-inductance of the structure will depend on how the current in the inner cylinder tends to be distributed. Investigate the following two extreme cases. (a) Let current in the inner conductor be distributed only on the surface and find the self-inductance. (b) Let current in the inner cylinder be distributed uniformly over its cross-section and find the self-inductance. Compare with your results in (a).

Exercise:

Problem:

In a damped oscillating circuit the energy is dissipated in the resistor. The Q-factor is a measure of the persistence of the oscillator against the dissipative loss. (a) Prove that for a lightly damped circuit the energy, U, in the circuit decreases according to the following equation.

$\frac{dU}{dt} = -2\beta U$, where $\beta = \frac{R}{2L}$.

(b) Using the definition of the Q-factor as energy divided by the loss over the next cycle, prove that Q-factor of a lightly damped oscillator as defined in this problem is

$$Q \equiv \frac{U_{\text{begin}}}{\Delta U_{\text{one cycle}}} = \frac{1}{2\pi R}\sqrt{\frac{L}{C}}.$$

(*Hint:* For (b), to obtain Q, divide E at the beginning of one cycle by the change ΔE over the next cycle.)

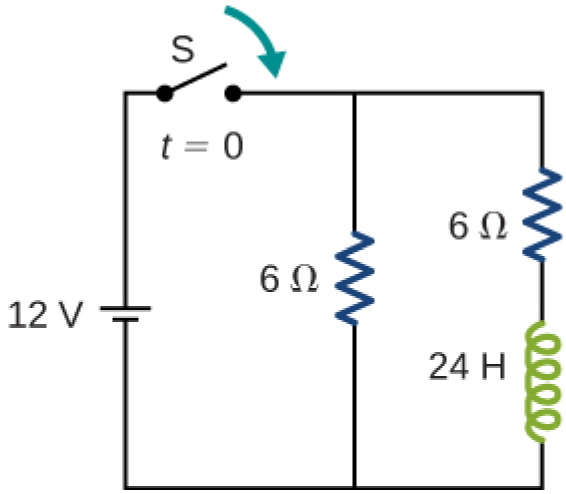
Solution:

proof

Exercise:

Problem:

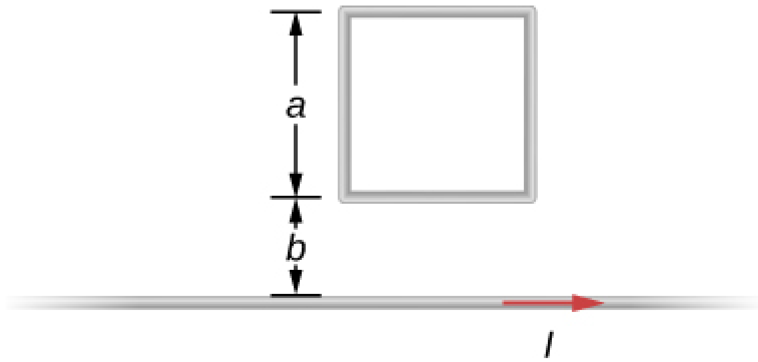
The switch in the circuit shown below is closed at $t = 0$ s. Find currents through (a) R_1, (b) R_2, and (c) the battery as function of time.



Exercise:

Problem:

A square loop of side 2 cm is placed 1 cm from a long wire carrying a current that varies with time at a constant rate of 3 A/s as shown below. (a) Use Ampère's law and find the magnetic field. (b) Determine the magnetic flux through the loop. (c) If the loop has a resistance of $3\,\Omega$, how much induced current flows in the loop?



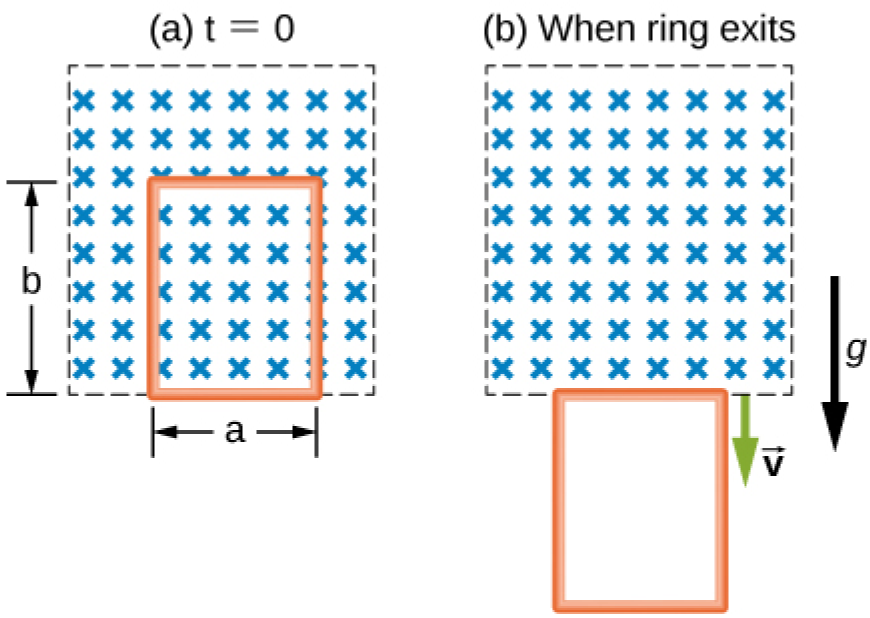
Solution:

a. $\frac{dB}{dt} = 6 \times 10^{-6}\,\mathrm{T/s}$; b. $\Phi = \frac{\mu_0 a I}{2\pi} \ln\left(\frac{a+b}{b}\right)$; c. 4.4 nA

Exercise:

Problem:

A rectangular copper ring, of mass 100 g and resistance $0.2\,\Omega$, is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface. The ring is let go from rest when it is at the edge of the nonzero magnetic field region (see below). (a) Find its speed when the ring just exits the region of uniform magnetic field. (b) If it was let go at $t = 0$, what is the time when it exits the region of magnetic field for the following values: $a = 25\,\text{cm},\ b = 50\,\text{cm},\ B = 3\,\text{T},$ and $g = 9.8\,\text{m/s}^2$? Assume the magnetic field of the induced current is negligible compared to 3 T.



Glossary

RLC circuit
: circuit with an ac source, resistor, inductor, and capacitor all in series.

Units

Quantity	**Common Symbol**	**Unit**	**Unit in Terms of Base SI Units**
Acceleration	$\vec{\mathbf{a}}$	m/s^2	m/s^2
Amount of substance	n	**mole**	mol
Angle	θ, ϕ	radian (rad)	
Angular acceleration	$\vec{\alpha}$	rad/s^2	s^{-2}
Angular frequency	ω	rad/s	s^{-1}
Angular momentum	$\vec{\mathbf{L}}$	$\text{kg} \cdot \text{m}^2/\text{s}$	$\text{kg} \cdot \text{m}^2/\text{s}$
Angular velocity	$\vec{\omega}$	rad/s	s^{-1}
Area	A	m^2	m^2
Atomic number	Z		
Capacitance	C	farad (F)	$\text{A}^2 \cdot \text{s}^4/\text{kg} \cdot \text{m}^2$
Charge	q, Q, e	coulomb (C)	$\text{A} \cdot \text{s}$
Charge density:			
Line	λ	C/m	$\text{A} \cdot \text{s}/\text{m}$
Surface	σ	C/m^2	$\text{A} \cdot \text{s}/\text{m}^2$
Volume	ρ	C/m^3	$\text{A} \cdot \text{s}/\text{m}^3$

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Conductivity	σ	$1/\Omega \cdot \mathrm{m}$	$\mathrm{A}^2 \cdot \mathrm{s}^3/\mathrm{kg} \cdot \mathrm{m}^3$
Current	I	**ampere**	A
Current density	$\vec{\mathbf{J}}$	$\mathrm{A/m^2}$	$\mathrm{A/m^2}$
Density	ρ	$\mathrm{kg/m^3}$	$\mathrm{kg/m^3}$
Dielectric constant	κ		
Electric dipole moment	$\vec{\mathbf{p}}$	$\mathrm{C} \cdot \mathrm{m}$	$\mathrm{A} \cdot \mathrm{s} \cdot \mathrm{m}$
Electric field	$\vec{\mathbf{E}}$	N/C	$\mathrm{kg} \cdot \mathrm{m}/\mathrm{A} \cdot \mathrm{s}^3$
Electric flux	Φ	$\mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}$	$\mathrm{kg} \cdot \mathrm{m}^3/\mathrm{A} \cdot \mathrm{s}^3$
Electromotive force	ε	volt (V)	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{A} \cdot \mathrm{s}^3$
Energy	E,U,K	joule (J)	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$
Entropy	S	J/K	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2 \cdot \mathrm{K}$
Force	$\vec{\mathbf{F}}$	newton (N)	$\mathrm{kg} \cdot \mathrm{m}/\mathrm{s}^2$
Frequency	f	hertz (Hz)	s^{-1}
Heat	Q	joule (J)	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$
Inductance	L	henry (H)	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{A}^2 \cdot \mathrm{s}^2$
Length:	ℓ, L	**meter**	m
Displacement	$\Delta x, \Delta \vec{\mathbf{r}}$		
Distance	d, h		
Position	$x, y, z, \vec{\mathbf{r}}$		

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Magnetic dipole moment	$\vec{\mu}$	$\mathrm{N \cdot J/T}$	$\mathrm{A \cdot m^2}$
Magnetic field	$\vec{\mathbf{B}}$	$\mathrm{tesla\,(T)} = \left(\mathrm{Wb/m^2}\right)$	$\mathrm{kg/A \cdot s^2}$
Magnetic flux	Φ_m	weber (Wb)	$\mathrm{kg \cdot m^2/A \cdot s^2}$
Mass	*m*, *M*	**kilogram**	kg
Molar specific heat	*C*	$\mathrm{J/mol \cdot K}$	$\mathrm{kg \cdot m^2/s^2 \cdot mol \cdot K}$
Moment of inertia	*I*	$\mathrm{kg \cdot m^2}$	$\mathrm{kg \cdot m^2}$
Momentum	$\vec{\mathbf{p}}$	$\mathrm{kg \cdot m/s}$	$\mathrm{kg \cdot m/s}$
Period	*T*	s	s
Permeability of free space	μ_0	$\mathrm{N/A^2} = (\mathrm{H/m})$	$\mathrm{kg \cdot m/A^2 \cdot s^2}$
Permittivity of free space	ε_0	$\mathrm{C^2/N \cdot m^2} = (\mathrm{F/m})$	$\mathrm{A^2 \cdot s^4/kg \cdot m^3}$
Potential	*V*	$\mathrm{volt\,(V)} = (\mathrm{J/C})$	$\mathrm{kg \cdot m^2/A \cdot s^3}$
Power	*P*	$\mathrm{watt\,(W)} = (\mathrm{J/s})$	$\mathrm{kg \cdot m^2/s^3}$
Pressure	*p*	$\mathrm{pascal\,(Pa)} = \left(\mathrm{N/m^2}\right)$	$\mathrm{kg/m \cdot s^2}$
Resistance	*R*	$\mathrm{ohm\,(\Omega)} = (\mathrm{V/A})$	$\mathrm{kg \cdot m^2/A^2 \cdot s^3}$
Specific heat	*c*	$\mathrm{J/kg \cdot K}$	$\mathrm{m^2/s^2 \cdot K}$
Speed	*v*	m/s	m/s
Temperature	*T*	**kelvin**	K
Time	*t*	**second**	s

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Torque	$\vec{\tau}$	$\mathrm{N \cdot m}$	$\mathrm{kg \cdot m^2/s^2}$
Velocity	$\vec{\mathbf{v}}$	m/s	m/s
Volume	V	m^3	m^3
Wavelength	λ	m	m
Work	W	$\text{joule}\,(\mathrm{J}) = (\mathrm{N \cdot m})$	$\mathrm{kg \cdot m^2/s^2}$

Units Used in Physics (Fundamental units in bold)

Conversion Factors

	m	cm	km
1 meter	1	10^2	10^{-3}
1 centimeter	10^{-2}	1	10^{-5}
1 kilometer	10^3	10^5	1
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}
1 foot	0.3048	30.48	3.048×10^{-4}
1 mile	1609	1.609×10^4	1.609
1 angstrom	10^{-10}		
1 fermi	10^{-15}		
1 light-year			9.460×10^{12}
	in.	**ft**	**mi**
1 meter	39.37	3.281	6.214×10^{-4}
1 centimeter	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	3.937×10^4	3.281×10^3	0.6214
1 inch	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	12	1	1.894×10^{-4}
1 mile	6.336×10^4	5280	1

Length

Area

$1\ \text{cm}^2 = 0.155\ \text{in.}^2$

$1\ \text{m}^2 = 10^4\ \text{cm}^2 = 10.76\ \text{ft}^2$

$1\ \text{in.}^2 = 6.452\ \text{cm}^2$

$1\ \text{ft}^2 = 144\ \text{in.}^2 = 0.0929\ \text{m}^2$

Volume

$1\ \text{liter} = 1000\ \text{cm}^3 = 10^{-3}\ \text{m}^3 = 0.03531\ \text{ft}^3 = 61.02\ \text{in.}^3$

$1 \text{ ft}^3 = 0.02832 \text{ m}^3 = 28.32 \text{ liters} = 7.477 \text{ gallons}$

$1 \text{ gallon} = 3.788 \text{ liters}$

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.944×10^{-4}	1.901×10^{-6}
1 hour	3600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^{4}	1440	24	1	2.738×10^{-3}
1 year	3.156×10^{7}	5.259×10^{5}	8.766×10^{3}	365.25	1

Time

	m/s	cm/s	ft/s	mi/h
1 meter/second	1	10^{2}	3.281	2.237
1 centimeter/second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot/second	0.3048	30.48	1	0.6818
1 mile/hour	0.4470	44.70	1.467	1

Speed

Acceleration

$1 \text{ m/s}^2 = 100 \text{ cm/s}^2 = 3.281 \text{ ft/s}^2$

$1 \text{ cm/s}^2 = 0.01 \text{ m/s}^2 = 0.03281 \text{ ft/s}^2$

$1 \text{ ft/s}^2 = 0.3048 \text{ m/s}^2 = 30.48 \text{ cm/s}^2$

$1 \text{ mi/h} \cdot \text{s} = 1.467 \text{ ft/s}^2$

	kg	**g**	**slug**	**u**
1 kilogram	1	10^3	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.661×10^{-27}	1.661×10^{-24}	1.138×10^{-28}	1
1 metric ton	1000			

Mass

	N	**dyne**	**lb**
1 newton	1	10^5	0.2248
1 dyne	10^{-5}	1	2.248×10^{-6}
1 pound	4.448	4.448×10^5	1

Force

	Pa	**dyne/cm^2**	**atm**	**cmHg**	**lb/in.^{2}**
1 pascal	1	10	9.869×10^{-6}	7.501×10^{-4}	1.450×1
1 dyne/centimeter^2	10^{-1}	1	9.869×10^{-7}	7.501×10^{-5}	1.450×1
1 atmosphere	1.013×10^5	1.013×10^6	1	76	14.70
1 centimeter mercury*	1.333×10^3	1.333×10^4	1.316×10^{-2}	1	0.1934
1 pound/inch^2	6.895×10^3	6.895×10^4	6.805×10^{-2}	5.171	1
1 bar	10^5				
1 torr				1 (mmHg)	
*Where the acceleration due to gravity is $9.80665\ \mathrm{m/s^2}$ and the temperature is $0\,^\circ\mathrm{C}$					

Pressure

	J	erg	ft.lb
1 joule	1	10^7	0.7376
1 erg	10^{-7}	1	7.376×10^{-8}
1 foot-pound	1.356	1.356×10^7	1
1 electron-volt	1.602×10^{-19}	1.602×10^{-12}	1.182×10^{-19}
1 calorie	4.186	4.186×10^7	3.088
1 British thermal unit	1.055×10^3	1.055×10^{10}	7.779×10^2
1 kilowatt-hour	3.600×10^6		
	eV	**cal**	**Btu**
1 joule	6.242×10^{18}	0.2389	9.481×10^{-4}
1 erg	6.242×10^{11}	2.389×10^{-8}	9.481×10^{-11}
1 foot-pound	8.464×10^{18}	0.3239	1.285×10^{-3}
1 electron-volt	1	3.827×10^{-20}	1.519×10^{-22}
1 calorie	2.613×10^{19}	1	3.968×10^{-3}
1 British thermal unit	6.585×10^{21}	2.520×10^2	1

Work, Energy, Heat

Power

$1\ \text{W} = 1\ \text{J/s}$

$1\ \text{hp} = 746\ \text{W} = 550\ \text{ft} \cdot \text{lb/s}$

$1\ \text{Btu/h} = 0.293\ \text{W}$

Angle

$1\ \text{rad} = 57.30^\circ = 180^\circ/\pi$

$1^\circ = 0.01745\ \text{rad} = \pi/180\ \text{rad}$

$1\ \text{revolution} = 360^\circ = 2\pi\ \text{rad}$

$1\ \text{rev/min (rpm)} = 0.1047\ \text{rad/s}$

Fundamental Constants

Quantity	Symbol	Value
Atomic mass unit	u	$1.660\ 538\ 782\ (83)\ \times\ 10^{-27}\ \mathrm{kg}$ $931.494\ 028\ (23)\ \mathrm{MeV}/c^2$
Avogadro's number	N_A	$6.02214076\ \times\ 10^{23}$ reciprocal mole$\left(\mathrm{mol}^{-1}\right)$
Bohr magneton	$\mu_\mathrm{B} = \frac{e\hbar}{2m_e}$	$9.274\ 009\ 15\ (23)\ \times\ 10^{-24}\ \mathrm{J/T}$
Bohr radius	$a_0 = \frac{\hbar^2}{m_e e^2 k_e}$	$5.291\ 772\ 085\ 9\ (36)\ \times\ 10^{-11}\ \mathrm{m}$
Boltzmann's constant	$k_\mathrm{B} = \frac{R}{N_\mathrm{A}}$	$1.380649\ \times\ 10^{-23}$ joule per kelvin$\left(\mathrm{J} \cdot \mathrm{K}^{-1}\right)$
Compton wavelength	$\lambda_\mathrm{C} = \frac{h}{m_e c}$	$2.426\ 310\ 217\ 5\ (33)\ \times\ 10^{-12}\ \mathrm{m}$
Coulomb constant	$k_e = \frac{1}{4\pi\varepsilon_0}$	$8.987\ 551\ 788...\ \times\ 10^9\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}^2$ (exact)
Deuteron mass	m_d	$3.343\ 583\ 20\ (17)\ \times\ 10^{-27}\ \mathrm{kg}$ $2.013\ 553\ 212\ 724\ (78)\ \mathrm{u}$ $1875.612\ 859\ \mathrm{MeV}/c^2$
Electron mass	m_e	$9.109\ 382\ 15\ (45)\ \times\ 10^{-31}\ \mathrm{kg}$ $5.485\ 799\ 094\ 3\ (23)\ \times\ 10^{-4}\ \mathrm{u}$ $0.510\ 998\ 910\ (13)\ \mathrm{MeV}/c^2$
Electron volt	eV	$1.602\ 176\ 487\ (40)\ \times\ 10^{-19}\ \mathrm{J}$

Quantity	Symbol	Value
Elementary charge	e	$1.602176634 \times 10^{-19}\,\text{C}$
Gas constant	R	$8.314\,472\ (15)\ \text{J/mol}\cdot\text{K}$
Gravitational constant	G	$6.674\,28\ (67) \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$
Neutron mass	m_n	$1.674\,927\,211\ (84) \times 10^{-27}\,\text{kg}$ $1.008\,664\,915\,97\ (43)\ \text{u}$ $939.565\,346\ (23)\ \text{MeV}/c^2$
Nuclear magneton	$\mu_n = \frac{e\hbar}{2m_p}$	$5.050\,783\,24\ (13) \times 10^{-27}\,\text{J/T}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A}\ (\text{exact})$
Permittivity of free space	$\varepsilon_0 = \frac{1}{\mu_0 c^2}$	$8.854\,187\,817... \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\ (\text{exact})$
Planck’s constant	h $\hbar = \frac{h}{2\pi}$	$6.62607015 \times 10^{-34}\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}$ $1.05457182 \times 10^{-34}\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}$
Proton mass	m_p	$1.672\,621\,637\ (83) \times 10^{-27}\,\text{kg}$ $1.007\,276\,466\,77\ (10)\ \text{u}$ $938.272\,013\ (23)\ \text{MeV}/c^2$
Rydberg constant	R_{H}	$1.097\,373\,156\,852\,7\ (73) \times 10^{7}\,\text{m}^{-1}$
Speed of light in vacuum	c	$2.997\,924\,58 \times 10^{8}\,\text{m/s}\ (\text{exact})$

Fundamental Constants*Note:* These constants are the values recommended in 2006 by CODATA, based on a least-squares adjustment of data from different measurements. The numbers in parentheses for the values represent the uncertainties of the last two digits.

Useful combinations of constants for calculations:

$hc = 12{,}400\ \mathrm{eV \cdot Å} = 1240\ \mathrm{eV \cdot nm} = 1240\ \mathrm{MeV \cdot fm}$

$\hbar c = 1973\ \mathrm{eV \cdot Å} = 197.3\ \mathrm{eV \cdot nm} = 197.3\ \mathrm{MeV \cdot fm}$

$k_e e^2 = 14.40\ \mathrm{eV \cdot Å} = 1.440\ \mathrm{eV \cdot nm} = 1.440\ \mathrm{MeV \cdot fm}$

$k_\mathrm{B}T = 0.02585\ \mathrm{eV}$ at $T = 300\ \mathrm{K}$

Astronomical Data

Celestial Object	**Mean Distance from Sun (million km)**	**Period of Revolution (d = days) (y = years)**	**Period of Rotation at Equator**	**Eccentricity of Orbit**
Sun	–	–	27 d	–
Mercury	57.9	88 d	59 d	0.206
Venus	108.2	224.7 d	243 d	0.007
Earth	149.6	365.26 d	23 h 56 min 4 s	0.017
Mars	227.9	687 d	24 h 37 min 23 s	0.093
Jupiter	778.4	11.9 y	9 h 50 min 30 s	0.048
Saturn	1426.7	29.5 6	10 h 14 min	0.054
Uranus	2871.0	84.0 y	17 h 14 min	0.047
Neptune	4498.3	164.8 y	16 h	0.009

Celestial Object	**Mean Distance from Sun (million km)**	**Period of Revolution (d = days) (y = years)**	**Period of Rotation at Equator**	**Eccentricity of Orbit**
Earth’s Moon	149.6 (0.386 from Earth)	27.3 d	27.3 d	0.055
Celestial Object	**Equatorial Diameter (km)**	**Mass (Earth = 1)**	**Density (g/cm^3)**	
Sun	1,392,000	333,000.00	1.4	
Mercury	4879	0.06	5.4	
Venus	12,104	0.82	5.2	
Earth	12,756	1.00	5.5	
Mars	6794	0.11	3.9	
Jupiter	142,984	317.83	1.3	
Saturn	120,536	95.16	0.7	
Uranus	51,118	14.54	1.3	
Neptune	49,528	17.15	1.6	
Earth’s Moon	3476	0.01	3.3	

Astronomical Data

Other Data:

Mass of Earth: 5.97×10^{24} kg

Mass of the Moon: 7.36×10^{22} kg

Mass of the Sun: 1.99×10^{30} kg

Mathematical Formulas

Quadratic formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Triangle of base b and height h	**Area** $= \frac{1}{2}bh$	
Circle of radius r	Circumference $= 2\pi r$	Area $= \pi r^2$
Sphere of radius r	Surface area $= 4\pi r^2$	Volume $= \frac{4}{3}\pi r^3$
Cylinder of radius r and height h	Area of curved surface $= 2\pi rh$	Volume $= \pi r^2 h$

Geometry

Trigonometry

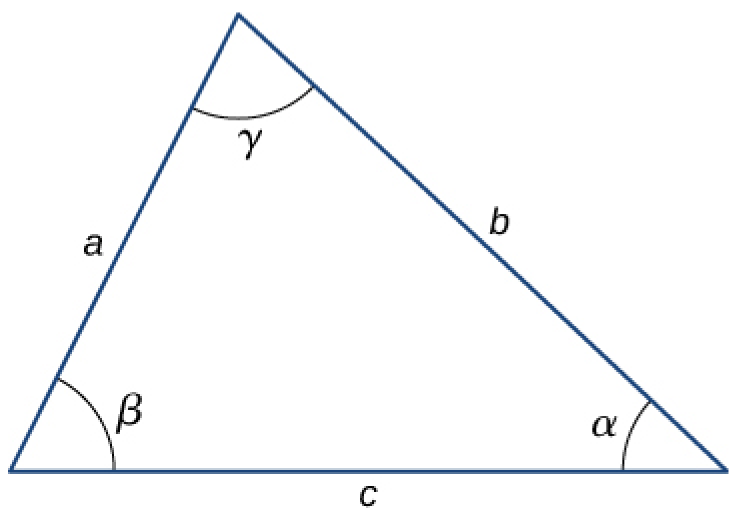
Trigonometric Identities

1. $\sin\theta = 1/\csc\theta$
2. $\cos\theta = 1/\sec\theta$
3. $\tan\theta = 1/\cot\theta$
4. $\sin\left(90^0 - \theta\right) = \cos\theta$
5. $\cos\left(90^0 - \theta\right) = \sin\theta$
6. $\tan\left(90^0 - \theta\right) = \cot\theta$
7. $\sin^2\theta + \cos^2\theta = 1$
8. $\sec^2\theta - \tan^2\theta = 1$

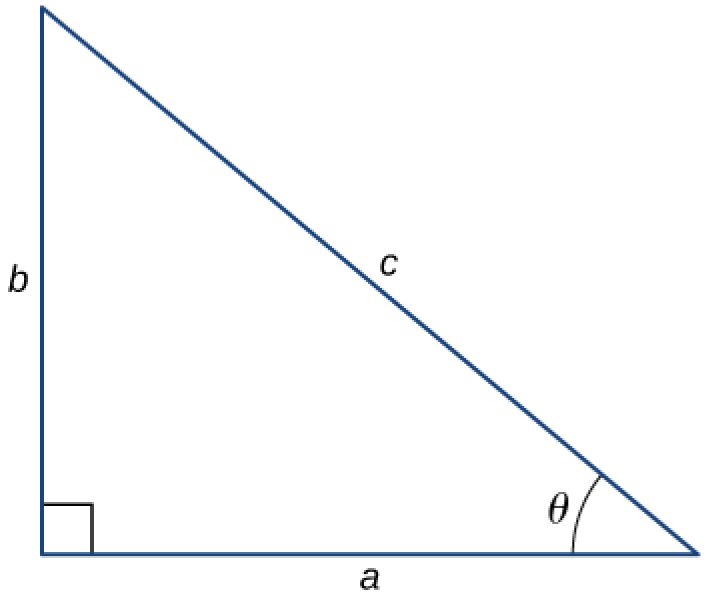
9. $\tan\theta = \sin\theta/\cos\theta$
10. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
11. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
12. $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
13. $\sin 2\theta = 2\sin\theta\cos\theta$
14. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
15. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
16. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

Triangles

1. Law of sines: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$
2. Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$



3. Pythagorean theorem: $a^2 + b^2 = c^2$



Series expansions

1. Binomial theorem:
   $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots$
2. $(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \cdots \left(x^2 < 1\right)$
3. $(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} \mp \cdots \left(x^2 < 1\right)$
4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$
7. $e^x = 1 + x + \frac{x^2}{2!} + \cdots$
8. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots (|x| < 1)$

Derivatives

1. $\frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x)$
2. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
3. $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
4. $\frac{d}{dx}f(u) = \left[\frac{d}{du}f(u)\right]\frac{du}{dx}$
5. $\frac{d}{dx}x^m = mx^{m-1}$
6. $\frac{d}{dx}\sin x = \cos x$

7. $\frac{d}{dx}\cos x = -\sin x$
8. $\frac{d}{dx}\tan x = \sec^2 x$
9. $\frac{d}{dx}\cot x = -\csc^2 x$
10. $\frac{d}{dx}\sec x = \tan x \sec x$
11. $\frac{d}{dx}\csc x = -\cot x \csc x$
12. $\frac{d}{dx}e^x = e^x$
13. $\frac{d}{dx}\ln x = \frac{1}{x}$
14. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\tan^{-1} x = -\frac{1}{1+x^2}$

Integrals

1. $\int af(x)dx = a\int f(x)dx$
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
3. $\begin{aligned}\int x^m dx &= \frac{x^{m+1}}{m+1}\ (m \neq -1) \\ &= \ln x (m = -1)\end{aligned}$
4. $\int \sin x\ dx = -\cos x$
5. $\int \cos x\ dx = \sin x$
6. $\int \tan x\ dx = \ln|\sec x|$
7. $\int \sin^2 ax\ dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$
8. $\int \cos^2 ax\ dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$
9. $\int \sin ax \cos ax\ dx = -\frac{\cos 2ax}{4a}$

10. $\int e^{ax}\,dx = \frac{1}{a}e^{ax}$

11. $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$

12. $\int \ln ax\,dx = x\ln ax - x$

13. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

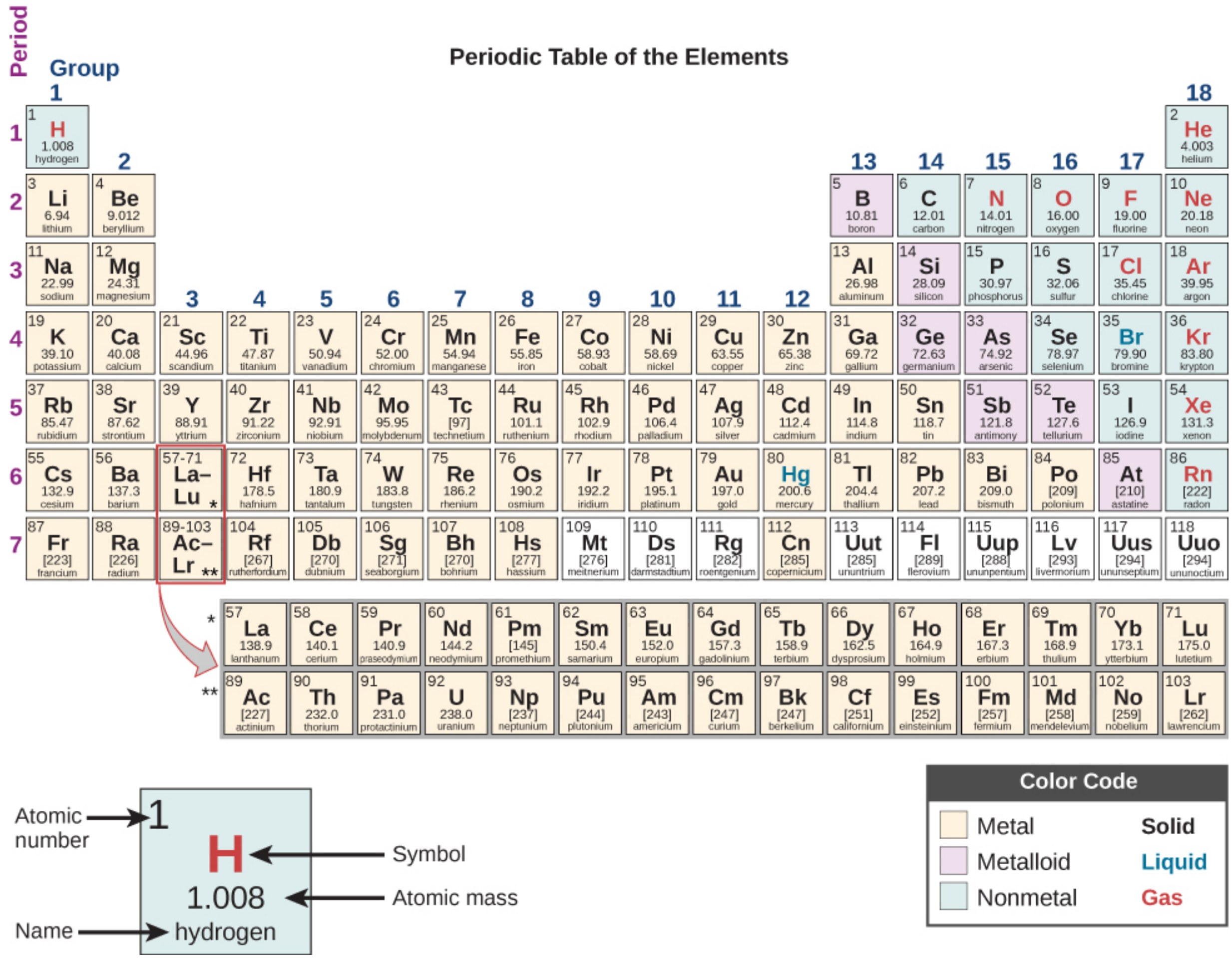
14. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$

15. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$

16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

17. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

18. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$


Periodic Table of the Elements
Period
Group
Atomic number
Symbol
Atomic mass
Name
1
H
1.008
hydrogen
Color Code
Metal
Metalloid
Nonmetal
Solid
Liquid
Gas

The Greek Alphabet

Name	**Capital**	**Lowercase**	**Name**	**Capital**	**Lowercase**
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi		ξ
Gamma		γ	Omicron	O	o
Delta		δ	Pi		π
Epsilon	E	ε	Rho	P	ρ
Zeta	Z	ζ	Sigma		σ
Eta	H	η	Tau	T	τ
Theta		θ	Upsilon	Υ	υ
lota	I	ι	Phi		ϕ
Kappa	K	κ	Chi	X	χ
Lambda		λ	Psi	ψ	ψ
Mu	M	μ	Omega		ω

The Greek Alphabet